Chapter 7

Linear Equations and Matrices

Topics :

1. Systems of linear equations
2. Matrices and Gaussian elimination
3. Matrix operations

Matrices are used to represent a variety of mathematical concepts. Matrices can be added and multiplied in ways similar to the ways in which numbers are added and multiplied, and these operations with matrices have far-reaching applications. For instance, matrices enable us to solve systems of linear equations and to do other computational problems in a *fast* and *efficient* manner.

7.1 Systems of linear equations

One of the most frequently recurring practical problems in many fields of study is that of solving a system of linear equations.

7.1.1 Definition. An equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ are (usually) real numbers, is called a **linear equation** in the **variables** (or **unknowns**) $x_1, x_2, \ldots, x_n$.

A **solution** to such a linear equation is an n-tuple of numbers $(s_1, s_2, \ldots, s_n)$ which *satisfy* the given equation.

7.1.2 Example. The triplet $(2, 3, -4)$ is a solution to the linear equation

$$6x_1 - 3x_2 + 4x_3 = -13$$

because

$$6 \cdot 2 - 3 \cdot 3 + 4 \cdot (-4) = -13.$$

7.1.3 Definition. A **system of m linear equations in n unknowns** (or, simply, a **linear system**) is a set of m linear equations in n unknowns.

A linear system can conveniently be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m\,. \end{cases}$$

Thus the ith equation is

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i.$$

A **solution** to such a linear system is an n-tuple of numbers $(s_1, s_2, \ldots, s_n)$ which *satisfy* each equation of the system.

7.1.4 Definition. If a linear system has no solution, it is said to be **inconsistent**. If it has a solution, it is called **consistent**.

If $b_1 = b_2 = \cdots = b_m = 0$, then the linear system is called **homogeneous**. The solution $x_1 = x_2 = \cdots = x_n = 0$ (that is, the n-tuple $(0, 0, \ldots, 0)$) to a homogeneous linear system is called the **trivial solution**. A solution to a homogeneous linear system in which not all of $x_1, x_2, \ldots, x_n$ are zero is called a **nontrivial solution**.

7.1.5 Definition. Two linear systems are **equivalent** provided they have *exactly* the same solutions.

7.1.6 Example. The linear system

$$\begin{cases} x_1 - 3x_2 = -7 \\ 2x_1 + x_2 = 7 \end{cases}$$

has only the solution $(2, 3)$. The linear system

$$\begin{cases} 8x_1 - 3x_2 = 7 \\ 3x_1 - 2x_2 = 0 \\ 10x_1 - 2x_2 = 14 \end{cases}$$

also has only the solution $(2, 3)$. Thus these linear systems are equivalent.

To find solutions to a linear system, we shall use a technique called the **method of elimination** ; that is, *we eliminate some variables by adding a multiple of one equation to another equation.* Elimination merely amounts to the development of a new linear system that is equivalent to the original system, but is much simpler to solve.

7.1.7 Example. Consider the linear system

$$\begin{cases} x_1 - 3x_2 = -3 \\ 2x_1 + x_2 = 8. \end{cases}$$

To eliminate x_1, we add (-2) times the first equation to the second one, obtaining

$$7x_2 = 14.$$

Thus we have eliminated the unknown x_1. Then solving for x_2, we have

$$x_2 = 2$$

and substituting into the first equation, we obtain

$$x_1 = 3.$$

Then $(3, 2)$ is the *only* solution to the given linear system.

7.1.8 Example. Consider the linear system

$$\begin{cases} x_1 & - & 3x_2 = -7 \\ 2x_1 & - & 6x_2 = 7. \end{cases}$$

Again, we decide to eliminate x_1. We add (-2) times the first equation to the second one, obtaining

$$0 = 21$$

which makes no sense. This means that the given linear system has *no* solution : it is inconsistent.

7.1.9 Example. Consider the linear system

$$\begin{cases} x_1 & + & 2x_2 & - & 3x_3 = -4 \\ 2x_1 & + & x_2 & - & 3x_3 = 4. \end{cases}$$

Eliminating x_1, we add (-2) times the first equation to the second equation, to obtain

$$-3x_2 + 3x_3 = 12.$$

We must now solve this equation. A solution is

$$x_2 = x_3 - 4$$

where x_3 can be any real number. Then from the first equation of the system,

$$x_1 = -4 - 2x_2 + 3x_3 = -4 - 2(x_3 - 4) + 3x_3 = x_3 + 4.$$

Thus a solution to the given linear system is

$$x_1 = x_3 + 4, \quad x_2 = x_3 - 4, \quad x_3 = \text{ any real number.}$$

We may write such a solution as follows

$$(\alpha + 4, \alpha - 4, \alpha), \quad \alpha \in \mathbb{R}.$$

This means that the linear system has *infinitely many* solutions. Every time we assign a value α we obtain another solution.

NOTE : These examples suggest that *a linear system may have a unique solution, no solution, or infinitely many solutions.*

GEOMETRIC INTERPRETATION. Consider next a linear system of two equations in the unknowns x_1 and x_2 :

$$\begin{cases} a_1x_1 & + & a_2x_2 = c_1 \\ a_2x_2 & + & a_2x_2 = c_2 \,. \end{cases}$$

The graph of each of these equations is a straight line, which we denote by $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Then exactly one of the following situations must occur :

- the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect at a single point ;
- the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are parallel lines ;
- the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ coincide - they actually are the same line.

If the pair (s_1, s_2) is a solution to the linear system, then the point (s_1, s_2) lies on both lines $\mathcal{L}_1$ and $\mathcal{L}_2$. Conversely, if the point (s_1, s_2) lies on both lines $\mathcal{L}_1$ and $\mathcal{L}_2$, then the pair (s_1, s_2) is a solution to the linear system. Thus we are led geometrically to the same three possibilities mentioned above.

If we examine the method of elimination more closely, we find that it involves three manipulations that can be performed on a linear system to convert it into an equivalent system. These manipulations, called **elementary operations**, are as follows :

- interchange the ith and j^{th} equations ;
- multiply an equation by a nonzero constant ;
- replace the ith equation by c times the jth equation plus the ith equation, $i \neq j$.

It is not difficult to prove that *performing these manipulations on a linear system leads to an equivalent system.*

7.2 Matrices and Gaussian elimination

If we examine the method of elimination described in the previous section, we make the following observation : only the numbers in front of the unknowns $x_1, x_2, \ldots, x_n$ and the numbers $b_1, b_2, \ldots, b_m$ on the right-hand side are being changed as we perform the steps in the method of elimination. Thus we might think of looking for a way of writing a linear system without having to carry along the unknowns. *Matrices* enable us to do this.

NOTE : Their use is not, however, merely that of a convenient notation. As any good definition should do, the notion of a matrix provides not only a new way of looking at old problems but also gives rise to a great many new questions.

7.2.1 DEFINITION. A **matrix** is a rectangular array of numbers denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

The ith **row** of A is

$$\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}, \quad 1 \le i \le m$$

while the jth **column** of A is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad 1 \le j \le n.$$

If a matrix A has m rows and n columns, we say that A is an $m \times n$ **matrix** (read "m by n"). If $m = n$, we say that A is a **square matrix** of order n. We refer to a_{ij} as the (i, j) **entry** or the (i, j) **element** and we often write

$$A = [a_{ij}].$$

7.2.2 Example. Consider the matrices :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 0 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Here, A is 3×3, B is 1×3, C is 4×1, D is 3×2, and E is 2×2. In A, $a_{23} = 0$; in B, $b_{12} = -1$; in C, $c_{41} = 3$; in D, $d_{11} = 2$, and in E, $e_{12} = e_{21} = -1$.

Consider again a general system of m linear equations in the variables

$x_1, x_2, \ldots, x_n$:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m\,. \end{cases}$$

NOTE : Such a linear system may be written, in compact form, as follows :

$$\sum_{j=1}^{n} a_{ij}x_j = b_i\,, \quad i = 1, 2, \ldots, m.$$

The **coefficient matrix** of such a linear system is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}.$$

Let us write

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

for the column of constants in the general system; this $m \times 1$ matrix is often called a **column vector**. When we adjoin the column vector b to the coefficient matrix A (as a final column), we get the matrix

$$[A \;\; b] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{bmatrix}.$$

This $m \times (n+1)$ matrix is called the **augmented coefficient matrix** (or, simply, the **augmented matrix**) of the given linear system.

7.2.3 Example. The augmented matrix of the linear system

$$\begin{cases} 2x_1 + 3x_2 - 7x_3 + 4x_4 = 6 \\ \phantom{2x_1 + {}} x_2 + 3x_3 - 5x_4 = 0 \\ -x_1 + 2x_2 \phantom{{}+3x_3} - 9x_4 = 17 \end{cases}$$

is the 3×5 matrix

$$\begin{bmatrix} 2 & 3 & -7 & 4 & 6 \\ 0 & 1 & 3 & -5 & 0 \\ -1 & 2 & 0 & -9 & 17 \end{bmatrix}.$$

In previous section we described the three elementary operations that are used in the method of elimination. To each of these corresponds an *elementary row operation* on the augmented matrix of the system.

7.2.4 Definition. The following are the three types of **elementary row operations** on the matrix A :

- interchange two rows of A ;
- multiply any (single) row of A by a nonzero constant ;
- add a constant multiple of one row of A to another row.

7.2.5 Example. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}.$$

Interchanging rows 1 and 3 of A, we obtain

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Multiplying the third row of A by $\frac{1}{3}$, we obtain

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix} \xrightarrow{(\frac{1}{3})R_3} \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}.$$

Adding (-2) times row 2 to row 3 of A, we obtain

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix} \xrightarrow{(-2)R_2+R_3} \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}.$$

It is not self-evident that a sequence of elementary row operations produces a linear system which is equivalent to the original system. To state the pertinent result concisely, we need the following definition.

7.2.6 Definition. Two matrices are called **row-equivalent** provided one can be obtained from the other by a (finite) sequence of elementary operations.

7.2.7 Example. The matrix

$$A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 1 & -1 & 2 & 3 \end{bmatrix}$$

is row equivalent to

$$B = \begin{bmatrix} 2 & 4 & 8 & 6 \\ 1 & -1 & 2 & 3 \\ 4 & -1 & 7 & 8 \end{bmatrix}$$

because

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 1 & -1 & 2 & 3 \end{bmatrix} \xrightarrow{(2)R_3+R_2} \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & -1 & 7 & 8 \\ 1 & -1 & 2 & 3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 4 & 3 \\ 1 & -1 & 2 & 3 \\ 4 & -1 & 7 & 8 \end{bmatrix}$$

$$\xrightarrow{(2)R_1} \begin{bmatrix} 2 & 4 & 8 & 6 \\ 1 & -1 & 2 & 3 \\ 4 & -1 & 7 & 8 \end{bmatrix}.$$

The following result holds.

7.2.8 Proposition. *If the augmented coefficient matrices of two linear systems are row-equivalent, then the two systems are equivalent.*

Up to this point we have been somewhat informal in our description of the method of elimination : its objective is *to transform, by elementary row operations, a given linear system into one for which back substitution leads easily to a solution.* The following definition tells what should be the appearance of the augmented matrix of the transformed system.

7.2.9 Definition. A matrix E is in **(row) echelon form** provided it satisfies the following conditions :

(1) every row of E that consists entirely of zeros (if any) lies *below* every row that contains a nonzero element ;

(2) in each row of E that contains a nonzero element, the *first* nonzero element lies strictly to the *right* of the first nonzero element in the preceding row (if there is a preceding row).

Property 1 says that if E has any all-zero rows, then they are grouped together at the bottom of the matrix. The first (from left) *nonzero* element in each of the other rows is called its **leading entry** (or **pivot**).

Note : One can always make the leading entries to be 1.

Property 2 says that the leading entries form a "descending staircase" pattern from upper left to lower right, as in the following echelon form matrix :

$$E = \begin{bmatrix} \mathbf{2} & -1 & 0 & 4 & 7 \\ 0 & \mathbf{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \mathbf{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Suppose that a linear system is in **echelon form** - its augmented matrix is in echelon form. Then those variables that correspond to *columns* containing

leading entries are called **leading** (or **basic**) **variables**; all the other (non-leading) variables are called **free variables**. The following algorithm describes the process of **back substitution** to solve such a system.

Algorithm : Back substitution

To solve a consistent linear system in echelon form by back substitution, carry out the following steps :

1. Set each free variable equal to an arbitrary parameter (a different parameter for each free variable).

2. Solve the final (nonzero) equation for its leading variables.

3. Substitute the result in the next-to-last equation and then solve for its leading variables.

4. Continuing in this fashion, work upward through the system of equations until all variables have been determined.

7.2.10 Example. The augmented matrix of the linear system

$$\begin{cases} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10 \\ \qquad\qquad\quad x_3 + \qquad\quad 2x_5 = -3 \\ \qquad\qquad\qquad\qquad x_4 - 4x_5 = 7 \end{cases}$$

is the echelon matrix

$$\begin{bmatrix} \mathbf{1} & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & \mathbf{1} & 0 & 2 & -3 \\ 0 & 0 & 0 & \mathbf{1} & -4 & 7 \end{bmatrix}.$$

The leading entries are in the first, third, and fourth columns. Hence, x_1, x_3 and x_4 are the leading variables, and x_2 and x_5 are the free variables. To solve the system by back substitution, we first write

$$x_2 = s \quad \text{and} \quad x_5 = t$$

where s and t are arbitrary parameters. Then the third equation gives

$$x_4 = 7 + 4x_5 = 7 + 4t\,;$$

the second equation gives

$$x_3 = -3 - 2x_5 = -3 - 2t\,;$$

finally, the first equation yields

$$x_1 = 10 + 2x_2 - 3x_3 - 2x_4 - x_5 = 5 + 2s - 3t.$$

Thus the linear system has infinitely many solutions $(x_1, x_2, x_3, x_4, x_5)$, given in terms of the two parameters s and t, as follows :

$$x_1 = 5 + 2s - 3t\,, \quad x_2 = s\,, \quad x_3 = -3 - 2t\,, \quad x_4 = 7 + 4t\,, \quad x_5 = t.$$

Alternatively, the *solution set* of the given linear system is

$$S = \{(5 + 2s - 3t, s, -3 - 2t, 7 + 4t, t) \,|\, s,\, t \in \mathbb{R}\}.$$

Because we can use back substitution to solve any linear system already in echelon form, it remains only to establish that we can transform any matrix (using elementary row operations) into an echelon form. This procedure is known as **Gaussian elimination** (named after the great German mathematician Carl F. Gauss (1777-1855)).

Algorithm : Gaussian elimination

To transforme a matrix A into an echelon form, carry out the following steps :

1. Locate the first column of A that contains a nonzero element.

2. If the first (top) entry in this column is zero, interchange the first row of A with a row in which the corresponding entry is nonzero.

3. Now the first entry in our column is nonzero. Replace the entries below it in the same column with zeroes by adding appropriate multiples of the first row of A to lower rows.

4. Perform steps 1-3 on the lower right matrix A_1.

5. Repeat the cycle of steps until an echelon form matrix is obtained.

7.2.11 EXAMPLE. To solve the linear system

$$\begin{cases} x_1 - 2x_2 + 3x_3 + 2x_4 + x_5 = 10 \\ 2x_1 - 4x_2 + 8x_3 + 3x_4 + 10x_5 = 7 \\ 3x_1 - 6x_2 + 10x_3 + 6x_4 + 5x_5 = 27 \end{cases}$$

we reduce its augmented coefficient matrix to echelon form as follows :

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 2 & -4 & 8 & 3 & 10 & 7 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix} \xrightarrow{(-2)R_1+R_2} \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 3 & -6 & 10 & 6 & 5 & 27 \end{bmatrix} \xrightarrow{(-3)R_1+R_3}$$

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 2 & -1 & 8 & -13 \\ 0 & 0 & 1 & 0 & 2 & -3 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 2 & -1 & 8 & -13 \end{bmatrix} \xrightarrow{(-2)R_2+R_3}$$

$$\begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & -1 & 4 & -7 \end{bmatrix} \xrightarrow{(-1)R_3} \begin{bmatrix} 1 & -2 & 3 & 2 & 1 & 10 \\ 0 & 0 & 1 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & -4 & 7 \end{bmatrix}.$$

Our result is the echelon matrix in EXAMPLE 7.2.10, so the infinite solution set of the given linear system is

$$S = \{(5 + 2s - 3t, s, -3 - 2t, 7 + 4t, t) \mid s,\, t \in \mathbb{R}\}.$$

7.2.12 EXAMPLE. To solve the linear system

$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ 3x_1 + 8x_2 + 7x_3 = 20 \\ 2x_1 + 7x_2 + 9x_3 = 23 \end{cases}$$

whose augmented matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix},$$

we carry the following sequence of elementary row operations :

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix} \xrightarrow{(-3)R_1+R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 2 & 7 & 9 & 23 \end{bmatrix} \xrightarrow{(-2)R_1+R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{bmatrix}$$

$$\xrightarrow{(\frac{1}{2})R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{bmatrix} \xrightarrow{(-3)R_2+R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

The final matrix here is the augmented matrix of the linear system

$$\begin{cases} x_1 + 2x_2 + x_3 = 4 \\ \qquad\quad x_2 + 2x_3 = 4 \\ \qquad\qquad\quad x_3 = 3\,, \end{cases}$$

whose *unique* solution (found by back substitution) is

$$x_1 = 5,\ x_2 = -2,\ x_3 = 3.$$

NOTE : EXAMPLES 7.2.11 and 7.2.12 illustrate the ways in which Gaussian elimination can result in either a unique solution or infinitely many solutions. If the reduction of the augmented matrix to echelon form leads to a row of the form

$$\begin{bmatrix} 0 & 0 & \dots & 0 & * \end{bmatrix}$$

where the asterisc denotes a nonzero entry in the last column, then we have an inconsistent equation,

$$0x_1 + 0x_2 + \cdots + 0x_n = *$$

and therefore the linear system has no solution.

The structure of linear systems

The result of the process of Gaussian elimination is not uniquely determined. That is, two different sequences of elementary row operations, both starting with the same matrix A, may yield two different echelon matrices (each of course still row-equivalent to A). A full understanding of the structure of systems of linear equations depends upon the definition of a special class of echelon matrices - the *reduced echelon matrices.*

7.2.13 Definition. A matrix E is in **reduced (row) echelon form** provided

- E is in row echelon form.
- Each pivot (leading entry) is 1.
- All entries above each pivot are 0.

One can prove that

7.2.14 Proposition. *Every matrix is row-equivalent to precisely one reduced echelon form matrix.*

The following result(s) then follow :

7.2.15 Proposition. *A linear system has either*

- *no solution (it is inconsistent),*
- *exactly one solution (if the system is consistent and all variables are basic), or*
- *infinitely many solutions (if the system is consistent and there are free variables).*

In particular, a homogeneous linear system either has only the trivial solution or has infinitely many solutions.

7.2.16 Corollary. *A linear system with more variables than equations has either no solution or infinitely many solutions. In particular, a homogeneous linear system with more variables than equations has infinitely many solutions.*

7.2.17 Example. The reduced row echelon forms of the augmented matrices of three systems are given below. How many solutions are there in each case ?

$$(1)\quad \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \quad (2)\quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}; \quad (3)\quad \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution : (1) Infinitely many solutions (the second variable is free).

(2) Exactly one solution (all variables are basic).

(3) No solutions (the third row represents the equation $0 = 1$).

Note : Let A be an $n \times n$ matrix. Then the homogeneous linear system with coefficient matrix A has only the trivial solution if and only if A is row-equivalent to the $n \times n$ identity matrix.

7.2.18 Example. The 3×3 matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 7 \\ 2 & 7 & 9 \end{bmatrix}$$

is row-equivalent to the 3×3 identity matrix (check !). Hence, the homogeneous linear system

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 3x_1 + 8x_2 + 7x_3 = 0 \\ 2x_1 + 7x_2 + 9x_3 = 0 \end{cases}$$

with coefficient matrix A, has *only* the trivial solution.

7.3 Matrix operations

Matrices can be added and multiplied in ways similar to the ways in which numbers are added and multiplied.

7.3.1 Definition. Two matrices A and B of the *same size* are called **equal**, and we write $A = B$, provided that each element of A is equal to the corresponding element of B.

Thus, two matrices of the same size (the same number of rows and the same number of columns) are equal if they are *elementwise equal*.

7.3.2 Example. If

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 8 \end{bmatrix}$$

then $A \neq B$ because $a_{22} = 6$, whereas $b_{22} = 7$, and $A \neq C$ because the matrices A and C are not of the same size.

7.3.3 Definition. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same size, their **sum** $A+B$ is the matrix obtained by adding corresponding elements of the matrices A and B. That is,

$$A + B := [a_{ij} + b_{ij}].$$

7.3.4 Example. If

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -7 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -3 & 6 \\ 9 & 0 & -2 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 3 & -2 \\ -1 & 6 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 7 & -3 & 5 \\ 11 & -7 & 3 \end{bmatrix}.$$

But the sum $A + C$ is not defined because the matrices A and C are not of the same size.

7.3.5 Definition. If $A = [a_{ij}]$ is a matrix and r is a number, than rA is the matrix obtained by multiplying each element of A by r. That is,

$$rA := [ra_{ij}].$$

Note : We also write

$$-A = (-1)A \quad \text{and} \quad A - B = A + (-B).$$

7.3.6 Example. If

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -7 & 5 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 4 & -3 & 6 \\ 9 & 0 & -2 \end{bmatrix}$$

then

$$3A = \begin{bmatrix} 9 & 0 & -3 \\ 6 & -21 & 15 \end{bmatrix}, \; -B = \begin{bmatrix} -4 & 3 & -6 \\ -9 & 0 & 2 \end{bmatrix}, \text{ and } 3A - B = \begin{bmatrix} 5 & 3 & -9 \\ -3 & -21 & 17 \end{bmatrix}.$$

We recall that a **column vector** (or simply **vector**) is merely an $n \times 1$ matrix. If

$$a = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

we can form such (linear) combinations as

$$3a + 2b = 3 \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ -6 \\ 9 \end{bmatrix}.$$

Note : A column vector should not be confused with a row vector. A **row vector** is a $1 \times n$ (rather than $n \times 1$) matrix. For instance,

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}.$$

Given a linear system

$$\sum_{j=1}^{n} a_{ij}x_j = b_i\,, \quad i = 1, 2, \ldots, n$$

we may regard a solution of this system as a *vector*

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

whose elements satisfy each of the equations of the system.

7.3.7 Example. Consider the homogeneous linear system

$$\begin{cases} x_1 + 3x_2 - 15x_3 + 7x_4 = 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 = 0 \\ 2x_1 + 5x_2 - 26x_3 + 11x_4 = 0. \end{cases}$$

We find that the (reduced) echelon form of the augmented coefficient matrix of this system is

$$\begin{bmatrix} 1 & 0 & -3 & -2 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, x_1 and x_2 are leading variables and x_3 and x_4 are free variables. We therefore see that the infinite solution set of the system is described by

$$x_4 = t,\ x_3 = s,\ x_2 = 4s - 3t,\ x_1 = 3s + 2t$$

in terms of the arbitrary parameters s and t. Now let us write the solution (x_1, x_2, x_3, x_4) in vector notation. We have

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s + 2t \\ 4s - 3t \\ s \\ t \end{bmatrix}$$

and "separating" the s and t parts gives

$$x = \begin{bmatrix} 3s \\ 4s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 2t \\ -3t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

NOTE : This equation expresses in **vector form** the general solution of the given linear system. (It says that the vector x is a solution if and only if x is a *linear combination* of two particular solutions.)

7.3.8 DEFINITION. If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, their **product** AB is defined by

$$AB := [c_{ij}], \quad \text{where} \quad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

NOTE : (1) The product AB is defined *only when* the number of columns of A is the same as the number of rows of B.

(2) The (i, j) entry in AB is obtained by using the ith row of A and the jth column of B. If $a_1, a_2, \ldots, a_m$ denote the m row vectors of A and $b_1, b_2, \ldots, b_n$ denote the n column vectors of B, then

$$AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \ldots & b_n \end{bmatrix} = \begin{bmatrix} a_i b_j \end{bmatrix}, \quad \text{where} \quad a_i b_j := \sum_{k=1}^{n} a_{ik} b_{kj}.$$

One might ask *why matrix equality and matrix addition are defined elementwise while matrix multiplication appears to be much more complicated.* Only a thorough understanding of the composition of functions and the relationship that exists between matrices and what are called *linear mappings* would show that the definition of multiplication given here is the *natural* one.

7.3.9 Example. Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} (1)(-2)+(2)(4)+(-1)(2) & (1)(5)+(2)(-3)+(-1)(1) \\ (3)(-2)+(1)(4)+(4)(2) & (3)(5)+(1)(-3)+(4)(1) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}.$$

7.3.10 Example. Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix} \quad \text{while} \quad BA = \begin{bmatrix} 1 & 7 \\ -1 & 3 \end{bmatrix}.$$

Thus $AB \neq BA$.

7.3.11 Definition. If $A = [a_{ij}]$ is an $m \times n$ matrix, the its **transpose** A^T is the $n \times m$ matrix defined by

$$A^T := [a_{ij}^T], \quad \text{where} \quad a_{ij}^T = a_{ji}.$$

Thus the transpose of A is obtained from A by interchanging the rows and columns of A.

7.3.12 Example. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 1 & -2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 2 & 0 & 3 \end{bmatrix}.$$

The following (algebraic) properties of matrix operations hold :

Properties of matrix addition : *If A, B and C are matrices of appropriate size, then* :

(1) $A + B = B + A$;

(2) $A + (B + C) = (A + B) + C$.

Properties of scalar multiplication : *If r and s are real numbers, and A and B are matrices of appropriate size, then* :

(1) $(r + s)A = rA + sA$;

(2) $r(A + B) = rA + rB$;

(3) $r(sA) = (rs)A$.

Properties of matrix multiplication : *If A, B and C are matrices of appropriate size, then* :

(1) $A(BC) = (AB)C$;

(2) $(A + B)C = AC + BC$;

(3) $C(A + B) = CA + CB$.

Properties of transpose : *If r is a real number, and A and B are matrices of appropriate size, then* :

(1) $(A^T)^T = A$;

(2) $(A + B)^T = A^T + B^T$;

(3) $(rA)^T = rA^T$;

(4) $(AB)^T = B^T A^T$.

NOTE : There are some differences between matrix multiplication and the multiplication of real numbers; for instance :

- AB need not equal BA;

- AB may be the *zero matrix* (the matrix with all entries zero) with $A \neq O$ and $B \neq O$;
- AB may equal AC with $B \neq C$.

We introduce now some special types of matrices.

7.3.13 Definition. An $n \times n$ matrix $A = [a_{ij}]$ is called

- **diagonal matrix** if $a_{ij} = 0$ for $i \neq j$;
- the $n \times n$ **identity matrix**, denoted I_n, if $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$;
- **upper triangular matrix** if $a_{ij} = 0$ for $i > j$;
- **lower triangular matrix** if $a_{ij} = 0$ for $i < j$;
- **symmetric matrix** if $A^T = A$;
- **skew symmetric matrix** if $A^T = -A$;
- **orthogonal matrix** if $AA^T = I_n$;
- **nonsingular matrix** (or **invertible matrix**) if there exists an $n \times n$ matrix B such that $AB = BA = I_n$; such a B is called an **inverse** of A. (Otherwise, A is called **singular** or **noninvertible**.)

7.3.14 Proposition. *The inverse of a matrix, if it exists, is unique.*

Proof : Let B and C be inverses of A. Then

$$AB = BA = I_n \quad \text{and} \quad AC = CA = I_n.$$

We then have

$$B = BI_n = B(AC) = (BA)C = I_nC = C$$

which proves that the inverse of a matrix, if it exists, is unique. □

Because of this uniqueness, we write the inverse of a nonsingular matrix A as A^{-1}. Thus

$$AA^{-1} = A^{-1}A = I_n.$$

7.3.15 Example. If

$$A = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 7 & -9 \\ -3 & 4 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 7 & -9 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I\,;$$

By a similar computation we get $BA = I$.

7.3.16 Example. Let

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If the matrix B had the property that $AB = BA = I$, then

$$AB = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-3c & b-3d \\ -2a+6c & -2b+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Upon equating corresponding elements of AB and the 2×2 identity matrix, we find that

$$\begin{cases} a - 3c = 1 \\ -2a + 6c = 0 \\ b - 3d = 0 \\ -2b + 6d = 1. \end{cases}$$

It is clear that this system of linear equations is inconsistent (Why ?). Thus, there can exist *no* 2×2 matrix B such that $AB = I$.

Note : *The* 2×2 *matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

7.3.17 Example. If

$$A = \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix}$$

then $ad - bc = 36 - 30 = 6 \neq 0$, so

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 9 & -6 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{5}{6} & \frac{2}{3} \end{bmatrix}.$$

Properties of nonsingular matrices : *If the matrices* A *and* B *of the same size are nonsingular, then :*

(1) A^{-1} *is nonsingular and* $(A^{-1})^{-1} = A$ *;*

(2) A^T *is nonsingular and* $(A^T)^{-1} = (A^{-1})^T$ *;*

(3) AB *is nonsingular and* $(AB)^{-1} = B^{-1}A^{-1}$.

Consider a system of n linear equations in variables $x_1, x_2, \ldots, x_n$

$$\sum_{j=1}^{n} a_{ij} x_j = b_i\,, \quad i = 1, 2, \ldots, n.$$

Recall that we can write this linear system in the compact form

$$A\,x = b$$

where

$$A = [a_{ij}], \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Suppose that A is nonsingular. Then A^{-1} exists and we can multiply $Ax = b$ by A^{-1} on both sides, obtaining

$$A^{-1}(Ax) = A^{-1}b$$

or

$$I_n x = x = A^{-1}b.$$

Moreover, $x = A^{-1}b$ is clearly a solution to the given linear system. Thus, if A is nonsingular, we have a *unique* solution.

7.3.18 Example. To solve the system

$$\begin{cases} 4x_1 & + & 6x_2 = 6 \\ 5x_1 & + & 9x_2 = 18 \end{cases}$$

we use the inverse of the coefficient matrix

$$A = \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix}.$$

We have

$$x = A^{-1}b = \begin{bmatrix} 4 & 6 \\ 5 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 18 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 \\ -\frac{5}{6} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 6 \\ 18 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \end{bmatrix}.$$

Thus $x_1 = -9$, $x_2 = 7$ is the unique solution.

The following result leads to a practical method for inverting matrices.

7.3.19 Proposition. *An $n \times n$ matrix is invertible if and only if it is row-equivalent to the $n \times n$ identity matrix I_n.*

Algorithm : Finding A^{-1}

To find the inverse A^{-1} of the nonsingular matrix A, carry out the following steps:

1. Find a sequence of elementary row operations that reduces A to the $n \times n$ identity matrix I_n.

2. Apply the same sequence of operations in the same order to I_n to transform it into A^{-1}.

NOTE : As a practical matter, it generally is more convenient to carry out the two reductions - from A to I_n, and from I_n to A^{-1} - in parallel.

7.3.20 EXAMPLE. Find the inverse of the 3×3 matrix

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix}.$$

SOLUTION : We want to reduce A to the 3×3 identity matrix I_3 while simultaneously performing the same sequence of row operations on I_3 to obtain A^{-1}. In order to carry out this process efficiently, we adjoin I_3 on the right of A to form the 3×6 matrix

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

We now apply the following sequence of elementary row operations to this 3×6 matrix (designed to transform its left half into the 3×3 identity matrix).

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-1)R_3+R_1} \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-1)R_3+R_2}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 0 & -1 \\ 2 & 1 & 1 & 0 & 1 & -1 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-2)R_1+R_2} \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 5 & 1 & -2 & 1 & 1 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-3)R_1+R_3}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 5 & 1 & -2 & 1 & 1 \\ 0 & 11 & 2 & -3 & 0 & 4 \end{bmatrix} \xrightarrow{(-2)R_2+R_3} \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 5 & 1 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & -2 & 2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 5 & 1 & -2 & 1 & 1 \end{bmatrix} \xrightarrow{(2)R_2+R_1} \begin{bmatrix} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 5 & 1 & -2 & 1 & 1 \end{bmatrix} \xrightarrow{(-5)R_2+R_3}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{bmatrix}.$$

Now that we have reduced the left half of the 3×6 matrix to I_3, we simply examine its right half to see that the inverse of A is

$$A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{bmatrix}.$$

7.4 Exercises

Exercise 91 Consider the linear system

$$\begin{cases} 2x_1 - x_2 = 5 \\ 4x_1 - 2x_2 = k. \end{cases}$$

(a) Determine a value of k so that the system is consistent.

(b) Determine a value of k so that the system is inconsistent.

(c) How many different values of k can be selected in part (b) ?

Exercise 92 Use elementary row operations to transform each augmented coefficient matrix to echelon form, and then solve the system by back substitution.

(a) $\begin{cases} 2x_1 + 8x_2 + 3x_3 = 2 \\ x_1 + 3x_2 + 2x_3 = 5 \\ 2x_1 + 7x_2 + 4x_3 = 8 \end{cases}$;

(b) $\begin{cases} 3x_1 - 6x_2 + x_3 + 13x_4 = 15 \\ 3x_1 - 6x_2 + 3x_3 + 21x_4 = 21 \\ 2x_1 - 4x_2 + 5x_3 + 26x_4 = 23 \end{cases}$;

(c) $\begin{cases} 4x_1 - 2x_2 - 3x_3 + x_4 = 3 \\ 2x_1 - 2x_2 - 5x_3 = -10 \\ 4x_1 + x_2 + 2x_3 + x_4 = 17 \\ 3x_1 + x_3 + x_4 = 12 \end{cases}$

Exercise 93 Determine for what values of k each system has (i) a unique solution; (ii) no solution; (iii) infinitely many solutions.

(a) $\begin{cases} 3x + 2y = 1 \\ 6x + 4y = k \end{cases}$;

(b) $\begin{cases} 3x + 2y = 0 \\ 6x + ky = 0 \end{cases}$;

(c) $\begin{cases} 3x + 2y = 1 \\ 7x + 5y = k \end{cases}$;

(d) $\begin{cases} x + 2y + z = 3 \\ 2x - y - 3z = 5 \\ 4x + 3y - z = k. \end{cases}$

Exercise 94 Under what condition on the constants a, b and c does the linear system

$$\begin{cases} 2x - y + 3z = a \\ x + 2y + z = b \\ 7x + 4y + 9z = c \end{cases}$$

have a unique solution ? No solution ? Infinitely many solutions ?

Exercise 95 Find an equation relating a, b and c so that the linear system

$$\begin{cases} x + 2y - 3z = a \\ 2x + 3y + 3z = b \\ 5x + 9y - 6z = c \end{cases}$$

is consistent for any values of a, b and c that satisfy that equation.

Exercise 96 For the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad E = [\,5\,],$$

determine which of the 25 matrix products AA, AB, AC, $\ldots$, ED, EE are defined, and compute those which are defined.

Exercise 97

(a) Find a value of r and a value of s so that

$$AB^T = 0\,, \quad \text{where} \quad A = [1 \;\; r \;\; 1] \quad \text{and} \quad B = [-2 \;\; 2 \;\; s]\,.$$

(b) If

$$\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix},$$

find a, b, c and d.

Exercise 98 TRUE or FALSE ? For two invertible $n \times n$ matrices A and B

(a) $(I_n - A)(I_n + A) = I_n - A^2$.

(b) $(A + B)^2 = A^2 + 2AB + B^2$.

(c) A^2 is invertible, and $(A^2)^{-1} = (A^{-1})^2$.

(d) $A + B$ is invertible, and $(A + B)^{-1} = A^{-1} + B^{-1}$.

(e) $(A - B)(A + B) = A^2 - B^2$.

(f) $ABB^{-1}A^{-1} = I_n$.

(g) $ABA^{-1} = B$.

(h) $(ABA^{-1})^3 = AB^3A^{-1}$.

(i) $(I_n + A)(I_n + A^{-1}) = 2I_n + A + A^{-1}$

(j) $A^{-1}B$ is invertible, and $(A^{-1}B)^{-1} = B^{-1}A$.

Exercise 99

(a) If $A = [a_{ij}]$ is a $n \times n$ matrix, then the **trace** of A is defined as the sum of all elements on the main diagonal of A; that is,

$$\operatorname{tr}(A) := \sum_{i=1}^{n} a_{ii}.$$

Prove :

i. $\operatorname{tr}(cA) = c \operatorname{tr}(A)$, where c is a real number.

ii. $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.

iii. $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

iv. $\operatorname{tr}(A^T) = \operatorname{tr}(A)$.

v. $\operatorname{tr}(A^T A) \geq 0$.

(b) Show that there are *no* 2×2 matrices A and B such that

$$AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Exercise 100 Prove or disprove :

(a) For any $n \times n$ *diagonal* matrices A and B, $AB = BA$.

(b) For any $n \times n$ matrix A, $AA^T = A^T A$.

Exercise 101 Let A and B be matrices of appropriate size. Show that :

(a) A is *symmetric* if and only if

$$a_{ij} = a_{ji} \quad \text{for all} \quad i, j.$$

(b) A is *skew symmetric* if and only if

$$a_{ij} = -a_{ji} \quad \text{for all} \quad i, j.$$

(c) If A is *skew symmetric*, then the elements on the main diagonal of A are zero.

(d) If A is *symmetric*, then A^T is *symmetric*.

(e) AA^T and $A^T A$ are *symmetric*.

(f) $A + A^T$ is *symmetric* and $A - A^T$ is *skew symmetric.*

(g) If A and B are *symmetric*, then $A + B$ is *symmetric.*

(h) AB is *symmetric* if and only if $AB = BA$.

Exercise 102 Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that

$$A^2 = (a + d)A - (ad - bc)I_2.$$

This formula can be used to compute A^2 without an explicit matrix multiplication. It follows that

$$A^3 = (a + d)A^2 - (ad - bc)A$$

without an explicit matrix multiplication,

$$A^4 = (a + d)A^3 - (ad - bc)A^2\,,$$

and so on. Use this method to compute $A^2, A^3, A^4,$ and A^5 given

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Exercise 103 Find :

(a) a 2×2 matrix A with each element 1 or -1, such that $A^2 = I_2$.

(b) a 2×2 matrix A with each main diagonal element 0, such that $A^2 = I_2$.

(c) a 2×2 matrix A with each main diagonal element 0, such that $A^2 = -I_2$.

(d) all 2×2 matrices X such that

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(e) all 2×2 matrices X which commute with $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(f) all 2×2 matrices X which commute with $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

(g) all 2×2 matrices X which commute with all 2×2 matrices.

Exercise 104 First find A^{-1} and then use A^{-1} to solve the linear system $Ax = b$.

(a) $A = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 6 \end{bmatrix};$

(b) $A = \begin{bmatrix} 7 & 9 \\ 5 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$

Exercise 105 Find the inverse A^{-1} of each given matrix A.

(a) $\begin{bmatrix} 1 & 5 & 1 \\ 2 & 5 & 0 \\ 2 & 7 & 1 \end{bmatrix};$

(b) $\begin{bmatrix} 1 & 4 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 1 \end{bmatrix};$

(c) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 3 & 2 & 4 & 1 \end{bmatrix}.$