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Lecture Notes CCR

AM4.4 - Geometric Control

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"Where there is matter, there is geometry".

J. Kepler

"Imagination is more important than knowledge".

A. Einstein

"Knowledge is power".

F. Bacon

"Power is nothing without control".

Anonymus

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Lie Groups

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1.1 Lie Groups: Definition and Examples

Lie groups form an important class of smooth (in fact, *analytic*) manifolds. (Their prototype is any finite-dimensional group of linear transformations on a vector space.) The key idea of a Lie group is that it is a group in the usual sense, but with the additional property that it is also a smooth manifold, and in such a way that the group operations are smooth. A good example is the circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Lie groups (and their Lie algebras) play a central role in geometry, topology, and analysis, as well as in modern theoretical physics. The precise definition is given below.

1.1.1 Definition. A (real) **Lie group** is a smooth manifold which is also a group such that the operations

$$G \times G \to G, \quad (g_1, g_2) \mapsto g_1 g_2 \qquad \text{and} \qquad G \to G, \quad g \mapsto g^{-1}$$

are smooth mapings.

1.1.2 Example. The vector space $\mathbb{R}^m$, when equipped with its natural smooth structure (i.e. viewed as the Euclidean space $\mathbb{E}^m$), is an m-dimensional (Abelian) Lie group.

1.1.3 Example. The *general linear group* $\mathsf{GL}(n, \mathbb{R})$ is evidently a Lie group. It is an open subset of (the vector space) $\mathbb{R}^{n \times n}$ (and hence a smooth submanifold of $\mathbb{E}^{n^2}$) and the group operations are given by rational functions of the coordinates.

Note : Let V be an n-dimensional vector space (over $\mathbb{R}$). Then the group $\mathsf{GL}(V)$ of all linear transformations on V is an n^2-manifold. Any choice of a basis in V

induces a linear isomorphism from $\mathsf{GL}(V)$ onto $\mathsf{GL}(n,\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ (an hence a global chart on $\mathsf{GL}(V)$). The coordinates of any product (composition) ST of elements in $\mathsf{GL}(V)$ are polynomial expressions of the coordinates of S and T, and the coordinates of S^{-1} are rational functions of the coordinates of S. It therefore follows that both group operations $(S,T) \mapsto ST$ and $S \mapsto S^{-1}$ are smooth (in fact, real analytic) mappings from $\mathsf{GL}(V) \times \mathsf{GL}(V)$ and $\mathsf{GL}(V)$, respectively, onto $\mathsf{GL}(V)$.

1.1.4 Example. The *special linear group* $\mathsf{SL}(n,\mathbb{R})$ and the *orthogonal group* $\mathsf{O}(n)$ are clearly Lie groups. Both subgroups $\mathsf{SL}(n,\mathbb{R})$ and $\mathsf{O}(n)$ are smooth submanifolds of (the Lie group) $\mathsf{GL}(n,\mathbb{R})$, hence smoothness of the group operations on $\mathsf{GL}(n,\mathbb{R})$ implies smoothness of their restrictions to $\mathsf{SL}(n,\mathbb{R})$ and $\mathsf{O}(n)$.

1.1.5 Example. The *complex general linear group* $\mathsf{GL}(n,\mathbb{C}) \subseteq \mathbb{R}^{2n^2}$ is a (real) Lie group. In particular, $\mathbb{C}^\times = \mathsf{GL}(1,\mathbb{C})$ is a Lie group. The unit circle $\mathbb{S}^1 \subseteq \mathbb{C}^\times$ is a subgroup and a (smoothly embedded) submanifold, hence also a Lie group.

1.1.6 Example. If G_1 and G_2 are Lie groups, then $G_1 \times G_2$ is a Lie group under the usual Cartesian group operations and the smooth product structure. In particular, the m-dimensional *torus*

$$\mathbb{T}^m = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$$

is a Lie group.

1.1.7 Example. Let $\mathbb{H}$ denote the division algebra of *quaternions*. The nonzero quaternions $\mathbb{H}^\times$ form a multiplicative group and a (smooth) manifold diffeomorphic to $\mathbb{R}^4 \setminus \{0\}$. It is clear that the group operations are smooth,

so $\mathbb{H}^\times$ is a Lie group. The 3-*sphere* $\mathbb{S}^3 \subseteq \mathbb{H}^\times$ consists of the unit length quaternions, hence it is closed under multiplication and passing to inverses. This gives a Lie group structure on $\mathbb{S}^3$.

Usually, the *identity element* of a Lie group will be denoted by e. (For matrix groups, however, the customary symbol for the identity is I.)

NOTE : In most of the literature, Lie groups are defined to be *real analytic*. That is, G is a manifold with a C^ω (real analytic) atlas and the group operations are real analytic. In fact, no generality is lost by this more restrictive definition. *Smooth Lie groups always support an analytic group structure*, and something even stronger is true. HILBERT'S FIFTH PROBLEM was to show that if G is only assumed to be a topological manifold with continuous group operations, then it is, in fact, a real analytic Lie group. This was finally proven by the combined work of A. GLEASON, D. MONTGOMERY, and L. ZIPPIN (195?).

1.2 Invariant Vector Fields

One of the most important features of a Lie group is the existence of an associated Lie algebra that encodes many of the properties of the group. The crucial property of a Lie group that enables this to occur is the existence of the left and right translations on the group.

Let G be a Lie group. For any $g \in G$, the mappings

$$L_g : G \to G, \quad x \mapsto gx \qquad \text{and} \qquad R_g : G \to G, \quad x \mapsto xg$$

are called the **left** and **right translation** (by g), respectively. For each $g \in G$, both L_g and R_g are smooth mappings on G.

Exercise 1 Verify that (for every $g_1, g_2, g, h \in G$)

(a) $L_{g_1} \circ L_{g_2} = L_{g_1 g_2}$.

(b) $R_{g_1} \circ R_{g_2} = R_{g_2 g_1}$.

(c) $L_e = R_e = id_G$ ($e \in G$ denotes the identity element).

(d) $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$. (Hence L_g and R_g are diffeomorphisms.)

(e) $L_g \circ R_h = R_h \circ L_g$.

NOTE : Given any admissible chart on G, one can construct an entire atlas on the Lie group G by use of left (or right) translations. Suppose, for example, that (U, ϕ) is an admissible chart with $e \in U$. Define a chart (U_g, ϕ_g) with $g \in U_g$ by letting

$$U_g := L_g(U) = \{L_g(x) \,|\, x \in U\}$$

and defining

$$\phi_g := \phi \circ L_{g^{-1}} : U_g \to \phi(U), \quad x \mapsto \phi(g^{-1}x).$$

The collection of charts $\{(U_g, \phi_g)\}_{g \in G}$ forms a (smooth) atlas provided one can show that the transition mappings

$$\phi_{g_2} \circ \phi_{g_1}^{-1} = \phi \circ L_{g_2^{-1} g_1} \circ \phi^{-1} : \phi_{g_1}(U_{g_1} \cap U_{g_2}) \to \phi_{g_2}(U_{g_1} \cap U_{g_2})$$

is smooth. But this follows from the smoothness of group multiplication and passing to inverse.

By the chain rule,

$$\left(L_{g^{-1}}\right)_{*,gh} \circ (L_g)_{*,h} = \left(L_{g^{-1}} \circ L_g\right)_{*,h} = id_G.$$

Thus the tangent mapping $(L_g)_{*,h}$ is invertible and so, in particular,

$$(L_g)_* = (L_g)_{*,e} : T_e G \to T_g G$$

is a linear isomorphism. Likewise, $(R_g)_{*,h}$ is invertible.

1.2.1 DEFINITION. A vector field X on G is called

- **left-invariant** if for every $g \in G$

$$(L_g)_* X(e) = X(g).$$

- **right-invariant** if for every $g \in G$

$$(R_g)_* X(e) = X(g).$$

It follows that a vector field (on G) that is either left- or right-invariant is determined by its value at the identity.

NOTE : Recall that smooth vector fields act as derivations on the space of smooth functions. (If X is a smooth vector field and f is a smooth function on M, then Xf denotes the (smooth) function $x \mapsto X(x)f$.) For any smooth vector fields X and Y, their *Lie bracket* $[X, Y]$ defined by

$$[X, Y]f = Y(Xf) - X(Yf)$$

is also a smooth vector field. The (vector) space $\mathfrak{X}(M)$ of all smooth vector space on M has the structure of a (real) Lie algebra, with the product given by the Lie bracket.

The set of all left-invariant (respectively, right-invariant) vector fields on a Lie group G is denoted $\mathfrak{X}_L(G)$ (respectively, $\mathfrak{X}_R(G)$). Clearly, both $\mathfrak{X}_L(G)$ and $\mathfrak{X}_R(G)$ are (real) vector spaces (under the pointwise vector addition and scalar multiplication).

NOTE : We defined the push forward $\Phi_{*,p} : T_pM \to T_{\Phi(p)}N$ induced by the (smooth) mapping $\Phi : M \to N$ (the so-called tangent mapping of Φ at $p \in M$). This is a linear mapping between the vector spaces T_pM and $T_{\Phi(p)}N$, and the question arises of whether it is similarly possible to define an induced mapping between the (vector) spaces of smooth vector fields $\mathfrak{X}(M)$ and $\mathfrak{X}(N)$. Given a vector field

$X \in \mathfrak{X}(M)$ and a smooth mapping $\Phi : M \to N$, a natural choice for an induced vector field $\Phi_* X \in \mathfrak{X}(N)$ might appear to be

$$\Phi_* X(\Phi(p)) = \Phi_{*,p}(X(p))$$

but this may fail to be well-defined for two reasons :

- If there are points $p_1, p_2 \in M$ such that $\Phi(p_1) = \Phi(p_2)$ (i.e. the mapping Φ is not one-to-one), then the "definition" above will be ambiguous when $\Phi_* X(p_1) \neq \Phi_* X(p_2)$.
- If Φ is not onto, then the defining equation does not specify the induced vector field outside the range of Φ.

Observe that if Φ is a diffeomorphism from M to N, then neither of these objections apply and an induced vector field $\Phi_* X$ *can* be defined via the above equation. However, it is possible that in certain cases the idea will work, even if Φ is not a diffeomeorphism, and this motivates the following definition : vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be Φ-*related* provided $\Phi_* X(p) = Y(\Phi(p))$ for all $p \in M$. We then write $\Phi_* X = Y$. It is not difficult to see that if $\Phi_* X_1 = Y_1$ and $\Phi_* X_2 = Y_2$, then $[X_1, X_2]$ is Φ-related to $[Y_1, Y_2]$ with

$$\Phi_*[X_1, X_2] = [\Phi_* X_1, \Phi_* X_2].$$

1.2.2 Proposition. *Let X and Y be any left-invariant (respectively, right-invariant) vector fields. Then $[X, Y]$ is a left-invariant (respectively, right-invariant) vector field.*

Proof : Let $X, Y \in \mathfrak{X}_L(G)$ and $g \in G$. Then (and only then) $(L_g)_* X = X$ and $(L_g)_* Y = Y$. Hence

$$(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y] = [X, Y]$$

and so $[X, Y] \in \mathfrak{X}_L(G)$. The case of right-invariant vector fields is similar. $\square$

Therefore, both $\mathfrak{X}_L(G)$ and $\mathfrak{X}_R(G)$ are Lie subalgebras of the (infinite dimensional) Lie algebra $\mathfrak{X}(G)$ of all smooth vector fields on G.

For each $A \in T_eG$, we define a (smooth) vector field X_A on G by letting

$$X_A(g) := (L_g)_{*,e}A.$$

Then

$$\begin{aligned}
(L_g)_* X_A(e) &= (L_g)_* \left((L_e)_* A\right) \\
&= (L_g)_* \circ (L_e)_* A \\
&= (L_{ge})_{*,e} A \\
&= (L_g)_{*,e} A \\
&= X_A(g)
\end{aligned}$$

which shows that X_A is left-invariant. Consider the mappings

$$\zeta_1 : \mathfrak{X}_L(G) \to T_eG, \qquad X \mapsto X(e)$$

and

$$\zeta_2 : T_eG \to \mathfrak{X}_L(G), \qquad A \mapsto X_A.$$

Exercise 2 Verify that ζ_1 and ζ_2 are linear mappings that satisfy

$$\zeta_1 \circ \zeta_2 = id_{T_e(G)} \qquad \text{and} \qquad \zeta_2 \circ \zeta_1 = id_{\mathfrak{X}_L(G)}.$$

(It is clear that ζ_2 is the inverse of ζ_1, and hence for a left-invariant vector field X

$$(L_g)_* X(e) = X(g) \qquad \text{and} \qquad (L_{g^{-1}})_* X_A(g) = A.)$$

Therefore, $\mathfrak{X}_L(G)$ *and* T_eG *are isomorphic (as vector spaces).* It follows that the dimension of the vector space $\mathfrak{X}_L(G)$ is equal to $\dim T_eG = \dim G$.

NOTE : Since, by assumption, G is a (finite-dimensional) manifold it follows that $\mathfrak{X}_L(G)$ is a finite-dimensional, nontrivial subalgebra of the Lie algebra of all (smoth) vector fields on G.

For any $A, B \in T_eG$, we define their *Lie product* (bracket) $[A, B]$ by

$$[A, B] := [X_A, X_B](e)$$

where $[X_A, X_B]$ is the Lie bracket of vector fields. This makes T_eG into a Lie algebra. We say that this defines a Lie product in T_eG via *left extension.*

NOTE : By construction,

$$[X_A, X_B] = X_{[A,B]}$$

for all $A, B \in T_eG$.

1.2.3 DEFINITION. The vector space T_eG with this Lie algebra structure is called the **Lie algebra** of G and is denoted by $\mathfrak{g}$.

Exercise 3 Let $\varphi : G \to H$ be a smooth homomorphism between the Lie groups G and H. Show that the induced mapping

$$d\varphi = \varphi_{*,e} : T_eG = \mathfrak{g} \to T_eH = \mathfrak{h}$$

is a homomorphism between the Lie algebras of the groups.

A similar construction to the above can be carried out with the Lie algebra $\mathfrak{X}_R(G)$ of right-invariant vector fields on G. In this case, for each $A \in T_eG$, the corresponding right-invariant vector field is defined by

$$Y_A(g) := (R_g)_{*,e}A.$$

We have (for $A, B \in T_eG$)

$$[Y_A, Y_B](e) = -[X_A, X_B](e).$$

Therefore, the Lie product $[\cdot,\cdot]^R$ in $\mathfrak{g}$ defined by *right extension* of elements of $\mathfrak{g}$:

$$[A,B]^R := [Y_A, Y_B](e)$$

is the *negative* of the one defined by left extension; that is,

$$[A,B]^R = -[A,B].$$

NOTE : There is a natural isomorphism between the (Lie algebras) $\mathfrak{X}_L(G)$ and $\mathfrak{X}_R(G)$. It is equal to the tangent mapping of $\Phi : G \to G, \quad x \mapsto x^{-1}$. In particular, we have (for $A \in \mathfrak{g} = T_eG$)

$$\Phi_* X_A = -Y_A.$$

Orbits of invariant vector fields

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1.3 The Exponential Mapping

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1.4 Matrix Groups as Lie Groups

We have seen that the matrix groups $\mathsf{GL}\,(n,\mathbb{k})$, $\mathsf{SL}\,(n,\mathbb{k})$, and $\mathsf{O}\,(n)$ are all Lie groups. These examples are typical of what happens for any matrix group that is a Lie subgroup of $\mathsf{GL}\,(n,\mathbb{R})$. The following important result holds.

1.4.1 THEOREM. *Let $G \leq \mathsf{GL}\,(n,\mathbb{R})$ be a matrix group. Then G is a Lie subgroup of $\mathsf{GL}\,(n,\mathbb{R})$.*

NOTE : In fact, a more general result also holds (but we will not give a proof) : *Every closed subgroup of a Lie group is a Lie subgroup.*

Our aim in this section is to prove THEOREM 4.5.1.

Let $G \leq \mathsf{GL}(n, \mathbb{R})$ be a matrix group, and let $\mathfrak{g} = T_I G$ denote its Lie algebra.

1.4.2 PROPOSITION. *Let*

$$\widetilde{\mathfrak{g}} := \{A \in \mathbb{R}^{n\times n} \mid \exp(tA) \in G \ \textit{for all} \ t\}.$$

Then $\widetilde{\mathfrak{g}}$ *is a Lie subalgebra of* $\mathbb{R}^{n\times n}$.

PROOF : By definition, $\widetilde{\mathfrak{g}}$ is closed under (real) scalar multiplication. If $U, V \in \widetilde{\mathfrak{g}}$ and $r \geq 1$, then the following are in G :

$$\exp\left(\frac{1}{r}U\right)\exp\left(\frac{1}{r}V\right), \quad \left(\exp\left(\frac{1}{r}U\right)\exp\left(\frac{1}{r}V\right)\right)^r,$$
$$\exp\left(\frac{1}{r}U\right)\exp\left(\frac{1}{r}V\right)\exp\left(-\frac{1}{r}U\right)\exp\left(-\frac{1}{r}V\right),$$
$$\left(\exp\left(\frac{1}{r}U\right)\exp\left(\frac{1}{r}V\right)\exp\left(-\frac{1}{r}U\right)\exp\left(-\frac{1}{r}V\right)\right)^{r^2}.$$

For $t \in \mathbb{R}$, by the LIE-TROTTER PRODUCT FORMULA we have

$$\exp(tU + tV) = \lim_{r\to\infty}\left(\exp\left(\frac{1}{r}tU\right)\exp\left(\frac{1}{r}tV\right)\right)^r$$

and by the COMMUTATOR FORMULA

$$\begin{aligned}\exp(t[U,V]) &= \exp([tU,V]) \\ &= \lim_{r\to\infty}\left(\exp\left(\frac{1}{r}tU\right)\exp\left(\frac{1}{r}V\right)\exp\left(-\frac{1}{r}tU\right)\exp\left(-\frac{1}{r}V\right)\right)^{r^2}.\end{aligned}$$

As these are both limits of elements of the closed subgroup $G \leq \mathsf{GL}(n, \mathbb{R})$, they are also in G. This shows that $\widetilde{\mathfrak{g}}$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n\times n}$.

□

1.4.3 Corollary. *$\widetilde{\mathfrak{g}}$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof : Let $U \in \widetilde{\mathfrak{g}}$. Then the curve

$$\gamma : \mathbb{R} \to G, \quad t \mapsto \exp(tU)$$

has $\gamma(0) = I$ and $\dot{\gamma}(0) = U$, hence $U \in \mathfrak{g}$. □

Note : Eventually we will see that $\widetilde{\mathfrak{g}} = \mathfrak{g}$.

We will require a technical result.

1.4.4 Lemma. *Let $(A_r)_{r\geq 1}$ and $(\lambda_r)_{r\geq 1}$ be sequences in $\exp^{-1}(G)$ and $\mathbb{R}$, respectively. If $\|A_r\| \to 0$ and $\lambda_r A_r \to A \in \mathbb{R}^{n\times n}$ as $r \to \infty$, then $A \in \widetilde{\mathfrak{g}}$.*

Proof : Let $t \in \mathbb{R}$. For each r, choose an integer $m_r \in \mathbb{Z}$ so that $|t\lambda_r - m_r| \leq 1$. Then

$$\begin{aligned} \|m_r A_r - tA\| &\leq \|(m_r - t\lambda_r)A_r\| + \|t\lambda_r A_r - tA\| \\ &= |m_r - t\lambda_r|\|A_r\| + \|t\lambda_r A_r - tA\| \\ &\leq \|A_r\| + |t|\|\lambda_r A_r - A\| \to 0 \end{aligned}$$

as $r \to \infty$, showing that $m_r A_r \to tA$. Since $\exp(m_r A_r) = \exp(A_r)^{m_r} \in G$ and G is closed in $\mathsf{GL}\,(n, \mathbb{R})$, we have

$$\exp(tA) = \lim_{r\to\infty} \exp(m_r A_r) \in G.$$

Thus every scalar multiple tA is in $\exp^{-1}(G)$, showing that $A \in \widetilde{\mathfrak{g}}$. □

Proof of Theorem 4.5.1 : Choose a complementary $\mathbb{R}$-subspace $\mathfrak{w}$ to $\widetilde{\mathfrak{g}}$ in $\mathbb{R}^{n\times n}$; that is, any vector subspace such that

$$\begin{aligned} \widetilde{\mathfrak{g}} + \mathfrak{w} &= \mathbb{R}^{n\times n} \\ \dim \widetilde{\mathfrak{g}} + \dim \mathfrak{w} &= \dim \mathbb{R}^{n\times n} = n^2. \end{aligned}$$

(The second of these conditions is equivalent to $\widetilde{\mathfrak{g}} \cap \mathfrak{w} = 0$.) This gives a a *direct sum decomposition* of $\mathbb{R}^{n\times n}$, so every element $X \in \mathbb{R}^{n\times n}$ has a unique decomposition of the form

$$X = U + V \qquad (U \in \widetilde{\mathfrak{g}},\, V \in \mathfrak{w}).$$

Consider the mapping

$$\Phi : \mathbb{R}^{n\times n} \to \mathsf{GL}\,(n, \mathbb{R}), \quad U + V \mapsto \exp(U)\exp(V).$$

Φ is a smooth mapping which maps O to I. Observe that the factor $\exp(U)$ is in G. Consider the derivative (at O)

$$D\Phi(O) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}.$$

To determine $D\Phi(O)\cdot(A+B)$, where $A \in \widetilde{\mathfrak{g}}$ and $B \in \mathfrak{h}$, we differentiate the curve $t \mapsto \Phi(t(A+B))$ at $t = 0$. Assuming that A and B small enough, for small $t \in \mathbb{R}$, there is a unique matrix $C(t)$ (depending on t) for which

$$\Phi(t(A+B)) = \exp(C(t)).$$

Then (by using the estimate in Proposition 3.5.6)

$$\|C(t) - tA - tB - \frac{t^2}{2}[A,B]\| \le 65|t|^3\left(\|A\| + \|B\|\right)^3.$$

From this we obtain

$$\begin{aligned} \|C(t) - tA - tB\| &\le \frac{t^2}{2}\|[A,B]\| + 65|t|^3\left(\|A\| + \|B\|\right)^3 \\ &= \frac{t^2}{2}\left(\|[A,B]\| + 130|t|\left(\|A\| + \|B\|\right)^3\right) \end{aligned}$$

and so

$$\begin{aligned} D\Phi(O)\cdot(A+B) &= \frac{d}{dt}\Phi(t(A+B))\Big|_{t=0} \\ &= \frac{d}{dt}\exp(C(t))\Big|_{t=0} \\ &= A + B. \end{aligned}$$

Hence $D\Phi(O)$ is the identity mapping on $\mathbb{R}^{n\times n}$, and by the Inverse Mapping Theorem, there exists an open neighborhood (and we may take this to be an open ball) $\mathcal{B}_{\mathbb{R}^{n\times n}}(O,\delta)$ of O such that the restriction

$$\Phi_1 := \Phi|_{\mathcal{B}(O,\delta)} : \mathcal{B}(O,\delta) \to \Phi\left(\mathcal{B}(O,\delta)\right)$$

is a smooth diffeomeorphism.

Now we must show that Φ maps some open subset (which we may assume to be an open ball) of $\mathcal{B}_{\mathbb{R}^{n\times n}}(O,\delta)\cap\widetilde{\mathfrak{g}}$ onto an open neighborhood of I in G. Suppose not. Then there is a sequence of elements $(U_r)_{r\geq 1}$ in G with $U_r \to I$ as $r\to\infty$ but $U_r \notin \Phi(\widetilde{\mathfrak{g}})$. For large enough r, $U_r \in \Phi(\mathcal{B}(O,\delta))$, hence there are unique elements $A_r \in \widetilde{\mathfrak{g}}$ and $B_r \in \mathfrak{w}$ with $\Phi(A_r+B_r)=U_r$. Notice that $B_r \neq O$ since otherwise $U_r \in \Phi(\widetilde{\mathfrak{g}})$. As Φ_1 is a diffeomorphism, $A_r+B_r \to O$ and this implies that $A_r \to O$ and $B_r \to O$. By definition of Φ,

$$\exp(B_r) = \exp(A_r)^{-1}U_r \in G.$$

Hence $B_r \in \exp^{-1}(G)$. Consider the elements $\bar{B}_r = \frac{1}{\|B_r\|}B_r$ of unit norm. Each $\bar{B}_r$ is in the unit sphere in $\mathbb{R}^{n\times n}$, which is compact hence there is a convergent subsequence of $(\bar{B}_r)_{r\geq 1}$. By renumbering this subsequence, we can assume that $\bar{B}_r \to B$, where $\|B\|=1$. Applying Lemma 4.5.4 to the sequences $(B_r)_{r\geq 1}$ and $\left(\frac{1}{\|B_r\|}\right)_{r\geq 1}$, we find that $B\in\widetilde{g}$. But each B_r (and hence $\bar{B}_r$) is in $\mathfrak{w}$, so B must be too. Thus $B\in\widetilde{g}\cap\mathfrak{w}$, which contradicts the fact that $B\neq O$.

So there must be an open ball

$$\mathcal{B}_{\widetilde{\mathfrak{g}}}(O,\delta_1) = \mathcal{B}_{\mathbb{R}^{n\times n}}(O,\delta_1)\cap\widetilde{\mathfrak{g}}$$

which is mapped by Φ onto an open neighborhood of I in G. So the restriction of Φ to this open ball is a local diffeomorphism at O. The inverse

mapping gives a local chart for G at I (and moreover $\mathcal{B}_{\widetilde{\mathfrak{g}}}(O, \delta_1)$ is then a smooth submanifold of $\mathbb{R}^{n\times n}$). We can use left translation to move this local chart to a new chart at any other point $U \in G$ (by considering $L_U \circ \Phi$).

So we have shown that $G \leq \mathsf{GL}\,(n, \mathbb{R})$ is a smooth submanifold. The matrix product $(A, B) \mapsto AB$ is clearly a smooth (in fact, analytic) function of the entries of A and B, and (in light of Cramer's rule) $A \mapsto A^{-1}$ is a smooth (in fact, analytic) function of the entries of A. Hence G is a Lie subgroup, proving THEOREM 4.5.1.

□

This is a fundamental result that can be usefully reformulated as follows : *A subgroup of* $\mathsf{GL}\,(n, \mathbb{R})$ *is a closed Lie subgroup if and only if it is a matrix subgroup.* (More generally, a subgroup of a Lie group G is a closed Lie subgroup if and only if is a closed subgroup.)

NOTE : Recall that the dimension of a matrix group G (as a manifold) is $\dim \widetilde{\mathfrak{g}}$. By COROLLARY 4.5.3, $\widetilde{\mathfrak{g}} \subseteq \mathfrak{g}$ and so $\dim \widetilde{\mathfrak{g}} \leq \dim \mathfrak{g}$. By definition of $\mathfrak{g} = T_I G$, these dimensions are in fact equal, giving

$$\widetilde{\mathfrak{g}} = \mathfrak{g}.$$

Combining with PROPOSITION 3.3.3, this gives the following result : *For a matrix group* $G \leq \mathsf{GL}\,(n, \mathbb{R})$*, the exponential mapping*

$$\exp : \mathfrak{g} \to \mathbb{R}^{n\times n}$$

has image in G*. Moreover,* $\exp_G$ *is a local diffeomorphism at the origin (mapping some open neighborhood of* 0 *onto an open neighborhood of* I *in* G*).*

It is a remarkable fact that most of the important examples of Lie groups are (or can easily be represented as) matrix groups. However, *not all Lie groups are matrix groups.* For the sake of completeness, we shall describe the simplest example of a Lie group which is *not* a matrix group.

Consider the matrix group (of *unipotent* 3×3 matrices)

$$\mathsf{H}(1) = \left\{ \gamma(x,y,t) = \begin{bmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x,y,t \in \mathbb{R} \right\} \leq \mathsf{GL}(3,\mathbb{R})$$

commonly referred to as the *Heisenberg group*. $\mathsf{H}(1)$ is a 3-dimensional Lie group.

NOTE : More generally, the *Heisenberg group* $\mathsf{H}(n)$ is defined by

$$\mathsf{H}(n) = \left\{ \gamma(x,y,t) = \begin{bmatrix} 1 & x^T & t \\ 0 & I_n & y \\ 0 & 0 & 1 \end{bmatrix} \mid (x,y) \in \mathbb{R}^{2n}, t \in \mathbb{R} \right\} \leq \mathsf{GL}(n+2,\mathbb{R}).$$

This (matrix) group is *isomorphic* to either one of the following groups :

- $\mathbb{R}^{2n+1}$ equipped with the group multiplication

$$(x,y,t) * (x',y',t') = (x+x', y+y', t+t'+x \bullet y').$$

- $\mathbb{R}^{2n+1}$ equipped with the group multiplication

$$(x,y,t)(x',y',t') = \left(x+x', y+y', t+t'+\frac{1}{2}(\Omega((x,y),(x',y'))) \right)$$

  where $\Omega((x,y),(x',y')) = x \bullet y' - x' \bullet y$ is the standard *symplectic form* on $\mathbb{R}^{2n}$.

The Lie algebra $\mathfrak{h}(n)$ of $\mathsf{H}(n)$ is given by

$$\mathfrak{h}(n) = \left\{ \Gamma(x,y,t) = \begin{bmatrix} 0 & x^T & t \\ 0 & O_n & y \\ 0 & 0 & 0 \end{bmatrix} \mid (x,y) \in \mathbb{R}^{2n}, t \in \mathbb{R} \right\}.$$

(The Lie algebra $\mathfrak{h}(1)$, which occurs throughout quantum physics, is essentially the same as the Lie algebra of operators on differentiable functions $f : \mathbb{R} \to \mathbb{R}$ spanned by the three operators $\mathbf{1}, \mathbf{p}, \mathbf{q}$ defined by

$$\mathbf{1}f(x) := f(x), \quad \mathbf{p}f(x) := \frac{d}{dx}f(x), \quad \mathbf{q}f(x) := xf(x).$$

The non-trivial commutator involving these three operators is given by the *canonical commutation relation* $[\mathbf{p}, \mathbf{q}] = \mathbf{pq} - \mathbf{qp} = \mathbf{1}$.)

Exercise 4 Determine the (group) commutator in $\mathsf{H}(1)$ (i.e. the product $\gamma\gamma'\gamma^{-1}\gamma'^{-1}$ for $\gamma, \gamma' \in \mathsf{H}(1)$) and hence deduce that the *centre* $Z(\mathsf{H}(1))$ of $\mathsf{H}(1)$ is

$$Z(\mathsf{H}(1)) = \{\gamma(0,0,t) \,|\, t \in \mathbb{R}\}\,.$$

Clearly, there is an isomorphism (of Lie groups) between $\mathbb{R}$ and $Z(\mathsf{H}(1))$, under which the subgroup $\mathbb{Z}$ of integers corresponds to the subgroup $\mathcal{Z}$ of $Z(\mathsf{H}(1))$. Thus

$$\mathcal{Z} = \{\gamma(0,0,t) \,|\, t \in \mathbb{Z}\}\,.$$

The subgroup $\mathcal{Z}$ is *discrete* and also *normal*.

NOTE : (1) By a *discrete* group Γ is meant a group with a countable number of elements and the discrete topology (every point is an open set). A discrete group is a 0-dimensional Lie group. Closed 0-dimensional Lie subgroups of a Lie group are usually called *discrete subgroups*. The following remarkable result holds : *If* Γ *is a discrete subgroup of a Lie group* G*, then the space of right (or left) cosets* G/Γ *is a smooth manifold* (and the natural projection $G \to G/\Gamma$ is a smooth mapping).

(2) A subgroup N of G is *normal* if for any $n \in N$ and $g \in G$ we have $gng^{-1} \in N$. A kernel of a homomorphism is normal. Conversely, if N is normal, we can define the *quotient group* G/N whose elements are equivalence classes $[g]$ of elements in G, and two elements g, h are equivalent if and only if $g = hn$ for some $n \in N$. The multiplication is given by $[g][h] = [gh]$ and the fact that N is normal says that this is well-defined. Thus *normal subgroups are exactly kernels of homomorphisms.*

Hence we can form the *quotient group*

$$\mathsf{H}(1)/\mathcal{Z}$$

which is in fact a (3-dimensional) Lie group. (Its Lie algebra is $\mathfrak{h}(1)$.)

The following result (which we will not prove) tells that *the Lie group* $\mathsf{H}(1)/\mathcal{Z}$ *cannot be realized as a matrix group.*

1.4.5 Proposition. *There are no continuous homomorphisms* $\varphi : \mathsf{H}(1)/\mathcal{Z} \to \mathsf{GL}(n, \mathbb{C})$ *with trivial kernel.*

1.5 Hamiltonian Vector Fields

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1.6 Lie-Poisson Reduction

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Problems and Further Results

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Chapter 2

Control Systems

Topics :

1. Control Systems: Definition and Examples
2. Invariant Systems on Matrix Lie Groups
3. Examples
4. Controllability
5. Linear Control Systems
6. Serret-Frenet Control Systems

2.1 Control Systems: Definition and Examples

There are many important problems (notably in engineering and physical sciences), envolving the study of various *control systems*, which cannot be treated satisfactory by "classical" (i.e. linear) control theory. This is the case essentially because the state space (of the control system under consideration) is *not* a vector space, but is, in a natural way, a much more sofisticated "non-linear" space, namely a *manifold*. Linearization often destroys the essence of the problem and new and different methods are needed (especially for treating global questions). It appears that differential-geometric methods, introduced in the 1970s, provide a very useful language and, at the same time, a powerful machinery for tackling most of these problems.

In what follows we shall restrict ourselves to the special, but very interesting case, when the state space is a *matrix Lie group*.

Matrix Lie groups arise naturally as the models for the *configuration space* of mechanical systems. For instance, the position and orientation of a rigid body in Euclidean 3-space can be completely characterized by the special Euclidean group $\mathsf{SE}(3)$. Control systems on matrix Lie groups thus find application in modeling and motion control of mechanical systems such as *robotic manipulators, wheeled robots, underwater vehicles*, and *spacecraft*.

Next to mechanical applications, matrix Lie groups also arise from physical *conservation principles* such as conservation of energy. For instance, *electrical networks* used for power conversion can be modeled as control systems evolving on the special orthogonal group $\mathsf{SO}(3)$, and so-called multilevel systems used to model molecular bonds in the context of *coherent control of quantum dynamics* can naturally be represented as control systems on the unitary group $\mathsf{U}(n)$.

Furthermore, matrix Lie groups arise in the study of the state transition matrix of a time-varying linear control system (on some Euclidean space).

From a theoretical point of view, control systems on matrix Lie grups are also an interesting subject of study since they form an important sub-class of *nonlinear* control systems. Their structure leads to simplifications which allows us to study the essence of various nonlinear control questions of more general formulations.

Control systems on matrix Lie groups were first introduced in 1972 by Roger Brockett who expressed notions such as (nonlinear) *controllability, observability*, and *realization theory* for (right-invariant) control systems evolving on matrix Lie groups. Velimir Jurdjevic and Héctor Sussmann further investigated the controllability properties of control systems on *abstract* Lie groups. One of the most important insights derived from this work was the recognition that questions about these kind of control systems on Lie groups can be reduced to questions about their associated Lie algebras. Since Lie algebras are (finite-dimensional) vector spaces, whereas Lie groups are *manifolds*, this reduction greatly simplifies the problem.

Constructive questions for control systems on matrix Lie groups such as deriving *optimal controls* for certain lower-dimensional control systems on matrix Lie groups were taken up by P.S. Krishnaprasad and Naomi Leonard in the early 1990s.

The study of (invariant) control systems on abstract Lie groups has been a subject of active research in mathematical control theory in the last three decades or so. The study is motivated both by important applications (in engineering and physical sciences) and by essential links with various branches of mathematics outside control theory (e.g. Lie groups and Lie algebras, dif-

ferential geometry, Lie semigroups, dynamical systems).

Control Systems

Roughly speaking, a *control system* (on a smooth manifold) is any system of ordinary differential equations in which control functions appear as parameters.

NOTE : A control system can be viewed as a (deterministic, smooth, finite dimensional) *dynamical system* whose dynamical laws are not entirely fixed but depend on parameters, called controls, that can vary and with which one can control the behaviour of the system.

From a geometric viewpoint, each control determines a vector field, and therefore *a control system can be viewed as a family* $\mathcal{F} = (F_u)_{u \in U}$ *of vector fields.* A *trajectory* of such a system is a (continuous) curve made up of finitely many segments of integral curves of vector fields in the family.

NOTE : More generally, let $\mathcal{F}$ be an arbitrary family of vector fields (on the smooth manifold M). For the sake of simplicity we shall assume that all the elements of $\mathcal{F}$ are *complete* vector fields. Then each element $X \in \mathcal{F}$ generates a one-parameter group of diffeomorphisms of M $(\exp tX)_{t \in \mathbb{R}}$. Let $G(\mathcal{F})$ denote the *group* of diffeomorphisms generated by $\bigcup_{X \in \mathcal{F}} (\exp tX)_{t \in \mathbb{R}}$. (The elements of $G(\mathcal{F})$ are precisely the diffeomorphisms Φ of M of the form

$$\Phi = (\exp t_k X_k) \circ (\exp t_{k-1} X_{k-1}) \circ \cdots \circ (\exp t_1 X_1)$$

for some $t_1, \ldots, t_k \in \mathbb{R}$ and $X_1, \ldots, X_k \in \mathcal{F}$.) $G(\mathcal{F})$ acts on M in the obvious way and partitions M into its orbits :

$$M = \bigcup_{p \in M} \mathcal{O}(p).$$

(The $G(\mathcal{F})$-*orbit* through the point $p \in M$ is $\mathcal{O}(p) = \{\Phi(p) \,|\, \Phi \in G(\mathcal{F})\}$.) The $G(\mathcal{F})$-orbits are referred to as the *orbits* of $\mathcal{F}$ and their structure is described in the following fundamental result :

(Orbit Theorem) *Every orbit $\mathcal{O}(p)$ of $\mathcal{F}$ is a connected, immersed submanifold of M. Moreover, the tangent space to $\mathcal{O}(p)$ at $q \in \mathcal{O}(p)$ is*

$$T_q\mathcal{O}(p) = \operatorname{span}\{\Phi_* X(q) \,|\, \Phi \in G(\mathcal{F}),\, X \in \mathcal{F}\}.$$

This result has a remarkable significance in geometric control theory.

Let $\mathsf{Lie}\,(\mathcal{F})$ denote the Lie algebra of vector fields generated by the family $\mathcal{F}$. ($\mathsf{Lie}\,(\mathcal{F})$ can be described as

$$\mathsf{Lie}\,(\mathcal{F}) = \operatorname{span}\{\operatorname{ad} X_1 \circ \operatorname{ad} X_2 \circ \cdots \circ \operatorname{ad} X_{k-1}(X_k) \,|\, X_1, \ldots, X_k \in \mathcal{F}\}$$

where $\operatorname{ad} X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the mapping $Y \mapsto \operatorname{ad} X(Y) := [X, Y]$.) For each $q \in M$, the *evaluation* of $\mathsf{Lie}_q(\mathcal{F})$ at q is (the vector space)

$$\mathsf{Lie}_q(\mathcal{F}) = \{X(q) \,|\, X \in \mathcal{F}\} \subseteq T_qM.$$

The following relation holds for every $q \in M$:

$$\mathsf{Lie}_q(\mathcal{F}) \subseteq T_q\mathcal{O}(p).$$

In many important cases (for instance, when M is a Lie group), this inclusion turns out to be an equality.

We make the following definition.

2.1.1 Definition. A **control system** is (given by) a mapping

$$F : M \times U \to TM, \quad (x, u) \mapsto F_u(x)$$

where

- M is a smooth m-dimensional manifold, called the *state space*;

- U is an arbitrary subset of (the Cartesian ℓ-space) $\mathbb{R}^\ell$, called the *control set*;
- TM is the tangent bundle of M (a smooth $2m$-dimensional manifold).

It is assumed that

- the mapping F is continuous;
- for each $u \in U$, the mapping

$$F_u = F(\cdot, u) : M \to TM$$

  is smooth. (F_u is a smooth vector field on M.)

Such a control system is usually written (in *classical notation*) as follows

$$\dot{x} = F(x, u), \qquad x \in M,\ u \in U \subseteq \mathbb{R}^\ell.$$

The variable x is the *state* and represents the "memory" of the system. The variable u is the *control* (or the input) and represents the external influence on the system. We define a **control** to be a U-valued mapping defined on some (compact) interval:

$$u(\cdot) : [a, b] \to U, \quad t \mapsto u(t) = (u_1(t), \ldots, u_\ell(t)) \in U.$$

Generally, a control must satisfy certain regularity conditions, in which case it is referrred to as an *admissible* control. For all geometric considerations *it is sufficient to consider only piecewise constant controls.*

NOTE : (1) The control functions, when regarded as an ℓ-tuple $u = (u_1, \ldots, u_\ell)$, are constrained to take value in a fixed subset U of $\mathbb{R}^\ell$, called the *control set*. Generally, U is assumed to be a *closed* subset of $\mathbb{R}^\ell$ (sometimes a *compact* or even *compact*

convex subset) with nonempty interior. Whenever $U = \mathbb{R}^\ell$, we may refer to the control system as an *unrestricted* control system.

(2) Although convenient for geometric considerations, piecewise constant controls are not particularly suitable for problems of optimal control. For such problems, the class of admissible controls $\mathcal{U}$ needs to be enlarged to accomodate more general controls (like *piecewise continuous* ones).

(3) Formally, a (nonlinear) control system is a 4-tuple

$$\Sigma = (M, U, \mathcal{U}, F)$$

where the manifold M is the *state space*, $U \subseteq \mathbb{R}^\ell$ is the *control set*, $\mathcal{U}$ is the class of admissible controls, and the mapping F is the *dynamics*. It is the dynamics, or the associated family of vector fields $\mathcal{F} = (F_u)_{u \in U}$, which provides a local in time description (i.e. the *state equation*) of Σ :

$$\dot{x} = F_u(x), \qquad x \in M,\ u = (u_1, \ldots, u_\ell) \in U.$$

The case when F_u is of the form

$$F_u = X_0 + u_1 X_1 + \cdots + u_\ell X_\ell, \quad u = (u_1, \ldots, u_\ell) \in U \subseteq \mathbb{R}^\ell$$

(i.e. each vector field F_u is an affine combination of some *fixed* vector fields $X_0, X_1, \ldots, X_\ell$) is of particular importance for applications. Such a **control-affine system** is usually written as follows

$$\dot{x} = X_0(x) + u_1 X_1(x) + \cdots + u_\ell X_\ell(x)$$

with *piecewise constant* control functions $u_1(\cdot), u_2(\cdot), \ldots, u_\ell(\cdot)$. The vector field X_0 is called the *drift*, and the remaining vector fields $X_1, \ldots, X_\ell$ are called the *controlled* vector fields. If $X_0 = 0$ and $0 \in \operatorname{int} U$, then we say that the system is *driftless* (or *homogeneous*).

The class of control-affine systems serves as a *kinematic model* for a wide range of problems relevant to mechanics, geometry, and control.

2.1.2 Example. (The Liénard control system) A general nonlinear *oscillator* with an external force $u(\cdot)$ is described by the (second-order differential) equation

$$\ddot{z} + a(z)\dot{z} + b(z) = u(t)$$

(known as the *Liénard equation*). This equation can be expressed as an equivalent first-order system (of diferential equations) in the *phase plane* by introducing the new variables $x_1 := z$ and $x_2 := \dot{z}$. Then

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -a(x_1)x_2 - b(x_1) + u.$$

If we set

$$X := \begin{bmatrix} x_2 \\ -a(x_1)x_2 - b(x_1) \end{bmatrix} \quad \text{and} \quad Y := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then we get (the state equation describing) a control-affine system (on $\mathbb{E}^2$)

$$\dot{x} = X(x) + uY(x), \qquad x \in \mathbb{E}^2,\ u \in U \subseteq \mathbb{R}.$$

The *external force* $u(\cdot)$ plays the role of a (scalar) control.

2.1.3 Example. (Mechanical system with damping controls) Consider the problem of controlling a *mechanical system*

$$\ddot{z} + u\dot{z} + f(z) = 0$$

by a *damping* control function $u(\cdot)$. The equivalent first-order system is given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -f(x_1) - ux_2.$$

For the sake of simplicity, assume that $f(z) = kz$ for same constant k. Then the forgoing system can be rewritten as

$$\begin{aligned} \dot{x} &= Ax + uBx \\ &= (A + uB)\,x, \qquad x \in \mathbb{E}^2,\ u \in U \subseteq \mathbb{R} \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have obtained (the state equation of) a control-affine system (on $\mathbb{E}^2$).

NOTE : A control-affine system of the form

$$\begin{aligned} \dot{x} &= Ax + u_1B_1x + \cdots + u_\ell B_\ell x \\ &= (A + u_1B_1 + \cdots + u_\ell B_\ell)\,x, \qquad x \in \mathbb{E}^m,\ u = (u_1, \ldots, u_\ell) \in U \subseteq \mathbb{R}^\ell \end{aligned}$$

where $A, B_i \in \mathbb{R}^{m\times m}$, is called a *bilinear* system (on the Euclidean m-space $\mathbb{E}^m$).

2.1.4 EXAMPLE. (LINEAR CONTROL SYSTEMS) A **linear control system** is a control-affine system on a Euclidean space $M = \mathbb{E}^m$ with a *linear* drift X_0 and each controlled vector field X_i *constant*.

Denoting the constant values of $X_1, \ldots, X_\ell$ by $b_1, \ldots, b_\ell$, and the drift term by a linear vector field A, $x \mapsto Ax$, the corresponding linear control system is given by

$$\begin{aligned} \dot{x} &= Ax + u_1b_1 + \cdots + u_\ell b_\ell \\ &= Ax + Bu, \qquad x \in \mathbb{E}^m,\ u = (u_1, \ldots u_\ell) \in U \subseteq \mathbb{R}^\ell \end{aligned}$$

where $B = \begin{bmatrix} b_1 & \ldots & b_\ell \end{bmatrix} \in \mathbb{R}^{m\times\ell}$. The case $\ell = 1$ is called the *single-input* case. Single-input linear control systems are intricately connected with m^{th}-order ODEs with constant coefficients.

Exercise 5 Verify that the mth-order ODE

$$z^{(m)} + a_1 z^{(m-1)} + \cdots + a_m z = u(t)$$

can be *converted* into its single-input linear control system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \cdots & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u, \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{E}^m.$$

NOTE : It is somewhat remarkable that *"almost all" single-input linear control systems are higher-order ODEs in disguise.* More precisly, if

$$\dot{x} = Ax + bu, \qquad x \in \mathbb{E}^m,\ u \in \mathbb{R}$$

is a single-input linear control system such that rank $\begin{bmatrix} b & Ab & \cdots & A^{m-1}b \end{bmatrix} = m$, than there exists a linear transformation (change of coordinates) $\widetilde{x} = Tx$ such that

$$\begin{aligned} \widetilde{A} &= TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_m & -a_{m-1} & -a_{m-2} & \cdots & -a_1 \end{bmatrix} \\ \widetilde{b} &= Tb = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Abstract Lie groups (in particular, matrix Lie groups) form an important class of smooth (in fact, *analytic*) manifolds. Henceforth, in this chapter, we shall consider only *control-affine systems on matrix Lie groups.*

2.2 Invariant Control Systems on Matrix Lie Groups

Invariant vector felds

Let $G \leq \mathsf{GL}\,(n, \mathbb{R})$ be an m-dimensional matrix Lie group with identity $e = I_n \in G$. Let $\mathfrak{g} \subseteq \mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n \times n}$ denote the Lie algebra of G (i.e. the tangent space $T_e G$ at the identity).

NOTE : Given an arbitrary element (matrix) $A \in \mathbb{R}^{n \times n}$, the *Cauchy problem* (i.e. ODE + initial condition), on the general linear group $\mathsf{GL}\,(n, \mathbb{R})$,

$$\dot{g} = gA, \quad g(0) = g_0 \qquad (g \in \mathsf{GL}\,(n, \mathbb{R}))$$

has a (unique) solution of the form $g(t) = g_0 \exp(tA)$ (see **Exercise 163**).

A similar problem, on the matrix Lie group G, fails to be well-defined, unless $A \in \mathfrak{g}$. (This is the case because $\exp(tA) \in G$ for all $t \iff A \in \mathfrak{g}$.)

Recall that

$$T_g G = \{\dot{\gamma}(0) \,|\, \gamma(t) \in G,\ \gamma(0) = g\}.$$

Exercise 6 Show that (for $g \in G$)

$$T_g G = g\, T_e G = \{gA \,|\, A \in \mathfrak{g}\}.$$

(The left translation L_g moves the tangent space at the identity to the tangent space at g.)

Thus for any element $A \in \mathfrak{g}$, the correspondence

$$g \in G \mapsto gA \in T_g G$$

defines a (smooth) vector field on (the matrix Lie group) G.

A vector field X on G is **left-invariant** if $X(g) = gA$ for some fixed (matrix) $A \in \mathfrak{g}$.

NOTE : (1) Recall that a vector field X on an *abstract* Lie group G is left-invariant if (and only if) for every $g \in G$

$$(L_g)_* X(e) = X(g).$$

The left translation $L_g : \mathsf{GL}(n, \mathbb{R}) \to \mathsf{GL}(n, \mathbb{R})$ is a *linear* transformation, hence $(L_g)_* = L_g : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ ($T_e \mathsf{GL}(n, \mathbb{R}) = \mathbb{R}^{n \times n}$; see also **Exercise 202**). When $G \leq \mathsf{GL}(n, \mathbb{R})$, it follows that (for $g \in G$)

$$\begin{aligned} X(g) &= (L_g)_* X(e) \\ &= L_g(A) \\ &= gA \end{aligned}$$

where $A = X(e) \in \mathfrak{g}$.

(2) The set $\mathfrak{X}_L(G)$ of all left-invariant vector fields on G has the structure of a vector space (in fact, a Lie algebra). The correspondence

$$X \in \mathfrak{X}_L(G) \mapsto X(e) \in T_e G = \mathfrak{g}$$

is an *isomorphism* (of Lie algebras) and so we can *identify* any left-invariant vector field on the matrix Lie group G with its value at the identity.

(3) Similarly, a vector field Y on G is *right-invariant* if $Y(g) = Bg$ for some fixed $B \in \mathfrak{g}$. Again, the space (Lie algebra) $\mathfrak{X}_R(G)$ of all right-invariant vector fields on G is isomorphic to the Lie algebra $\mathfrak{g}$ of G (and thus to $\mathfrak{X}_L(G)$).

Henceforth we shall not distinguish – notationwise – between an element (matrix) $A \in \mathfrak{g}$ and its corresponding left-invariant vector field $g \mapsto gA$.

It follows that the ODE, on the matrix Lie group G,

$$\dot{g} = gA \qquad (g \in G)$$

is well-defined and has solutions $g(t) = g(0) \exp(tA)$. Equivalently, in geometric language, the integral curve of the left-invariant vector field $A = \mathfrak{X}_L(G) =$

$\mathfrak{g}$ through $g_0 \in G$ is

$$t \mapsto g_0 \exp(tA).$$

NOTE : We can write

$$(\exp tA)(g) = g \exp(tA).$$

Caution : The left-hand side represents the flow – in "exponential notation" – of the vector field A, $g \mapsto gA$, whereas the right-hand side represents the product (matrix multiplication) of g with the matrix exponential of $A \in \mathfrak{g}$.

It follows that *any left-invariant vector field on G is complete.*

NOTE : More generally, let Fl^X be the flow corresponding to the left-invariant vector field X on the *abstract* Lie group G, and let $\gamma_e : t \mapsto \mathrm{Fl}^X(t, e)$ denote the integral curve through the group identity $e \in G$. Then the curve

$$\gamma_g : J_g \to G, \qquad t \mapsto g\,\gamma_e(t) \ (= L_g(\gamma_e(t)))$$

is the integral curve of X through g, which furthermore satisfies (for each $g \in G$ and each $t \in \mathbb{R}$)

$$\mathrm{Fl}^X(t, g) = g\,\mathrm{Fl}^X(t, e).$$

This equality has several implications :

(i) *The integral curve γ_e is defined for all $t \in \mathbb{R}$.* One can easily verify that the set $\{\gamma_e(t) \,|\, t \in \mathbb{R}\}$ is an Abelian subgroup of G. Now, if the curve γ_e is defined for a particular value of t, then γ_e must be defined for $t + \epsilon$ (where ϵ is independent of t) since $\gamma_e(t + \epsilon) = \gamma_e(t)\gamma_e(\epsilon)$. Therefore γ_e is defined for all $t \in \mathbb{R}$.

(ii) *X is a complete vector field*, because $\mathrm{Fl}^X(t, g) = g\,\gamma_e(t)$, and therefore $t \mapsto \mathrm{Fl}^X(t, g)$ is defined for all $t \in \mathbb{R}$.

(iii) If X were right-invariant, then its flow Fl^X would satisfy $\mathrm{Fl}^X(t, g) = \mathrm{Fl}^X(t, e)\,g$. It therefore follows that implications (i) and (ii) are also true for the right-invariant vector fields.

(iv) In particular, if a left-invariant vector field and a right-invariant vector field are equal to each other at the identity, then their integral curves through the identity are the same.

We further observe the following important fact : *the left translation* L_h *maps an integral curve into an integral curve.* Indeed, if $t \mapsto g(0)\exp(tA)$ is an integral curve of $A \in \mathfrak{X}_L(G) = \mathfrak{g}$, then

$$\begin{aligned} h(t) &= hg(t) \\ &= hg(0)\exp(tA) \\ &= h(0)\exp(tA). \end{aligned}$$

Hence its left-translation is also an integral curve of A. (This explains the title "left-invariant" for vector fields of the form $g \mapsto gA$.)

The Lie bracket of left-invariant vector fields is also a left-invariant vector field (see also Proposition 4.4.9). More precisely, the following result holds.

2.2.1 Proposition. *Let* $A, g \mapsto gA$ *and* $B, g \mapsto gB$ *be left-invariant vector fields on the matrix Lie group* G*. Then (for* $g \in G$*)*

$$[A, B](g) = g(AB - BA).$$

Proof : We shall give a "direct" proof based on the following characterization of the Lie bracket of two (arbitrary) vector fields :

$$[X, Y](p) = \frac{d}{dt}\gamma(\sqrt{t})\Big|_{t=0}$$

where the curve $t \mapsto \gamma(t)$ is defined by

$$\gamma(t) = (\exp -tY) \circ (\exp -tX) \circ (\exp tY) \circ (\exp tX)\,(p).$$

The flows (in "exponential notation") of $A, B \in \mathfrak{X}_L(G) = \mathfrak{g}$ are given by

$$(\exp\, tA)(g) = g \exp(tA) \quad \text{and} \quad (\exp\, tB) = g \exp(tB).$$

Then (by computing the low-order terms of the curve γ)

$$\begin{aligned}
\gamma(t) &= (\exp\, -tB) \circ (\exp\, -tA) \circ (\exp\, tB) \circ (\exp\, tA)\,(g) \\
&= g \exp(tA) \exp(tB) \exp(-tA) \exp(-tB) \\
&= g \left(I + tA + \frac{t^2}{2} A^2 + \cdots \right) \left(I + tB + \frac{t^2}{2} B^2 + \cdots \right) \\
&\qquad \left(I - tA + \frac{t^2}{2} A^2 + \cdots \right) \left(I - tB + \frac{t^2}{2} B^2 + \cdots \right) \\
&= g \left(I + t(A+B) + \frac{t^2}{2}(A^2 + 2AB + B^2) + \cdots \right) \\
&\qquad \left(I - t(A+B) + \frac{t^2}{2}(A^2 + 2AB + B^2) + \cdots \right) \\
&= g \left(I + t^2(AB - BA) + \cdots \right)
\end{aligned}$$

hence

$$\gamma(\sqrt{t}) = g\left(I + t(AB - BA) + \cdots \right).$$

Thus

$$\begin{aligned}
[A, B](g) &= \frac{d}{dt} \gamma(\sqrt{t}) \Big|_{t=0} \\
&= g(AB - BA).
\end{aligned}$$

□

NOTE : For right-invariant vector fields A, $g \mapsto Ag$ and B, $g \mapsto Bg$ the following "less convenient" formula holds (for $g \in G$)

$$[A, B](g) = (BA - AB)g.$$

Let $\mathcal{A} \subseteq \mathbb{R}^{n \times n}$ be an arbitrary family of matrices. Each element $A \in \mathcal{A}$ may be viewed as a left-invariant vector field on $\mathsf{GL}(n, \mathbb{R})$. By the *Orbit*

Theorem, the orbit $\mathcal{O}(e)$ of $\mathcal{A}$ through the identity is a connected, immersed submanifold of $\mathsf{GL}\,(n,\mathbb{R})$. Moreover,

$$\begin{aligned}\mathcal{O}(e) &= \{(\exp\, t_k A_k)\circ\cdots\circ(\exp\, t_1 A_1)\,(e)\,|\,t_i\in\mathbb{R}, A_i\in\mathcal{A}, k\in\mathbb{Z}^+\}\\ &= \{\exp\,(t_1A_1)\cdots\exp\,(t_kA_k)\,|\,t_i\in\mathbb{R}, A_i\in\mathcal{A}, k\in\mathbb{Z}^+\}.\end{aligned}$$

Consequently, the orbit $\mathcal{O}(e)$ is a subgroup of $\mathsf{GL}\,(n,\mathbb{R})$. This subgroup is in fact a *Lie subgroup* of $\mathsf{GL}\,(n,\mathbb{R})$. We also have

$$\begin{aligned}T_e\mathcal{O}(e) &= \mathrm{Lie}\,(\mathcal{A})\\ T_g\mathcal{O}(e) &= g\mathrm{Lie}\,(\mathcal{A})\\ \mathcal{O}(g) &= \{g\exp\,(t_1A_1)\cdots\exp\,(t_kA_k)\,|\,t_i\in\mathbb{R}, A_i\in\mathcal{A}, k\in\mathbb{Z}^+\}\\ &= g\mathcal{O}(e).\end{aligned}$$

In particular, by restricting to Lie subalgebras $\mathcal{A}=\mathrm{Lie}\,(\mathcal{A})\subseteq\mathbb{R}^{n\times n}$, we get the following result : to any Lie subalgebra $\mathcal{A}\subseteq\mathbb{R}^{n\times n}$ there corresponds a connected Lie subgroup G of $\mathsf{GL}\,(n,\mathbb{R})$ such that $T_eG=\mathcal{A}$. (Here $G=\mathcal{O}(e)$.) The converse is also true. (This can be proved by using arguments based, again, on the *Orbit Theorem*.) Hence we get the following classical result (due to Sophus Lie): *there exists a one-to-one correspondence between Lie subalgebras $\mathcal{A}\subseteq\mathbb{R}^{n\times n}$ and connected Lie subgroups G of $\mathsf{GL}\,(n,\mathbb{R})$ such that $T_eG=\mathcal{A}$.*

Note : A remarkable and very deep result, due to Igor Ado, states that *every finite-dimensional Lie algebra is (isomorphic to) a Lie algebra of matrices.* This is in contrast to the situation for Lie groups, where most but not all Lie groups are matrix Lie groups.

Invariant control systems

Let $G \leq \mathsf{GL}\,(n, \mathbb{R})$ be an m-dimensional matrix Lie group with its Lie algebra $\mathfrak{g}$. A control-affine system (on G) determined by left-invariant vector fields is said to be *left-invariant*. We make the following definition.

2.2.2 Definition. A **left-invariant control system** on the matrix Lie group G is (given by) a collection Γ of elements in $\mathfrak{g}$ of the form

$$\Gamma = \left\{ A_u = A_0 + u_1 A_1 + \cdots + u_\ell A_\ell \,|\, u = (u_1, \ldots, u_\ell) \in U \subseteq \mathbb{R}^\ell \right\}$$

for some fixed $A_0, A_1, \ldots, A_\ell \in \mathfrak{g} = \mathfrak{X}_L(G)$.

Note : $\Gamma \subseteq \mathfrak{g}$ is in fact a collection of *matrices* (in $\mathbb{R}^{n \times n}$).

In *classical notation*, a left-invariant control system on G is written as

$$\begin{aligned} \dot{g} &= g\,(A_0 + u_1 A_1 + \cdots + u_\ell A_\ell) \\ &= g A_u(t), \qquad g \in G,\ u = (u_1, \ldots, u_\ell) \in U \subseteq \mathbb{R}^\ell \end{aligned}$$

where $g(\cdot)$ is a curve in the matrix Lie group G and $A_u(\cdot)$ is a curve in the associated Lie algebra $\mathfrak{g} = \mathfrak{X}_L(G)$.

Note : We may assume that $\ell \leq m$ and also that $A_1, \ldots, A_\ell$ are *linearly independent* elements (matrices) of $\mathfrak{g}$ which can be completed such that $\{A_1, \ldots, A_\ell, A_{\ell+1}, \ldots, A_m\}$ is a *basis* for $\mathfrak{g}$.

A **trajectory** of a left-invariant control system (given by) Γ on G is a continuous curve $t \mapsto g(t)$ in G, defined on an interval $[0, T] \subset \mathbb{R}$ so that there exists a *partition* $0 = t_0 < t_1 < \cdots < t_N = T$ and elements (left-invariant vector fields) $X_1, \ldots, X_N \in \Gamma$ such that the restriction of $g(\cdot)$ to each open interval (t_{i-1}, t_i) is smooth and (for $t \in (t_{i-1}, t_i)$)

$$\dot{g}(t) = X_i(g(t)), \quad i = 1, 2, \ldots, N.$$

NOTE : Because the elements of Γ are parametrized by controls, it follows that each left-invariant vector field X_i is equal to A_{u_i} for some u_i. Hence $g(\cdot)$ is the integral curve of the *time-varying* vector field $(t, g) \mapsto A(g, u(t)) := gA_u(t)$, with $u(\cdot)$ equal to the piecewise constant control, which takes constant value u_i in each interval $[t_{i-1}, t_i]$, and $t \mapsto g(t)$ can be visualized as a "broken" continuous curve consisting of pieces of integral curves of vector fields corresponding to different choices of control values.

Similarly, a **right-invariant control system** on G can be written as

$$\begin{aligned} \dot{g} &= (A_0 + u_1 A_1 + \cdots + u_\ell A_\ell)\, g \\ &= A_u(t) g, \qquad g \in G,\ u = (u_1, \ldots, u_\ell) \in U \subseteq \mathbb{R}^\ell \end{aligned}$$

where $g(\cdot)$ is a curve in the matrix Lie group G and $t \mapsto A_u(t) := A_0 + u_1(t)A_1 + \cdots + u_\ell(t)A_\ell$ is a curve in the associated Lie algebra $\mathfrak{g} = \mathfrak{X}_R(G)$.

We focus on left-invariant control systems on matrix Lie graoups, but analogue results can be derived for right-invariant control systems. In fact, given a right-invariant control system written as

$$\dot{g} = A_u(t) g, \qquad g \in G,\ u = (u_1, \ldots, u_\ell) \in U \subseteq \mathbb{R}^\ell$$

we can always convert it into a left-invariant control system by considering $t \mapsto g^{-1}(t)$ as our *state trajectory.*

Exercise 7 Show that if the curve $t \mapsto g(t)$ in G satisfies the condition

$$\dot{g} = A_u(t) g$$

then the curve $t \mapsto h(t) := g^{-1}(t)$ satisfies

$$\dot{h} = -h A_u(t).$$

Thus, *there is no loss of generality in specializing to left-invariant control systems.*

Consider the general affine group

$$\mathsf{GA}\,(n,\mathbb{R}) = \left\{ g = \begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix} \mid c \in \mathbb{R}^n,\ X \in \mathsf{GL}\,(n,\mathbb{R}) \right\}.$$

Embedding $\mathbb{E}^n$ into $\mathbb{E}^{n+1}$ as the *hyperplane*

$$\{1\} \times \mathbb{E}^n = \{(1,p) \mid p \in \mathbb{E}^n\} \subset \mathbb{E}^{n+1}$$

we obtain the affine transformation on $\mathbb{E}^n$ defined by an element $g \in \mathsf{GA}\,(n,\mathbb{R})$:

$$x = \begin{bmatrix} 1 \\ x \end{bmatrix} \mapsto gx = \begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ Ax + c \end{bmatrix} = Ax + c.$$

That is, the group $\mathsf{GA}\,(n,\mathbb{R})$ *acts* on (the Euclidean space) $\mathbb{E}^n$ as follows :

$$(g,x) \mapsto gx := Ax + c.$$

The Lie algebra of $\mathsf{GA}\,(n,\mathbb{R})$ is

$$\mathfrak{ga}(n,\mathbb{R}) = \left\{ \bar{A} = \begin{bmatrix} 0 & 0 \\ a & A \end{bmatrix} \mid a \in \mathbb{R}^n,\ A \in \mathbb{R}^{n\times n} \right\}.$$

Every element (matrix) $\bar{A} \in \mathfrak{ga}(n,\mathbb{R})$ *induces* a vector field on $\mathbb{R}^n$:

$$x \mapsto Ax + a.$$

Now let $G \leq \mathsf{GA}\,(n,\mathbb{R})$ be a *connected* matrix subgroup of $\mathsf{GA}\,(n,\mathbb{R})$ (that acts *transitively* on $\mathbb{R}^n$); for instance, $\mathsf{GA}^+\,(n,\mathbb{R})$ or $\mathsf{SE}\,(n)$.

A right-invariant control system on the matrix Lie group G written as

$$\dot{g} = \left(\bar{A} + u_1\bar{B}_1 + \cdots + u_\ell\bar{B}_\ell\right) g, \qquad g \in G$$

with

$$\bar{A} = \begin{bmatrix} 0 & 0 \\ a & A \end{bmatrix}, \ \bar{B}_i = \begin{bmatrix} 0 & 0 \\ b_i & B_i \end{bmatrix} \in \mathfrak{ga}(n, \mathbb{R}), \quad i = 1, 2, \ldots, \ell$$

induces the following *affine control system* on $\mathbb{E}^n$:

$$\dot{x} = Ax + a + u_1(B_1 x + b_1) + \cdots + u_\ell(B_\ell x + b_\ell), \qquad x \in \mathbb{E}^n.$$

Particular cases of such control systems (on $\mathbb{E}^n$) are

- the *bilinear control systems*

$$\begin{aligned} \dot{x} &= Ax + u_1 B_1 x + \cdots + u_\ell B_\ell x \\ &= (A + u_1 B_1 + \cdots + u_\ell B_\ell)x, \qquad x \in \mathbb{E}^n \end{aligned}$$

  (obtained for $a = b_1 = \cdots = b_\ell = 0$)

- the *linear control systems*

$$\begin{aligned} \dot{x} &= Ax + u_1 b_1 + \cdots + u_\ell b_\ell \\ &= Ax + Bu, \qquad x \in \mathbb{E}^n \end{aligned}$$

  (obtained for $a = 0,\ B_1 = \cdots = B_\ell = 0$).

NOTE : An abstract Lie group G is said to *act* (from the left) on the (analytic) manifold M if there exists an (analytic) mapping $\theta : G \times M \to M, \quad (g, x) \mapsto \theta(g, x) = gx$ that satisfies (for $g_1, g_2 \in G$ and $x \in M$)

$$(g_2 g_1)x = g_2(g_1 x) \quad \text{and} \quad ex = x.$$

For each $g \in G$, consider the (analytic) *diffeomorphism* $\theta_g : M \to M, \quad x \mapsto \theta_g(x) = gx$ (the inverse of θ_g is $\theta_{g^{-1}}$). The mapping $g \mapsto \theta_g$ is called the (left) *action* of G on M. Any action is a *homomorphism* from the group G to the group of (analytic)

diffeomorhisms of M. For any element $A \in \mathfrak{g}$, $\theta_{\exp(tA)}$ is a one-parameter group of diffeomorphisms of M with the *generator* $\theta_*(A)$ – an (analytic) vector field on M :

$$\theta_*(A)(x) := \frac{d}{dt}\theta_{\exp(tA)}(x)\bigg|_{t=0}, \qquad x \in M,\ A \in \mathfrak{g}.$$

Such vector fields $\theta_*(A)$, $A \in \mathfrak{g}$ are called *subordinated to the action* θ. They form a (finite-dimensional) Lie algebra $\theta_*(\mathfrak{g})$. A collection of vector fields $\mathcal{F}$ on M is called *subordinated* to the action θ if $\mathcal{F} \subseteq \theta_*(\mathfrak{g})$. If $\mathcal{F} = \theta_*(\Gamma)$ for some right-invariant control system (determined by) $\Gamma \subseteq \mathfrak{g}$, then $\mathcal{F}$ is called *induced* by Γ.

A Lie group G is said to act *transitively* on (the manifold) M if for any $x \in M$ the *orbit* $\{\theta_g(x) \,|\, g \in G\}$ coincides with the whole M. A manifold that admits a transitive action of a Lie group is called a *homogeneous space* (of this Lie group). Homogeneous spaces are exactly manifolds that can be represented as quotients of Lie groups.

Given a right-invariant control system on a Lie group that acts on (the manifold) M, one can construct the control system (on M) induced by Γ. In particular, for G either $\mathsf{GA}^+(n,\mathbb{R})$ or $\mathsf{SE}(n)$ (or, more generally, any connected matrix subgroup of the general affine group $\mathsf{GA}(n,\mathbb{R})$ that acts transitively on $\mathbb{E}^n$), one obtain bilinear and affine control systems on $\mathbb{E}^n$.

Control systems on homogeneous spaces subordinated to a group action (in particular, bilinear and affine control systems) were among the most important motivations for the study of (righ-)invariant control systems on (matrix) Lie groups.

2.3 Examples

We will give some interesting examples of invariant control systems on matrix Lie groups.

The Brockett system

The so-called *Brockett system* is a simple (nonlinear) control system on $\mathbb{E}^3$ defined (after a change of variables) as

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1. \end{aligned}$$

Exercise 8 Verify that the Brockett system is a driftless, control-affine system on $\mathbb{E}^3$.

NOTE : In the literature, the Brockett system is also referred to as the *Brockett integrator* (or the *nonholonomic integrator* or even the *Heisenberg system*); it usually appears in the following form

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 - x_1 u_2. \end{aligned}$$

Since its appearance in the early 80's, the Brockett integrator has attracted the interest of several researchers. It is the simplest control system with *nonholonomic constraint* as well as the first example of a (globally) controllable nonlinear system which is not (smoothly) stabilizable. Despite its simplicity, the Brockett integrator presents challenging problems, many of them not yet solved. It arises in numerous applications and moreover has an educational relevance (it is a useful example to approach and understand difficult mathematical and control theoretic issues).

The Brockett system can be expressed as a *driftless, left-invariant control system* on the Heisenberg group (consisting of unipotent 3×3 matrices)

$$\mathsf{H}(1) = \left\{ g = \begin{bmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix} \mid x_1, x_2, x_3 \in \mathbb{R} \right\} \leq \mathsf{GL}(3, \mathbb{R}).$$

Consider the associated Lie algebra (consisting of all 3×3 strictly upper triangular matrices)

$$\mathfrak{h}(1) = \left\{ A = \begin{bmatrix} 0 & a_2 & a_3 \\ 0 & 0 & a_1 \\ 0 & 0 & 0 \end{bmatrix} \mid a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Define a basis $\{A_1, A_2, A_3\}$ for this Lie algebra, where

$$A_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with the following table for the Lie bracket (commutator) :

$[\cdot,\cdot]$	A_1	A_2	A_3
A_1	0	$-A_3$	0
A_2	A_3	0	0
A_3	0	0	0

(This means, for instance, that $[A_1, A_2] = -[A_2, A_1] = -A_3$.)

A simple computation shows that the Brockett's system can be written as

$$\dot{g} = g\,(u_1 A_1 + u_2 A_2)\,, \qquad g \in H(1),\ u = (u_1, u_2) \in \mathbb{E}^2.$$

Indeed,

$$\begin{aligned}
\dot{g} &= \begin{bmatrix} 0 & \dot{x}_2 & \dot{x}_3 \\ 0 & 0 & \dot{x}_1 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & u_2 & u_1 x_2 \\ 0 & 0 & u_1 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & u_2 & 0 \\ 0 & 0 & u_1 \\ 0 & 0 & 0 \end{bmatrix} \\
&= g(u_1 A_1 + u_2 A_2).
\end{aligned}$$

Unicycle

Let us consider a simplified model of a *unicycle*, where we just model the wheel which is assumed to roll without slipping on a plane with the wheel axis always parallel to the plane. The *configuration space* is

$$\mathbb{R}^2 \times \mathbb{S}^1 = \left\{ (x_1, x_2, \theta) \,|\, x_1, x_2 \in \mathbb{R},\ \theta \in \mathbb{S}^1 \right\}$$

where (x_1, x_2) describes the *position* of the unicycle on the plane (relative to an orthonormal *inertial frame* (r_1, r_2)) and θ describes the *orientation* of the unicycle (specifically, the *angle* between the tangent to the wheel and the r_1-axis). Further we assume that we have control over the *forward velocity* as well as the *steering velocity*, which describes the angular velocity of the wheel. So, with $u_1 = \dot{\theta}$ (steering speed) and $u_2 = v$ (rolling speed) as controls, the control system (i.e. the motion of the unicycle) can be described by (the scalar

state equations)

$$\begin{aligned} \dot{x}_1 &= u_2 \cos\theta \\ \dot{x}_2 &= u_2 \sin\theta \\ \dot{\theta} &= u_1. \end{aligned}$$

This control-affine system (on the *manifold* $\mathbb{R}^2 \times \mathbb{S}^1$) can be viewed as a *driftless, left-invariant control system on the special Euclidean group* $\mathsf{SE}(2)$. Indeed, let

$$\mathsf{SE}(2) = \left\{ g = \begin{bmatrix} 1 & 0 & 0 \\ x_1 & \cos\theta & -\sin\theta \\ x_2 & \sin\theta & \cos\theta \end{bmatrix} \mid x_1, x_2 \in \mathbb{R},\ \theta \in [0, 2\pi) \right\}.$$

Then its Lie algebra

$$\mathfrak{se}(2) = \left\{ A = \begin{bmatrix} 0 & 0 & 0 \\ a_2 & 0 & -a_1 \\ a_3 & a_1 & 0 \end{bmatrix} \mid a_1, a_2, a_3 \in \mathbb{R} \right\}$$

is generated by the elements (matrices)

$$A_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

with the following table for the Lie bracket (commutator) :

$[\cdot,\cdot]$	A_1	A_2	A_3
A_1	0	A_3	$-A_2$
A_2	$-A_3$	0	0
A_3	A_2	0	0

Again, a simple computation shows that the unicycle control system can be written as

$$\dot{g} = g\left(u_1 A_1 + u_2 A_2\right), \qquad g \in \mathsf{SE}\,(2),\ u = (u_1, u_2) \in \mathbb{R}^2.$$

Spacecraft

Let us consider a spacecraft free to move in the Euclidean 3-space $\mathbb{E}^3$. In the *attitude control problem* we restrict our attention to the *orientation* of the spacecraft (satellite) with respect to a reference frame (r_1, r_2, r_3). Let $\underline{b} = (b_1, b_2, b_3)$ be an orthonormal frame fixed on the body and assume that the origins of the two frames coincide. We assume that the *actuators* of the satellite (i.e. thrusters or momentum wheels) are fixed to the body such that resulting *angular velocity* vectors are alingned with the body frame $\underline{b}$. Further we make the idealizing assumption that we have direct control over the angular velocities (resulting from the actuators).

We define $g(t) \in \mathsf{SO}\,(3)$ such that

$$r_i = g(t) b_i, \quad i = 1, 2, 3$$

(i.e. $g(t)$ determines the *attitude* of the spacecraft at time t). Hence the *configuration space* is the special orthogonal group $\mathsf{SO}\,(3)$. Its associated Lie algebra

$$\mathfrak{so}\,(3) = \left\{A \in \mathbb{R}^{3\times 3} \,|\, A^T + A = 0\right\}$$

(consisting of all 3×3 skew-symmetric matrices) is customarily identified with the Lie algebra $\mathbb{R}^3$, via the canonical mapping

$$x = (x_1, x_2, x_3) \mapsto \widehat{x} := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

Define

$$A_i := \widehat{e}_i, \quad i = 1, 2, 3$$

where e_1, e_2, e_3 are the standard vectors in $\mathbb{R}^3$. That is,

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Exercise 9 Compute the corresponding table for the Lie bracket (commutator).

Then $\{A_1, A_2, A_3\}$ is a (standard) *basis* for $\mathfrak{so}(3)$ and $g(t)$ satisfies :

$$\begin{aligned} \dot{g} &= g\widehat{\omega} \\ &= g(\omega_1 A_1 + \omega_2 A_2 + \omega_3 A_3) \end{aligned}$$

where

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \in \mathbb{R}^{3\times 1}$$

is the *angular velocity* of the spacecraft in the body-fixed coordinates. If we let

$$u_i := \omega_i, \quad i = 1, 2, 3$$

(i.e. if we interpret the components of the angular velocity as controls), then the *kinematics* of the spacecraft can be described by (the state equation) :

$$\dot{g} = g(u_1 A_1 + u_2 A_2 + u_3 A_3), \qquad g \in \mathsf{SO}(3),\ u = (u_1, u_2, u_3) \in \mathbb{R}^3.$$

This is a *driftless, left-invariant control system on the special orthogonal group* $\mathsf{SO}(3)$.

An interesting particular case is when only two components of the angular velocity can be controlled (due, for instance, to a failure). Without loss of generality we may assume that $u_3 = 0$ and then $g(t)$ satisfies

$$\dot{g} \quad = \quad g\left(u_1 A_1 + u_2 A_2\right), \qquad g \in \mathsf{SO}\,(3),\ u = (u_1, u_2) \in \mathbb{R}^2.$$

NOTE : Any control configuration can be represented by choosing the appropriate basis for $\mathfrak{so}\,(3)$. For example, suppose there are only two independent control inputs defined by

$$u_1 := \omega_1 + \omega_2 \quad \text{and} \quad u_2 := \omega_2 + \omega_3.$$

Then the (left-invariant) control system is described by

$$\dot{g} \quad = \quad g\left(u_1 A_1' + u_2 A_2'\right), \qquad g \in \mathsf{SO}\,(3),\ u = (u_1, u_2) \in \mathbb{R}^2$$

where

$$A_1' := A_1 + A_2, \quad A_2' := A_2 + A_3, \quad A_3' := A_3.$$

Underwater vehicle

Consider an autonomous underwater vehicle (AUV) and let $\underline{b} = (b_1, b_2, b_3)$ be an orthonormal frame fixed on the vehicle. The configuration of the vehicle is modeled as the *position* and *orientation* of the body-fixed frame $\underline{b}$ with respect to an inertial frame (r_1, r_2, r_3). We assume that the individual *actuators* are configured such that the resulting *angular* and *translational velocities* are aligned with the body frame $\underline{b}$. We define

$$g(t) = \begin{bmatrix} 1 & 0 \\ x(t) & R(t) \end{bmatrix} \in \mathsf{SE}\,(3)$$

such that

$$\begin{bmatrix} 1 \\ r_i \end{bmatrix} = g(t) \begin{bmatrix} 1 \\ b_i \end{bmatrix}, \quad i = 1, 2, 3.$$

NOTE : This is essentially the same condition as

$$r_i = x(t) + R(t)b_i, \quad i = 1, 2, 3.$$

Thus $g(t)$ describes the position and orientation of the AUV at time t. Let

$$A_i := \begin{bmatrix} 0 & 0 \\ 0 & \widehat{e}_i \end{bmatrix}, \quad A_{i+3} := \begin{bmatrix} 0 & 0 \\ e_i & 0 \end{bmatrix}, \quad i = 1, 2, 3.$$

Then $\{A_1, A_2, A_3, A_4, A_5, A_6\}$ defines a *basis* for the Lie algebra $\mathfrak{se}(3)$ associated with (the configuration space) $\mathsf{SE}(3)$.

Exercise 10 Compute the corresponding table for the Lie bracket (commutator).

Now let

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^{3\times 1}$$

define the *angular velocity* and the *translational velocity* of the vehicle (in the body-fixed coordinates), respectively. Then $g(t)$ satisfies

$$\dot{g} \quad = \quad g\left(\omega_1 A_1 + \omega_2 A_2 + \omega_3 A_3 + v_1 A_4 + v_2 A_5 + v_3 A_6\right).$$

If we let

$$u_i := \omega_i, \; u_{i+3} := v_i, \quad i = 1, 2, 3$$

(i.e. if we interpret the components of the angular and translational velocities as controls), then the *kinematics* of the AUV can be described by (the state equation):

$$\dot{g} \quad = \quad g\left(u_1 A_2 + u_2 A_2 + \cdots + u_6 A_6\right), \qquad g \in \mathsf{SE}(3), \; u = (u_1, \ldots, u_6) \in \mathbb{R}^6.$$

This is a *driftless, left-invariant control system on the special Euclidean group* $\mathsf{SE}\,(3)$.

As in the spacecraft attitude control problem, we are interested in the case when fewer than $m\,(=6)$ control components are available (i.e. $\ell < 6$). For example, suppose that we can control angular velocity about b_1, b_2, b_3 and translational velocity along b_1. Then $g(t)$ satisfies

$$\dot{g} \;=\; g\,(u_1 A_1 + u_2 A_2 + u_3 A_3 + u_4 A_4)\,, \qquad g \in \mathsf{SE}\,(3),\; u = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4.$$

NOTE : The AUV is *controllable* with as few as *two* controls.

Kinematic car

Let us consider a simple *kinematic* model for a front-wheel drive car of length l. The front-wheel pair and the rear-wheel pair are each modelled as a single wheel located at the midpoint of each axle. We assume that only the front wheels are allowed to turn. The car, like the unicycle, is a *nonholonomic* system if we assume that the wheels do not slip.

NOTE : *Holonomic systems* are mechanical systems that are subject to constraints that limit their possible configurations. The word *holonomic* is comprised of the Greek words *holos* and *nomos* meaning "integral" (or "whole") and "law", respectively, and refers to the fact that such constraints, given as constraints on the velocity, may be integrated and reexpressed as constraints on the configuration variables. Examples of holonomic constraints are *length constraints* for simple pendula and *rigidity constraints* for rigid body motion.

Nonholonomic mechanics describes the motion of systems constrained by *nonintegrable constraints* (i.e. constraints on the system velocities that do not arise from constraints on the configurations alone). Classic examples are *rolling* and *skating* motion.

Nonholonomic mechanics fits uneasily into the classical mechanics, since it is not variational in nature : it is neither Lagrangian nor Hamiltonian in the strict sense of the word. It is important however, for the theory of optimal control. (There is a close link between *nonholonomic constraints* and *controllability* of nonlinear systems. Nonholonomic constraints are given by nonintegrable distributions – that is, taking the Lie bracket of two vector fields in such a distribution may give rise to a vector field not contained in this distribution. It is precisely this property that one wants in nonlinear control systems so that we can drive the system to as large a part of the state space as possible.)

The *configuration space* is

$$\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1 = \left\{(x, y, \theta, \varphi) \,|\, x_1, x_2 \in \mathbb{R},\ \theta, \varphi \in \mathbb{S}^1\right\}$$

where (x, y) describes the car's *position* on a plane (relative to an inertial frame (r_1, r_2)). On the other hand, (b_1, b_2) is an orthonormal frame fixed on the car. θ denotes the *orientation* of the car (i.e. the angle between the b_1-axis of the car and the r_1-axis), and φ denotes the *steering angle* (i.e. the angle betweeen the b_1-axis of the car and the front wheels). Assuming that we can control $u_1 = \dot{\varphi}$ (steering speed) and $u_2 = v$ (rolling speed), then the *kinematic state equations* are :

$$\begin{aligned}
\dot{x} &= u_2 \cos\theta \\
\dot{y} &= u_2 \sin\theta \\
\dot{\varphi} &= u_1 \\
\dot{\theta} &= u_2 \frac{1}{l} \tan\varphi.
\end{aligned}$$

NOTE : This control affine system (on the *manifold* $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$) can be viewed as a nonlinear control system on the matrix Lie group $\mathsf{SE}(2) \times \mathsf{SO}(2)$. Indeed,

the configuration of the car is more naturally described by the matrix Lie group $\mathsf{SE}(2) \times \mathsf{SO}(2)$. $\mathsf{SE}(2)$ describes the position and orientation of the car (as in the unicycle case) and $\mathsf{SO}(2) = \mathbb{S}^1$ describes the angular position of the front wheel. Let

$$g(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ x(t) & \cos\theta & -\sin\theta & 0 & 0 \\ y(t) & \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & 0 & \sin\varphi & \cos\varphi \end{bmatrix} \in \mathsf{SE}(2) \times \mathsf{SO}(2)$$

describe the configuration of the car at time t. Define

$$A_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; A_2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; A_3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\{A_1, A_2, A_3, [A_3, A_2]\}$ is a *basis* for the associated Lie algebra $\mathfrak{se}(2) \times \mathfrak{so}(2)$, and $g(t)$ satisfies

$$\dot{g} \;=\; g\left(u_1 A_1 + u_2 A_2 + \tan(\tilde{u}_1) u_2 A_3\right)$$

where $\tilde{u}_1 := \int_0^t u_1(\tau)\, d\tau$. This is a "left-invariant" control system on $\mathsf{SE}(2) \times \mathsf{SO}(2)$, *nonlinear* in the controls u_1, u_2.

An alternative way of describing the kinematics of the car is to convert the state equations into *chained form*.

Exercise 11 Verify that, by making the change of variables

$$v_1 = u_2 \cos\theta, \quad v_2 = u_1, \quad \alpha = \sin\theta$$

the kinematic state equations of the car become

$$\begin{aligned}
\dot{x} &= v_1 \\
\dot{\varphi} &= v_2 \\
\dot{\alpha} &= v_1 \frac{1}{l} \tan\varphi \\
\dot{y} &= v_1 \frac{\alpha}{\sqrt{1-\alpha^2}}.
\end{aligned}$$

If we take (for the sake of simplicity) $l = 1$ and make the following *approximations*

$$\tan\varphi \approx \varphi, \qquad \frac{\alpha}{\sqrt{1-\alpha^2}} \approx \alpha$$

the equations take the form :

$$\begin{aligned}
\dot{x} &= v_1 \\
\dot{\varphi} &= v_2 \\
\dot{\alpha} &= v_1\varphi \\
\dot{y} &= v_1\alpha.
\end{aligned}$$

This system (of equations) is in *chained form* and we shall write it as follows (for $x_1 := x,\ x_2 := \varphi,\ x_3 := \alpha,\ x_4 := y$) :

$$\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{x}_2 &= v_2 \\
\dot{x}_3 &= v_1 x_2 \\
\dot{x}_4 &= v_1 x_3.
\end{aligned}$$

This chained form control system can be expressed as a *driftless, left-invariant control system on a matrix Lie group* G_4 *of unipotent* 4×4 *matrices*

(see **Exercise 127**). Indeed, let

$$\mathsf{G}_4 = \left\{ g = \begin{bmatrix} 1 & x_2 & x_3 & x_4 \\ 0 & 1 & x_1 & \frac{x_1^2}{2} \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mid x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}.$$

Then its (nilpotent) Lie algebra $\mathfrak{g}_4$ is generated by the elements (matrices)

$$B_1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Exercise 12 Check that $\{B_1, B_2, B_3, B_4\} = \{B_1, B_2, [B_2, B_1], [[B_2, B_1], B_1]\}$ is a *basis* for $\mathfrak{g}_4$.

A simple computation shows that the kinematic car control system (under simplifying conditions) can be written as

$$\dot{g} \;=\; g\,(v_1 B_1 + v_2 B_2)\,, \qquad g \in \mathsf{G}_4,\ v = (v_1, v_2) \in \mathbb{R}^2.$$

NOTE : Other more general *chained form* control systems can equivalently be described as driftless, left-invariant control systems on some matrix Lie groups of unipotent matrices. For example, consider the (two-input) chained form system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 x_2 \\ \dot{x}_4 &= u_1 x_3 \\ &\vdots \\ \dot{x}_k &= u_1 x_{k-1}. \end{aligned}$$

It can be shown that *the kinematic state equations of a car with* $k-3$ *trailers can be converted into this form.* Such a control system can be expressed as a driftless, left-invariant control system on a matrix Lie group (of unipotent $k \times k$ matrices) G_k.

2.4 Controllability

Let G be an m-dimensional matrix Lie group with associated Lie algebra $\mathfrak{g} = T_eG = \mathfrak{X}_L(G)$. Consider a left-invariant control system on G written as

$$\dot{g} \;=\; g\,(A_0 + u_1A_1 + \cdots + u_\ell A_\ell)\,, \qquad g \in G,\; u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell$$

where $A_0, A_1, \ldots, A_\ell \in \mathfrak{g}$ and $\ell \leq m$. $A_1, \ldots, A_\ell$ are assumed to be linearly independent. (For simplicity, the control set U coincides with $\mathbb{R}^\ell$.) Henceforth, in this chapter, any such (left-invariant) control system on G will be identified with the corresponding collection

$$\Gamma = \left\{ A_0 + u_1A_1 + \cdots + u_\ell A_\ell \,|\, u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell \right\}$$

of elements (matrices) in $\mathfrak{g}$.

NOTE : $\Gamma \subseteq \mathfrak{g}$ is an *affine subspace* (i.e. translation of a vector subspace) in $\mathfrak{g}$.

Reachable sets and orbits

Let Γ be a left-invariant control system on G and let $\mathrm{Traj}\,(\Gamma)$ denote the set of all trajectories of Γ.

For any $T \geq 0$ and any point $g \in G$, the **time** T **reachable set from** g is the set

$$\mathbf{A}(g,T) := \{g(T) \,|\, g(\cdot) \in \mathrm{Traj}\,(\Gamma),\;\; g(0) = g\}\,.$$

That is, $\mathbf{A}(g,T)$ is the set of all points (in G) that can be reached from (the initial point) g in exactly T units of time. We also define

$$\mathbf{A}(g,\leq T) := \bigcup_{0\leq t\leq T} \mathbf{A}(g,t).$$

The **reachable** (or **attainable**) **set from** g is the set $\mathbf{A}(g)$ of all terminal points $g(T)$, $T \geq 0$ of all trajectories $g(\cdot)$ starting at (the initial point) g. That is,

$$\mathbf{A}(g) := \bigcup_{T\geq 0} \mathbf{A}(g,T).$$

2.4.1 Definition. The left-invariant control system Γ is called (completely) **controllable** if, for any $g \in G$,

$$\mathbf{A}(g) = G.$$

In other words, Γ is (completely) controllable if, given any pair of points $g_0, g_1 \in G$, the point g_1 can be reached from g_0 (along a trajectory of Γ) for a nonnegative time T : $g_1 \in \mathbf{A}(g_0,T)$.

Note : (1) The weaker property of accessibility is essential for the description of reachable sets : Γ is called *accessible* at a point $g \in G$ if the reachable set $\mathbf{A}(g)$ has nonempty interior (in G).

(2) There are various controllability concepts, all of which involve reachable sets being "very large" in some sense (e.g. complete controllability, controllability from a point, local controllability, or small-time local controllability). In general, *controllability theory is the study of the structure of reachable sets.* One major concern is to determine "reasonable" (and, if possible, "effectively computable") conditions for the various controllability (and accesibility) conditions.

(3) All these considerations and concepts can be extended to the more general case of *control-affine* systems on *manifolds.* In particular, they are valid for *linear control*

systems. For the linear control system (with unrestricted controls)

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{E}^m,$$

the reachable set from the origin is

$$\mathbf{A}(0,T) = \left\{ \int_0^T \exp((T-\tau)A)Bu(\tau)\, d\tau \mid u(\cdot) \in \mathcal{U} \right\}.$$

This reachable set is a *linear subspace* of $\mathbb{E}^m$. The control system is *controllable from the origin* if (for every $T > 0$) $\mathbf{A}(0,T) = \mathbb{E}^m$. This immediately implies that also (for any $T > 0$ and $x \in \mathbb{E}^m$)

$$\mathbf{A}(x,T) = \mathbb{E}^m.$$

Given $A \in \mathfrak{g} = \mathfrak{X}_L(G)$, its integral curve through $g \in G$ is $t \mapsto g \exp(tA)$. One can use this simple fact to obtain a (very useful) description of an endpoint of a trajectory.

2.4.2 Lemma. *Let* $g(\cdot) \in \mathrm{Traj}\,(\Gamma)$ *with* $g(0) = g_0$. *Then there exist* $t_1, \ldots, t_k > 0$ *and* $X_1, \ldots, X_k \in \Gamma$ *such that*

$$g(T) = g_0 \exp(t_1 X_1) \cdots \exp(t_k X_k), \quad t_1 + \cdots + t_k = T.$$

Proof : Let $g(\cdot) : [0,T] \to G$ be a trajectory of Γ with initial point g_0. Then there exist a partition $0 = \tau_0 < \tau_1 < \cdots < \tau_k = T$ and elements $X_1, \ldots, X_k \in \Gamma$ such that

$$t \in (\tau_{i-1}, \tau_i) \quad \Rightarrow \quad \dot{g} = g(t)X_i \qquad (i = 1, 2, \ldots k).$$

For $i = 1$:

$$t \in (0, \tau_1) \quad \Rightarrow \quad \dot{g} = g(t)X_1, \;\; g(0) = g_0.$$

It follows that

$$g(t) = g_0 \exp(tX_1)$$

and also (by continuity)

$$g(\tau_1) = g_0 \exp(\tau_1 X_1).$$

For $i = 2$:

$$t \in (\tau_1, \tau_2) \quad \Rightarrow \quad \dot{g} = g(t)X_2, \;\; g(\tau_1) = g \exp(\tau_1 X_1).$$

It follows that

$$\begin{aligned} g(t) &= g_0 \exp(\tau_1 X_1) \exp((t - \tau_1)X_2) \\ g(\tau_2) &= g_0 \exp(\tau_1 X_1) \exp((\tau_2 - \tau_1)X_2), \quad t_1 = \tau_1, \; t_2 = \tau_2 - \tau_1 \end{aligned}$$

and so on. Finally, we get (for $i = k$) :

$$g(T) = g(\tau_k) = g_0 \exp(t_1 X_1) \cdots \exp(t_k X_k)$$

where $t_k = \tau_k - \tau_{k-1}, \ldots, t_2 = \tau_2 - \tau_1, t_1 = \tau_1 \quad (t_1 + \cdots + t_k = T)$.

□

Now we can derive a description, as well as some elementary properties, of reachable sets.

2.4.3 Proposition. *Let Γ be a left-invariant control system on G and let $g \in G$ be an arbitrary point. Then*

(RS1) $\mathbf{A}(g) = \{g \exp(t_1 X_1) \cdots \exp(t_k X_k) \mid X_i \in \Gamma, t_i \geq 0, k \in \mathbb{N}\}$.

(RS2) $\mathbf{A}(g) = g\mathbf{A}(e)$.

(RS3) $\mathbf{A}(e)$ *is a subsemigroup of* G.

(RS4) $\mathbf{A}(g)$ *is a path-connected subset of* G.

PROOF : (RS1) and (RS2) follow immediately from LEMMA 5.4.2.

(RS3) Since

$$\mathbf{A}(e) = \{\exp(t_1 X_1) \cdots \exp(t_k X_k) \,|\, X_i \in \Gamma, t_i \geq 0, k \in \mathbb{N}\}$$

it follows that for any $g_1, g_2 \in \mathbf{A}(e)$, $g_1 g_2 \in \mathbf{A}(e)$.

(RS4) Any point in $\mathbf{A}(e)$ is connected with the initial point g by a path $g(\cdot) \in \mathrm{Traj}\,(\Gamma)$.

□

2.4.4 COROLLARY. *The left-invariant control system $\Gamma \subseteq \mathfrak{g}$ is controllable if and only if $\mathbf{A}(e) = G$.*

PROOF : By definition, Γ is controllable if (and only if) $\mathbf{A}(g) = G$ for every $g \in G$. Since $\mathbf{A}(g) = g\mathbf{A}(e)$, it follows that controllability is equivalent to the condition $\mathbf{A}(e) = G$.

□

The **orbit** through the point $g \in G$ is denoted by $\mathcal{O}(g)$ and is defined as the set

$$\mathcal{O}(g) := \{g(T) \,|\, g(\cdot) \in \mathrm{Traj}\,(\Gamma),\ g(0) = g,\ T \in \mathbb{R}\}\,.$$

This set is defined analogously to the reachable set $\mathbf{A}(g)$ but the *terminal time* T may take both positive and negative values. The structure of orbits is simpler than that of reachable sets. Clearly (for $g \in G$),

$$\mathbf{A}(g) \subseteq \mathcal{O}(g).$$

2.4.5 PROPOSITION. *Let Γ be a left-invariant control system on G and let $g \in G$ be an arbitrary point. Then*

(O1) $\quad \mathcal{O}(g) = \{g \exp(t_1 X_1) \cdots \exp(t_k X_k) \,|\, X_i \in \Gamma, t_i \in \mathbb{R}, k \in \mathbb{N}\}.$

(O2) $\mathcal{O}(g) = g\mathcal{O}(e)$.

(O3) $\mathcal{O}(e)$ *is the connected Lie subgroup of* G *with the Lie algebra* $\mathrm{Lie}\,(\Gamma)$.

PROOF : (O1) and (O2) follow immediately from (RS1), (RS2) and the definition of an orbit.

(O3) The orbit $\mathcal{O}(e)$ is a subgroup of G. Indeed, if $g_1, g_2 \in \mathcal{O}(e)$, then $g_1 g_2^{-1} \in \mathcal{O}(e)$. For the Lie subalgebra $\mathrm{Lie}\,(\Gamma) \subseteq \mathfrak{g}$, by the *Orbit Theorem*, the orbit $\mathcal{O}(e) \subseteq G$ is a connected, immersed submanifold such that $T_e\mathcal{O}(e) = \mathrm{Lie}\,(\Gamma)$. Then $\mathcal{O}(e)$ is a connected Lie subgroup of G with the Lie algebra $\mathrm{Lie}\,(\Gamma)$ (see the Lie correspondence).

□

Since all essential properties of reachable sets (including controllability) are expressed in terms of the reachable set from the identity $\mathbf{A}(e)$, in the sequel we restrict ourselves to this set and denote it by $\mathbf{A}$. Likewise, we denote the orbit (through the identity) $\mathcal{O}(e)$ simply by $\mathcal{O}$.

Basic controllability conditions

Let $\Gamma \subseteq \mathfrak{g}$ be a left-invariant control system on the matrix Lie group G. We can see that *a necessary condition for* Γ *to be controllable is that* G *be connected.* Henceforth, all matrix Lie groups are assumed to be connected, unless otherwise stated.

We denote by $\mathrm{Lie}\,(\Gamma)$ the Lie algebra *generated* by $\Gamma \subseteq \mathfrak{g}$ (i.e. the smallest subalgebra of $\mathfrak{g}$ containing Γ). It follows that $\mathrm{Lie}\,(\Gamma)$ is the smallest vector subspace S of $\mathfrak{g}$ that also satisfies (for any $X \in \mathfrak{g}$)

$$[X, S] \subseteq S.$$

$\mathrm{Lie}\,(\Gamma)$ can also be described in terms of the following notation : for each $X \in \mathfrak{g}$, let $\mathrm{ad}\,X : \mathfrak{g} \to \mathfrak{g}$ denote the mapping $\mathrm{ad}\,X(Y) := [X, Y]$ for $Y \in \mathfrak{g}$ ($\mathrm{ad} : X \mapsto \mathrm{ad}\,X$ is the *adjoint representation* of the Lie algebra $\mathfrak{g}$). Then $\mathrm{Lie}\,(\Gamma)$ is equal to the smallest vector subspace S of $\mathfrak{g}$ for which $\mathrm{ad}\,X_1 \circ \mathrm{ad}\,X_2 \circ \cdots \circ \mathrm{ad}\,X_{k-1}(X_k) \in S$ for any finite set of elements $X_1, \ldots, X_k \in \Gamma$. That is,

$$\mathrm{Lie}\,(\Gamma) = \mathrm{span}\,\{\mathrm{ad}\,X_1 \circ \mathrm{ad}\,X_2 \circ \cdots \circ \mathrm{ad}\,X_{k-1}(X_k) \mid X_1, \ldots, X_k \in \Gamma\}\,.$$

2.4.6 Proposition. *If* $\Gamma \subseteq \mathfrak{g}$ *is controllable, then* $\mathrm{Lie}\,(\Gamma) = \mathfrak{g}$.

Proof : If $\mathbf{A} = G$, then $\mathcal{O} = G$ and hence (by Proposition 5.4.5)

$$\mathrm{Lie}\,(\Gamma) = \mathfrak{g}.$$

□

The condition that Γ generates $\mathfrak{g}$ as a Lie algebra (i.e. $\mathrm{Lie}\,(\Gamma) = \mathfrak{g}$) is referred to as the *Lie algebra rank condition* (LARC). A left-invariant control system Γ satisfying (LARC) is said to have *full rank*.

Note : If a point $h \in G$ is reachable (or accessible) from a point $g \in G$, then there exist elements $X_1, \ldots, X_k \in \Gamma$ and $t_1, \ldots, t_k \in \mathbb{R}$ (with $t_i > 0$) such that

$$h = g \exp(t_1 X_1) \cdots \exp(t_k X_k).$$

The following stronger concept turns out to be important in the study of *topological* properties of reachable sets (and hence of controllability). A point $h \in G$ is said to be *normally accessible* from a point $g \in G$ if there exists elements $X_1, \ldots, X_k \in \Gamma$ and $t_1, \ldots, t_k \in \mathbb{R}$ (with $t_i > 0$) such that the mapping

$$\Psi : \mathbb{R}^k \to G, \quad (s_1, \ldots, s_k) \mapsto g \exp(s_1 X_1) \cdots \exp(s_k X_k)$$

satisfies the following conditions :

(i) $\Psi((t_1, \ldots, t_k)) = h$.

(ii) The rank of Ψ at $t = (t_1, \ldots, t_k)$ is equal to m (the dimension of G).

(The rank of Ψ at $t \in \mathbb{R}^k$ is the rank of the *differential* $D\Psi(t)$.) We say that the point h is normally accessible from the point g by $X_1, \ldots, X_k$. It can be proved that *if* $\Gamma \subseteq \mathfrak{g}$ *has full rank, then in any neighborhood* N *of the identity* $e \in G$ *there are points normally accessible from* e.

Exercise 13 Show that if $\Gamma \subseteq \mathfrak{g}$ has full rank (i.e. $\mathrm{Lie}\,(\Gamma) = \mathfrak{g}$), then

(a) for any neighborhood N of e, the set $\mathrm{int}\,(\mathbf{A}) \cap N$ is *nonempty.*

(b) the reachable set $\mathbf{A}$ has nonempty interior (i.e. Γ is accessible at the identity).

In general, the Lie algebra rank condition (LARC) is *not* sufficient for controllability, but is equivalent to accessibility.

2.4.7 Proposition. *The left-invariant control system* $\Gamma \subseteq \mathfrak{g}$ *is accessible at the identity (and thus at any point* $g \in G$*) if and only if* $\mathrm{Lie}\,(\Gamma) = \mathfrak{g}$.

Proof : ($\Leftarrow$) If $\mathrm{Lie}\,(\Gamma) = \mathfrak{g}$, then (by **Exercise 252**) $\mathrm{int}\,(\mathbf{A})$ is nonempty (in G); that is, Γ is accessible at the identity. Since the left translation L_g is a homeomorphism, by Proposition 5.4.3, it follows that

$$\mathrm{int}\,(\mathbf{A}(g)) = \mathrm{int}\,(g\mathbf{A}) \neq \emptyset.$$

Thus Γ is accessible at $g \in G$.

($\Rightarrow$) Let $\mathrm{Lie}\,(\Gamma) \neq \mathfrak{g}$. Then

$$\dim \mathcal{O} = \dim \mathrm{Lie}\,(\Gamma) < \dim \mathfrak{g} = \dim G.$$

Thus $\mathrm{int}\,(\mathcal{O}) = \emptyset$ and so

$$\mathrm{int}\,(\mathbf{A}) = \emptyset.$$

□

2.4.8 THEOREM. (GROUP TEST) *The left-invariant control system* $\Gamma \subseteq \mathfrak{g}$ *is controllable if and only if*

(i) The reachable set $\mathbf{A}$ *is a subgroup of* G.

(ii) $\mathrm{Lie}\,(\Gamma) = \mathfrak{g}$.

PROOF : ($\Rightarrow$) Condition (i) is obvious, and condition (ii) follows from the Lie algebra rank condition (LARC).

($\Leftarrow$) If $\mathbf{A} \subseteq G$ is a subgroup, then for any $g \in \mathbf{A}$, its inverse g^{-1} also belongs to $\mathbf{A}$. For any exponential $\exp(tX) \in \mathbf{A}$, its inverse

$$(\exp(tX))^{-1} = \exp(-tX) \in \mathbf{A}.$$

Thus the reachable set $\mathbf{A}$ coincides with the orbit $\mathcal{O}$. But $\mathcal{O} \subseteq G$ is a connected Lie subgroup with Lie algebra $\mathrm{Lie}\,(\Gamma) = \mathfrak{g}$. Then (by the Lie correspondence) $\mathcal{O} = G$ and hence

$$\mathbf{A} = \mathcal{O} = G.$$

□

NOTE : A control system (on manifold M) is called *locally controllable* at a point $p \in M$ if $p \in \mathrm{int}\,(\mathbf{A}(p))$. For such general control systems, the local controllability property is weaker than the global controllability property. However, for left-invariant control systems on matrix Lie groups, these two notions coincide. Hence the following result holds :

(LOCAL CONTROLLABILITY TEST) *The left-invariant control system* $\Gamma \subseteq \mathfrak{g}$ *is controllable if and only if the group identity* e *is contained in the interior of* $\mathbf{A}$.

This particular result can be used to derive another interesting test :

(Closure Test) *The left-invariant control system* $\Gamma \subseteq \mathfrak{g}$ *is controllable if and only if the (topological) closure of the reachable set* $\mathbf{A}$ *is the entire group* G *:* $\mathrm{cl}\,(\mathbf{A}) = G$.

This means that in the study of controllability one can replace the reachable set $\mathbf{A}$ by its closure $\mathrm{cl}\,(\mathbf{A})$. This fact has far-reaching consequences.

Other controllability criteria

Let $\Gamma \subseteq \mathfrak{g}$ be a *driftless* (or homogeneous) left-invariant control system on G; that is,

$$\begin{aligned} \Gamma &= \left\{ u_1 A_1 + u_2 A_2 + \cdots + u_\ell A_\ell \,|\, u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell \right\} \\ &= \mathrm{span}\,\{A_1, \ldots, A_\ell\} \subseteq \mathfrak{g} \end{aligned}$$

where $A_1, \ldots, A_\ell$ are assumed to be linearly independent. (Again, for the sake of simplicity, the system is assumed to be unconstained : $U = \mathbb{R}^\ell$.)

Note : If $\Gamma \subseteq \mathfrak{g}$ is driftless, then together with any element X, it contains also the *negative* $-X$:

$$X \in \Gamma \;\Rightarrow\; -X \in \Gamma.$$

(This fact can also be expressed by saying that the "symmetry condition" : $\Gamma = -\Gamma$ is satisfied.)

Exercise 14 Show that if $\Gamma \subseteq \mathfrak{g}$ is a *driftless* left-invariant control system on G, then its reachable set $\mathbf{A}$ is a subgroup of G and coincides with the orbit $\mathcal{O}$.

Thus deciding controllability for a driftless left-invariant control system $\Gamma \subseteq \mathfrak{g}$ reduces to verifying the algebraic condition of coincidence of the (connected) matrix Lie groups $\mathcal{O}$ and G.

Exercise 15 Show that a driftless left-invariant control system $\Gamma \subseteq \mathfrak{g}$ is controllable if and only if $\mathrm{Lie}\,(\Gamma) = \mathfrak{g}$.

2.4.9 THEOREM. *Consider a driftless left-invariant control system*

$$\Gamma = \left\{ u_1 A_1 + \cdots + u_\ell A_\ell \,|\, u = (u_1, \ldots, u_\ell) \in \mathbb{R}^\ell \right\} \subset \mathfrak{g}$$

on a (not necessarily connected) matrix Lie group G. *Then :*

(i) The reachable set $\mathbf{A}$ *coincides with the orbit* $\mathcal{O}$ *(i.e. the connected matrix Lie subgroup of* G *with associated Lie algebra* $\mathrm{Lie}\,(\Gamma)$*).*

(ii) Any point of $\mathbf{A}$ *can be reached from the identity* $e \in G$ *in an arbitrary time :*

$$\mathbf{A}(e, T) = \mathbf{A} = \mathcal{O} \quad \textit{for any } T > 0.$$

(iii) If G *is connected, then* Γ *is controllable if and only if*

$$\mathrm{Lie}\,(\{A_1, \ldots, A_\ell\}) = \mathfrak{g}.$$

PROOF : (i) and (iii) follow immediately from **Exercise 253** and **Exercise 254**, respectively.

To prove (ii), choose any $T > 0$. Let a point $g \in G$ be reachable from e in some time $T_1 > 0$:

$$g = \exp(t_1 X_1) \cdots \exp(t_k X_k), \quad t_1 + \cdots + t_k = T_1$$

where $t_1, \ldots, t_k > 0$ and $X_1, \ldots, X_k \in \Gamma$. The elements (vector fields)

$$\widehat{X}_i := \alpha X_i, \quad i = 1, 2, \ldots, k$$

belong to Γ for $\alpha = \frac{T_1}{T}$. Thus g can be reached from the identity e in time T :

$$g = \exp(s_1 \widehat{X}_1) \cdots \exp(s_k \widehat{X}_k), \quad s_1 + \cdots + s_k = T$$

where $s_i = \frac{1}{\alpha} t_i, \;\; i = 1, 2, \ldots, k.$ □

NOTE : For *compact* (connected) matrix Lie groups, the following result holds : *The left-invariant control system* $\Gamma \subseteq \mathfrak{g}$ *is controllable if and only if* $\mathrm{Lie}\,(\Gamma) = \mathfrak{g}$. (Moreover, if Γ is controllable, then there exists $T > 0$ such that, for every two points $g_0, g_1 \in G$, there is a control $u(\cdot)$ that steers g_0 into g_1 in no more than T units of time.)

Exercise 16 Let A_0, A_1 be any two linearly independent 3×3 real skew-symmetric matrices (i.e. two linearly independent elements of the Lie algebra $\mathfrak{so}\,(3)$). Show that the left-invariant control system

$$\Gamma = A_0 + \mathrm{span}\,\{A_1\} \subseteq \mathfrak{so}\,(3)$$

(or, in classical notation,

$$\dot{g} = g\,(A_0 + uA_1)\,, \qquad g \in \mathsf{SO}\,(3),\ u \in \mathbb{R}\,)$$

is controllable.

Exercise 17 Investigate for *controllability* each of the following driftless left-invariant control systems on a specific (connected) matrix Lie group G :

(a) The Brockett system on $G = \mathsf{H}\,(1)$.

(b) The unicycle on $G = \mathsf{SE}\,(2)$.

(c) The spacecraft on $G = \mathsf{SO}\,(3)$.

(d) The autonomous underwater vehicle (AUV) on $G = \mathsf{SE}\,(3)$.

(e) The kinematic car on $G = \mathsf{G}_4$.

2.5 Linear Control Systems

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2.6 Serret-Frenet Control Systems

The arc length parametrization of a (geometric) curve describing the *path* of (the center of mass of) a *rigid body* in Euclidean 3-space can be used to express the state equation of the "motion" of this (left-invariant) control system (on the special Euclidean group $\mathsf{SE}(3)$).

Consider a *unit-speed* curve $x(\cdot)$ in $\mathbb{E}^3$.

NOTE : The map $t \mapsto x(t) \in \mathbb{E}^3$ is assumed to be *smooth*. For the sake of convenience, we use the variable t (instead of s) for the *arc length parameter* of the curve.

The *Serret-Frenet frame* (T, N, B) along the curve $x(\cdot)$ is described by the (unipotent orthogonal) matrix

$$R(t) := \begin{bmatrix} T(t) & N(t) & B(t) \end{bmatrix} \in \mathsf{SO}(3)$$

that relates this frame to the natural frame (e_1, e_2, e_3) in $\mathbb{E}^3$ (we have omitted any notational distinctions between tangent vectors and parallel vector fields) and that further satisfies the following differential equation (in matrices) :

$$\dot{R} = R \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}$$

where $\kappa(\cdot)$ and $\tau(\cdot)$ represent the *curvature* and *torsion* function, respectively.

NOTE : $R(\cdot)$ is the *attitude matrix* of the frame field (T, N, B) and the differential equation satisfied by $R(\cdot)$ represents the *Serret-Frenet formulas*. Clearly,

$$R(t)e_1 = T, \quad R(t)e_2 = N, \quad R(t)e_3 = B.$$

The curve $x(\cdot)$ (i.e.

$$t \mapsto x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \in \mathbb{R}^{3\times 1}\,)$$

and the rotation matrix $R(t) \in \mathsf{SO}\,(3)$ can be expressed as (the curve)

$$g(t) = \begin{bmatrix} 1 & 0 \\ x(t) & R(t) \end{bmatrix} \in \mathsf{SE}\,(3)$$

in the (matrix Lie) group of proper rigid motions on the Euclidean 3-space $\mathbb{E}^3$. Since

$$\dot{x} = T = R(t)e_1,$$

we get

$$\begin{aligned} \dot{g} &= g \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & -\tau \\ 0 & 0 & \tau & 0 \end{bmatrix} \\ &= g\,(X_0 + \kappa X_1 + \tau X_2)\,, \qquad g \in \mathsf{SE}\,(3) \end{aligned}$$

where

$$X_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

NOTE : Recall that the matrices $A_1, A_2, \ldots, A_6$ defined by

$$A_i := \begin{bmatrix} 0 & 0 \\ 0 & \widehat{e}_i \end{bmatrix}, \quad A_{i+3} := \begin{bmatrix} 0 & 0 \\ e_i & 0 \end{bmatrix}, \qquad i = 1, 2, 3$$

form a *basis* for the Lie algebra $\mathfrak{se}(3)$ associated with the special Euclidean group $\mathsf{SE}(3)$. We can see that

$$X_0 = \begin{bmatrix} 0 & 0 \\ e_1 & 0 \end{bmatrix} = A_4, \quad X_1 = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{e}_3 \end{bmatrix} = A_3, \quad X_2 = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{e}_1 \end{bmatrix} = A_1.$$

2.6.1 Definition. The left-invariant control system on the special Euclidean group $\mathsf{SE}(3)$, written as

$$\dot{g} \;=\; g\,(X_0 + \kappa X_1 + \tau X_2)\,, \qquad g \in \mathsf{SE}(3)$$

with the curvature and torsion functions playing the role of controls, is called the **Serret-Frenet control system** (on $\mathsf{SE}(3)$).

For $\tau(\cdot) = 0$, we obtain a left-invariant control system (on $\mathsf{SE}(2)$), described by (the state equation)

$$\begin{aligned} \dot{g} &= g \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -\kappa \\ 0 & \kappa & 0 \end{bmatrix} \\ &= g\left(X_0' + \kappa X_1'\right), \qquad g \in \mathsf{SE}(2),\ \kappa \geq 0. \end{aligned}$$

Consider now the special case where the torsion function $\tau(\cdot)$ is *constant*. This assumption reduces the number of controls to a single control ($u := \kappa$) and introduces a *drift term* in the rotational part of the equation (corresponding to the constant torsion).

Under this assumption, the differential equation satisfied by the rotation matrix $R(\cdot)$ can be written

$$\dot{R} \;=\; R\,(A + uB)\,, \qquad R \in \mathsf{SO}(3),\ u \geq 0$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

NOTE : We have

$$A = \tau\, \widehat{e}_1, \quad B = \widehat{e}_2$$

where $\{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}$ is the *standard basis* for the Lie algebra $\mathfrak{so}\,(3)$.

We call the forgoing (left-invariant) control system the **stiff Serret-Frenet control system** (on $\mathsf{SO}\,(3)$).

NOTE : Writing $h(t)$ for the matrix $R(t)^{-1}$, turns this left-invariant control system into a right-invariant control system

$$\dot{h} \;=\; -\,(A + uB)\,h, \qquad h \in \mathsf{SO}\,(3),\ u \geq 0.$$

The matrix Lie group $\mathsf{SO}\,(3)$ is *compact* and *connected*, and hence *the stiff Serret-Frenet control system is controllable if and only if the set* $\Gamma = \{A + uB \,|\, u \geq 0\}$ *generates* $\mathfrak{so}\,(3)$ *as a Lie algebra*; that is,

$$\mathrm{Lie}\,(\{A, B\}) = \mathfrak{so}\,(3).$$

Exercise 18 Show that if the fixed torsion τ is *nonzero* in the expression for A, then the Lie algebra generated by A and B equals the Lie algebra $\mathfrak{so}\,(3)$.

Consider the (left-invariant) Serret-Frenet control system on $\mathsf{SE}\,(2)$

$$\dot{g} \;= g\,(X + uY)\,, \qquad g \in \mathsf{SE}\,(2),\ u \in \mathbb{R}$$

where

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Exercise 19 Calculate the Lie algebra generated by $\{X, Y\}$.

NOTE : For a left-invariant control system on a connected (but *not* compact) matrix Lie group, the Lie algebra rank condition (LARC) is *only* a necessary condition. The Serret-Frenet control system on (the *noncompact*) matrix Lie group $\mathsf{SE}(2)$ is, in fact, *controllable.*

Consider now the matrix Lie group $\mathsf{SE}(n)$. Recall that an arbitrary element $g \in \mathsf{SE}(n)$ can be expressed as a matrix $\begin{bmatrix} 1 & 0 \\ c & R \end{bmatrix} \in \mathsf{GL}(n+1, \mathbb{R})$ with $c \in \mathbb{R}^{n \times 1}$ and $R \in \mathsf{SO}(n)$. We may denote such an element by $(c, R) \in \mathbb{R}^n \times \mathsf{SO}(n)$.

NOTE : The group product in $\mathbb{R}^n \times \mathsf{SO}(n)$ is defined by

$$(c_1, R_1) \cdot (c_2, R_2) := (c_1 + R_1 c_2,\, R_1 R_2).$$

We say that (the group) $\mathsf{SE}(n)$ is the *semidirect product* of (the vector space) $\mathbb{R}^n$ and (the group) $\mathsf{SO}(n)$ and write $\mathsf{SE}(n) = \mathbb{R}^n \ltimes \mathsf{SO}(n)$.

Likewise, the Lie algebra $\mathfrak{se}(n)$ of the special Euclidean group $\mathsf{SE}(n)$ is the *semidirect sum* $\mathbb{R}^n \rightthreetimes \mathfrak{so}(n)$; that is, the vector space $\mathfrak{se}(n)$ is the direct sum of the vector spaces $\mathbb{R}^n$ and $\mathfrak{so}(n)$, and the Lie bracket is as follows :

$$[(a, A), (b, B)] := (Ab - Ba, [A, B]).$$

Exercise 20 Verify that the commutator of the matrices

$$M_1 = \begin{bmatrix} 0 & 0 \\ a_1 & A_1 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & 0 \\ a_2 & A_2 \end{bmatrix}$$

is

$$[M_1, M_2] = \begin{bmatrix} 0 & 0 \\ A_1 a_2 - A_2 a_1 & A_1 A_2 - A_2 A_1 \end{bmatrix}.$$

(Observe that for $a \in \mathbb{R}^{n\times 1}$ and $A \in \mathfrak{so}(n)$, the matrix $\begin{bmatrix} 0 & 0 \\ a & A \end{bmatrix}$ is an element of $\mathfrak{se}(n)$.)

Let

$$\pi : \mathsf{SE}(n) \to \mathsf{SO}(n), \qquad (c, R) \mapsto R$$

denote the *projection* (on the second factor). Projection π is a Lie homomorphism, and hence the derivative

$$d\pi = \pi_* : \mathfrak{se}(n) \to \mathfrak{so}(n)$$

is a Lie algebra homomorphism (see THEOREM 3.4.17).

2.6.2 THEOREM. *A left-invariant control system* $\Gamma \subseteq \mathfrak{se}(n)$ *on the special Euclidean group* $\mathsf{SE}(n)$ *is controllable if and only if*

$$\mathrm{Lie}(\Gamma) = \mathfrak{se}(n).$$

PROOF : ($\Rightarrow$) : The Lie algebra rank condition $\mathrm{Lie}(\Gamma) = \mathfrak{se}(n)$ is necessary for controllability (see PROPOSITION 5.4.6).

($\Leftarrow$) : Assume that $\mathrm{Lie}(\Gamma) = \mathfrak{se}(n)$. Then the left-invariant control system $\Gamma_{\mathsf{SO}(n)} := \pi_*(\Gamma) \subset \mathfrak{so}(n)$ on $\mathsf{SO}(n)$ is controllable since $\mathsf{SO}(n)$ is compact and connected (see the note after THEOREM 5.4.9). That is,

$$\pi(\mathbf{A}) = \mathsf{SO}(n).$$

It follows (see COROLLARY 5.4.4) that it is sufficient to show that the group identity $e = (0, I) \in \mathsf{SE}(n) = \mathbb{R}^n \ltimes \mathsf{SO}(n)$ is contained in the *interior* of $\mathbf{A}$.

Let $(x, g) \in \mathrm{int}(\mathbf{A}) \neq \emptyset$. There exists $y \in \mathbb{R}^n$ such that $(y, g^{-1}) \in \mathbf{A}$. Then $(x, g) \cdot (y, g^{-1}) = (x + gy, I)$, and this product is in the interior of $(\mathbf{A})$.

Denote $x + gy$ by v. Let Ω be a neighborhood of I in $\mathsf{SO}(n)$ such that $(v, \Omega) \subset \operatorname{int}(\mathbf{A})$. For any $h \in \Omega$ and $r \in \mathbb{N}$, the element

$$(v, h)^r = (v + hv + \cdots + h^{r-1}v, h^r)$$

is contained in $\operatorname{int}(\mathbf{A})$. If $h^r = I$, and if $v = hw - w$ for some $w \in \mathbb{R}^n$, then $v + hv + \cdots + h^{r-1}v = 0$ and $(0, I) \in \operatorname{int}(\mathbf{A})$.

To finish the proof, we need to show that for any $v \in \mathbb{R}^n$ and any neighborhood Ω of I in $\mathsf{SO}(n)$, there exists an element $h \in \Omega$ such that

- $v = hw - w$ for some $w \in \mathbb{R}^n$
- $h^s = I$ for some $s \in \mathbb{N}$.

(We *outline* a proof. Let P denote a plane in $\mathbb{R}^n$ $(n \geq 2)$ that contains a given point $v \in \mathbb{R}^n$. Then for any neighborhood Ω of I in $\mathsf{SO}(2, P)$, there exists a rotation $R \in \Omega$ such that $R - I$ is nonsingular and $R^s = I$ for some $s \in \mathbb{N}$. Then R can be extended to $\mathbb{R}^n$ by defining it equal to the identity on the orthogonal complement $P^\perp$ of P in $\mathbb{R}^n$. Hence

$$v \in \operatorname{im}(R - I) \quad \text{and} \quad R^s = I.)$$

□

NOTE : A more general result holds : *Let K be a compact connected Lie group which acts linearly on a (real) vector space V, and suppose that V admits no nonzero fixed points (with respect to K). Then a left-invariant control system $\Gamma \subseteq \mathfrak{g}$ on the Lie group $G = V \ltimes K$ is controllable if and only if* $\operatorname{Lie}(\Gamma) = \mathfrak{g}$.

Besides the case $G = \mathbb{R}^n \ltimes \mathsf{SO}(n)$, another interesting case (in applications) is $G = \mathbb{R}^{2n} \ltimes \mathsf{U}(2n)$.

THEOREM 5.5.2 has far-reaching implications (in the theory of curves), as the following examples illustrate.

2.6.3 EXAMPLE. The Serret-Frenet system associated with a curve $x(\cdot)$ in $\mathbb{E}^3$ is given by

$$\dot{x} = R(t)e_1 \qquad \text{and} \quad \dot{R} = R \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}.$$

If both the curvature κ and the torsion τ are constant, then

$$\omega = \begin{bmatrix} \tau \\ 0 \\ \kappa \end{bmatrix}$$

is the axis of rotation for

$$A = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}.$$

Then $\exp(tA)$ is the rotation about ω through the angle $t\sqrt{\tau^2 + \kappa^2}$, and $x(\cdot)$ is a *helix* (along ω).

2.6.4 EXAMPLE. Suppose now that we consider curves whose curvature $\kappa =$ constant $(\neq 0)$ and whose torsion can take only two distinct values : τ_1 and τ_2. Such curves are concatenations of helices along

$$\omega_1 = \begin{bmatrix} \tau_1 \\ 0 \\ \kappa \end{bmatrix} \quad \text{and} \quad \omega_2 = \begin{bmatrix} \tau_2 \\ 0 \\ \kappa \end{bmatrix}.$$

The corresponding family of left-invariant vector fields on the special Euclidean group $\mathsf{SE}(3) = \mathbb{R}^3 \ltimes \mathsf{SO}(3)$ is

$$\Gamma = \{(e_1, A), (e_1, B)\} \subset \mathfrak{se}(3) = \mathbb{R}^3 \rightthreetimes \mathfrak{so}(3)$$

with

$$A = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau_1 \\ 0 & \tau_1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & -\tau_2 \\ 0 & \tau - 2 & 0 \end{bmatrix}.$$

It follows that $\mathrm{Lie}\,(\Gamma) = \mathbb{R}^3 \rtimes \mathfrak{so}\,(3)$ because of the following calculations :

$$(e_1, A) - (e_1, B) = (\tau_1 - \tau_2)(0, A_1)$$

and

$$[(e_1, A), (e_1, B)] = (\tau_1 - \tau_2)(0, A_2)$$

where we denote

$$A_1 := E_{23}, \quad A_2 := E_{13}, \quad \text{and} \quad A_3 := E_{12}.$$

(see Proposition 3.4.9). Then $[(0, A_1), (0, A_2)] = (0, A_3)$, and therefore $(0, \mathfrak{so}\,(3)) \subset \mathrm{Lie}\,(\Gamma)$. Hence $(e_1, 0) \in \mathrm{Lie}\,(\Gamma)$, and then $[(e_1, 0), (0, \mathfrak{so}\,(3))] = (\mathbb{R}^3, 0) \subset \mathrm{Lie}\,(\Gamma)$. Thus

$$\mathrm{Lie}\,(\Gamma) = \mathbb{R}^3 \rtimes \mathfrak{so}\,(3) = \mathfrak{se}\,(3).$$

According to Theorem 5.5.2, *any initial point* $x_0 \in \mathbb{E}^3$ *and any initial frame at* x_0 *can be connected to any terminal point* $x_1 \in \mathbb{E}^3$ *and any terminal frame at* x_1 *along the integral curves of the left-invariant family* $\Gamma = \{X_A, X_B\}$ *in* $\mathsf{SE}\,(3) = \mathbb{R}^3 \ltimes \mathsf{SO}\,(3)$ (with X_A and X_B equal to the left-invariant vector fields that coincide with (e_1, A) and (e_1, B) at the group identity, respectively).

Problems and Further Results

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Chapter 3

Optimal Control

Topics :

1. Optimal Control Problems
2. Pontryagin's Maximum Principle
3. Simple Examples
4. The Linear Time-Optimal Problem
5. The Linear-Quadratic Problem
6. Optimal Control on Matrix Lie Groups

3.1 Optimal Control Problems

Optimal control theory, recognized initially as an engineering subject, reveals a distinct relationship to classic forebears : the *calculus of variations, differential geometry*, and *mechanics*. This distinctive character of optimal control theory can be traced back to the mathematical problems of the subject in the mid 1950s dealing with inequality constraints. Faced with the practical, time-optimal control problems of that period, mathematicians (and engineers) looked to the calculus of variations for answers, but soon discovered that the answers to their problems were outside the scope of the classic theory (and would require different mathematical tools). That realization initiated a search for new necessary conditions for optimality suitable for control problems. That search, further intensified by the space programme and the race to the moon, eventually led to the *"maximum principle"* (1959), due to the Russian mathematician Lev S. Pontryagin (1908-1988) and his co-workers.

Note : Optimal control is significantly richer and broader than the calculus of variations, from which it differs in some fundamental ways. The calculus of variations deals mainly with optimization problems of the following "standard" form :

$$\mathcal{J} = \int_{t_0}^{t_1} L(x(t), \dot{x}(t), t)\, dt \;\to\; \min$$

subject to

$$x(t_0) = x_0 \quad \text{and} \quad x(t_1) = x_1$$

or, equivalently, of the form

$$\mathcal{J} = \int_{t_0}^{t_1} L(x(t), u(t), t)\, dt \;\to\; \min$$

subject to

$$x(t_0) = x_0, \quad x(t_1) = x_1, \quad \text{and} \quad \dot{x}(t) = u(t) \;\text{ for } t_0 \le t \le t_1.$$

The distinct feature of these problems is that the minimization takes place in the space of "all" curves, so nothing interesting happens on the level of the set of curves under consideration, and all the nontrivial features of the problem arise because of the *Lagrangian* L. Optimal control problems, by contrast, involve a minimization over a set $\mathcal{C}$ of curves which is itself determined by some dynamical constraints. For example, $\mathcal{C}$ might be the set of all curves $t \mapsto x(t)$ that satisfy a differential equation

$$\dot{x} = F(x, u, t)$$

for some choice of the "control function" $t \mapsto u(t)$. (Even more precisely, since it may happen that a member of $\mathcal{C}$ does not uniquely determine the control $u(\cdot)$ that generates it, we should be talking about trajectory-control pairs $(x(\cdot), u(\cdot))$). So in an optimal control problem there are at least two objects that give the situation interesting structure, namely, the *dynamics* F and the functional $\mathcal{J}$ to be minimized. In particular, optimal control theory contains at the opposite extreme from the calculus of variations, problems where the "Lagrangian" L is (identically) 1 (i.e. completely trivial), and therefore the interesting action occurs because of the dynamics F. Such problems, in which it is desired to minimize *time* (i.e. the integral of $\mathcal{J}$ with $L \equiv 1$) among all curves $t \mapsto x(t)$ that satisfy endpoint constraints and are solutions of (time-dependent) differential equations for some control $t \mapsto u(t)$, are called *time-optimal problems*. It is in these problems that the difference between optimal control and the calculus of variatons is most clearly seen, and it is no accident that these were the problems that propelled the development of optimal control in the early 1960s, and that time-optimal control is prominently represented in today's research.

Problem statement

Consider a *control system* of the form

$$\dot{x} = F(x, u), \qquad x \in M,\ u \in U \subseteq \mathbb{R}^m$$

where the *state space* M is a smooth manifold and the *control set* U is an arbitrary subset of $\mathbb{R}^m$. We shall assume that

- for each $u \in U$, the mapping $F_u = F(\cdot, u) : M \to TM$ is a smooth vector field on M
- the mapping $F : M \times U \to TM$ is continuous (or, most often, smooth).

The class of admissible controls $\mathcal{U}$ is the set of all (essentially) *bounded measurable* U-valued mappings (defined on some compact interval $[t_0, t_1]$). (For simplicity , one can consider *piecewise continuous* controls.)

NOTE : Let J be an *interval* in $\mathbb{R}$ and U an arbitrary subset of $\mathbb{R}^m$.

(a) A *piecewise constant* mapping $\omega : J \to U$ is one that is constant in each element J_i of a finite partition of J into subintervals.

(b) A mapping $u : J \to U$ is *measurable* if there exists some sequence $(\omega_r)_{r\geq 1}$ of piecewise constant mappings so that $\omega_r \to u$ almost everywhere (i.e. the set $\{t \in J \,|\, \omega_r(t) \not\to u(t)\}$ has measure zero). Clearly, *piecewise continuous mappings are measurable* (and, in general, $\varphi \circ u$ is measurable if u is measurable and φ is continuous).

(c) A mapping $u : J \to U$ is (essentially) *bounded* if it is measurable and there exists a compact subset $K \subseteq U$ such that $u(t) \in K$ for almost all $t \in J$. *Piecewise continuous mappings (with J compact) are (essentially) bounded.*

If $u(\cdot)$ is an admissible control, there is always a sequence $(\omega_r)_{r\geq 1}$ of piecewise constant mappings, converging almost everywhere to $u(\cdot)$. (Often one can obtain approximations by more regular controls. For instance, if U is *convex*, then each piecewise constant control can be approximated almost everywhere by continuous controls, and hence *every (essentially) bounded measurable control can be approximated by a sequence of continuous controls.* If, in addition, U is *open*, then one can

approximate (as long as the interval J is finite) by analytic, and even polynomial, controls.)

We shall use $\mathcal{F}$ to denote the family of (smooth) vector fields $\mathcal{F} = \{F_u \,|\, u \in U\}$ generated by F. A continuous curve $x(\cdot) : [t_0, t_1] \to M$ is called a **trajectory** of $\mathcal{F}$ if there exists a partition $t_0 = \tau_0 < \tau_1 < \cdots < \tau_m = t_1$ and vector fields $X_1, \ldots, X_m$ in $\mathcal{F}$ such that the restriction of $x(\cdot)$ to each open interval (τ_{i-1}, τ_i) is smooth and (for $t \in (\tau_{i-1}, \tau_i)$)

$$\dot{x}(t) = X_i(x(t)), \qquad i = 1, 2, \cdots m.$$

NOTE : Because the elements of $\mathcal{F}$ are parametrized by controls, it follows that each X_i is equal to F_{u_i} for some $u_i \in U$. Hence $x(\cdot) = x_u(\cdot)$ is an integral curve of the *time-varying* vector field (on M) $(t, x) \mapsto F(x, u(t))$, with $u(\cdot)$ equal to the piecewise constant control which takes constant value u_i in each subinterval $[\tau_{i-1}, \tau_i]$ (see DEFINITION 4.3.2).

We shall refer to a trajectory-control pair $(x(\cdot), u(\cdot))$ as a **controlled trajectory**. (In some cases, a trajectory $x(\cdot)$ cannot arise from more than one control $u(\cdot)$, so it is not necessary to distinguish between "trajectories" and "controlled trajectories".)

In order to compare admissible controls one with another (on an interval $[t_0, t_1]$), introduce a *cost functional*

$$\mathcal{J}(u) := \int_{t_0}^{t_1} L(x(t), u(t))\, dt.$$

(The integrand $L : M \times U \to \mathbb{R}$, called the **Lagrangian**, satisfies the same regularity assumptions as F.) Let $x_0, x_1 \in M$. We formulate the following problem :

"MINIMIZE THE COST FUNCTIONAL $\mathcal{J}$ IN THE CLASS OF ALL CONTROLLED TRAJECTORIES $(x(\cdot), u(\cdot))$ SUCH THAT

$$x(t_0) = x_0 \quad \text{AND} \quad x(t_1) = x_1."$$

A controlled trajectory $(x(\cdot), u(\cdot)) : [t_0, t_1] \to M \times U$ such that $x(t_0) = x_0$ and $x(t_1) = x_1$ is said to *transfer* (or steer) the initial point (state) x_0 to the final point (state) x_1 over the time interval $[t_0, t_1]$.

We shall refer to this problem as the **optimal control problem** (OCP) :

$$\dot{x} = F(x, u), \qquad x \in M,\ u \in U$$

$$x(t_0) = x_0, \quad x(t_1) = x_1 \qquad (x_0, x_1 \text{ fixed})$$

$$\mathcal{J} = \int_{t_0}^{t_1} L(x(t), u(t))\, dt \ \to \ \min.$$

NOTE : The length of time required to transfer x_0 to x_1 is *not* fixed in advance. On the other hand, if the controlled trajectory $(x(\cdot), u(\cdot))$ transfers x_0 to x_1 over the interval $[t_0, t_1]$, then the "time-shifted" controlled trajectory $(\bar{x}(\cdot), \bar{u}(\cdot))$ with $\bar{x}(t) = x(t + t_0)$ and $\bar{u}(t) = u(t + t_0)$, transfers x_0 to x_1 over the interval $[0, t_1 - t_0]$, and the cost of the transfer along $(\bar{x}(\cdot), \bar{u}(\cdot))$ is the same as the cost of transfer along $(x(\cdot), u(\cdot))$. Hence, *the initial time* t_0 *can always be taken to be* 0.

One study two types of problems : with *fixed* (final time) t_1, and with *free* t_1. A solution $(\bar{x}(\cdot), \bar{u}(\cdot))$ of the OCP is called **optimal**; the admissible control $\bar{u}(\cdot)$ is called and *optimal control*, and the corresponding trajectory (curve) $\bar{x}(\cdot)$ is the *optimal trajectory.*

NOTE : The optimal control problem (OCP) is the minimization problem for $\mathcal{J}(u)$ with constraints on u given by the (state equation of the) control system and the

fixed endpoints conditions. These constraints cannot usually be resolved w.r.t. u, hence *solving optimal control problems requires special techniques.*

Existence of optimal solutions

Optimal control problems on the state space M can be essentially reduced to the study of *attainable sets* of some auxiliary control systems. Indeed, the integrant L of the cost functional $\mathcal{J}$ (to be minimized) and the dynamics F of the control system can be viewed as (defining) an *extended control system* on $M_{\text{ext}} := \mathbb{R} \times M$:

$$\begin{aligned} \dot{\xi} &= L(x,u) \\ \dot{x} &= F(x,u), \qquad (\xi, x) \in M_{\text{ext}}, u \in U. \end{aligned}$$

Then trajectories $x_{\text{ext}}(\cdot)$ of the extended control system (with initial conditions $x_{\text{ext}}(0) = (0, x_0)$) are expressed through trajectories of the initial control system as

$$x_{\text{ext}}(t) = \left(\int_0^t L(x(\tau), u(\tau))\, d\tau,\ x(t) \right).$$

(Trajectories of the extended control systems are curves $t \mapsto (\xi(t), x(t))$ in $\mathbb{R} \times M$ parametrized by the control functions $u(\cdot)$.)

It turns out that optimal trajectories of the OCP on M (more precisely, their *lift* to the extended state space M_{ext}) must come to the boundary of the attainable set $\mathbf{A}_{\text{ext}}((0, x_0), t_1)$. Hence, in order to find optimal trajectories, we find first those coming to the boundary of $\mathbf{A}_{\text{ext}}((0, x_0), t_1)$, and then select the optimal ones among them.

NOTE : The first step is much more important than the second one, so solving OCPs essentially reduces to the study of (dynamics of boundary of) attainable sets.

Due to the reduction of OCPs to the study of attainable sets, *existence of optimal solutions to OCPs is reduced to compactness of attainable sets.* For control systems

$$\dot{x} = F(x, u), \qquad x \in M, u \in U$$

sufficient conditions for compactness of attainable sets are given in the following proposition (given without proof), due to the Russian mathematician Alexei F. Filippov (1923-).

3.1.1 Proposition. *Let the control set $U \subset \mathbb{R}^m$ be compact. Assume that :*

(i) There exists a compact set $K \subset M$ such that $F(x, u) = 0$ for $x \notin K$ and $u \in U$.

(ii) The velocity sets

$$F_U(x) := \{F(x, u) \,|\, u \in U\} \subseteq T_x M, \qquad x \in M$$

are convex.

Then the attainable sets $\mathbf{A}(x_0, t)$ and $\mathbf{A}(x_0, \leq T)$ are compact for all $x_0 \in M$, and $T > 0$.

Note : In *Filippov's theorem*, the hypothesis of common support of the vector fields (in the right-hand side) is essential to ensure the *completeness* of vector fields (and also the uniform boundedness of velocities). On a manifold, sufficient conditions for completeness of vector fields cannot be given in terms of boundedness of the vector field and its derivatives: a constant (parallel) vector field is *not* complete on a bounded domain in $\mathbb{R}^n$. Nevertheless, *one can prove compactness of attainable sets for many control systems without the assumption of common compact support.* (If for such a system we have a priori bounds on the solution, then we can multiply its

right-hand side by a "cut-off" function, and obtain a control system with vector fields having compact support. We can apply Filippov's theorem to this new system; since trajectories of the initial and new control systems coincide in a domain of interest for us, we obtain a conclusion on compactness of attainable sets for the initial system.)

For control systems on $M = \mathbb{E}^n$, there exist well-known sufficient conditions for completeness of vector fields : if the right-hand side grows at infinity not faster than a linear one (i.e., for some constant C,

$$\|F(x,u)\| \leq C\,(1 + \|x\|)\,, \qquad x \in \mathbb{E}^n, u \in U)$$

then the (time-varying) vector fields F_u are complete. These conditions provide an a priori bound for solutions : any solution $x(\cdot)$ of the control system

$$\dot{x} = F(x,u), \qquad x \in \mathbb{E}^n, u \in U$$

with the right-hand side satisfying the above condition, admits the bound

$$\|x(t)\| \leq e^{2Ct}\,(\|x(0)\| + 1)\,, \qquad t \geq 0.$$

Filippov's theorem (plus the preceding remark) implies the following sufficient condition for compactness of attainable sets for control systems on $\mathbb{E}^n$.

3.1.2 Corollary. *Let the control set $U \subset \mathbb{R}^m$ be compact. Assume that :*

(i) There exists a constant C such that

$$\|F(x,u)\| \leq C\,(1 + \|x\|)\,, \qquad x \in \mathbb{E}^n, u \in U.$$

(ii) The velocity sets

$$F_U(x) := \{F(x,u) \,|\, u \in U\} \subseteq T_x\mathbb{E}^n, \qquad x \in \mathbb{E}^n$$

are convex.

Then the attainable sets $\mathbf{A}(x_0, t)$ *and* $\mathbf{A}(x_0, \leq T)$ *are compact for all* $x_0 \in M$*, and* $T > 0$.

3.2 Pontryagin's Maximum Principle

The optimal control problem (OCP) is to find an optimal solution (i.e. an optimal controlled trajectory), assuming that the latter exists. An indirect approach to this problem consists in first determining some properties of optimal trajectories that will be sufficiently distinctive to narrow the class of candidates for optimal solutions to a small class of curves. *Pontryagin's Maximum Principle (PMP) provides a list of necessary conditions that an optimal trajectory must fulfill.* We begin with an initial formulation of the maximum principle for optimal control problems in $\mathbb{E}^n$.

NOTE : Rather than seeking the most general conditions under which the principle is valid, we shall follow the original presentation of LEV S. PONTRYAGIN and his co-workers, namely V.G. BOLTYANSKII, R.V. GAMKRELIDZE, and E.F. MISHCHENKO. This level of generality is sufficient for many important applications and is, at the same time, relatively free of the technicalities that could obscure its geometric content.

The maximum principle in $\mathbb{E}^n$

Let

$$F = (F_1, F_2, \ldots, F_n) : M \times U \to \mathbb{E}^n$$

be a given mapping (of $n + m$ variables $x_1, \ldots, x_n, u_1, \ldots, u_m$), where M is an *open* subset of (the Euclidean n-space) $\mathbb{E}^n$ and U is an arbitrary subset of $\mathbb{R}^m$. Consider the *control system* (on M) described by (the state equation)

$$\dot{x} = F(x, u), \qquad x \in M, u \in U.$$

NOTE : The Euclidean space $\mathbb{E}^n$ is viewed as a smooth n-manifold, whereas the Cartesian n-space $\mathbb{R}^m$ is viewed here only as a topological space. The state space M is a smooth submanifold of $\mathbb{E}^n$ (of dimension n).

We shall assume that

- for each $u \in U$, the mapping $F_u = F(\cdot, u) : M \to \mathbb{E}^n$ is smooth
- the mappings $F, \frac{\partial F}{\partial x_i} : M \times U \to \mathbb{E}^n$ are continuous (with respect to the canonical topology on $\mathbb{E}^n$ and the induced topology on $M \times U \subset \mathbb{E}^n \times \mathbb{R}^m$).

By an *admissible control* $u(\cdot)$ we mean a U-valued mapping defined on some compact interval $[t_0, t_1]$ that is (essentially) bounded and measurable on $[t_0, t_1]$. When an admissible control $u(\cdot) : [t_0, t_1] \to U$ is substituted in the (right-hand side of the) state equation, a system of (time-varying) ODEs

$$\dot{x}_i = F_i(x(t), u(t)), \qquad i = 1, 2, \ldots, n$$

results. CARATHÉODORY's Existence and Uniqueness Theorem guarantees that, *for any admissible control* $u(\cdot)$ *and any* $x_0 \in M$ *: (i) there exists, on some interval* $J \subseteq [t_0, t_1]$ *such that* $t_0 \in J$*, a solution curve through* x_0 (i.e. an absolutely continuous mapping $x(\cdot) : J \to M$ such that $\dot{x}_i(t) = F_i(x(t), u(t))$ for almost all $t \in J$ and $x(t_0) = x_0$; *(ii) if* $x_1(\cdot) : J_1 \to M$ *and* $x_2(\cdot) : J_2 \to M$ *are two such solution curves, then they coincide on* $J_1 \cap J_2$.

NOTE : A mapping $\xi : J = [a, b] \to \mathbb{R}^n$ is said to be *absolutely continuous* if it satisfies the following property : for each $\varepsilon > 0$ there is a $\delta > 0$ such that, for every finite sequence of points

$$a \leq a_1 < b_1 < a_2 < \cdots < a_k < b_k \leq b$$

so that $\sum_{i=1}^{k}(b_i - a_i) < \delta$, it holds that $\sum_{i=1}^{k} \|\xi(b_i) - \xi(a_i)\| < \varepsilon$. The mapping ξ is absolutely continuous if and only if it is an indefinite integral (i.e. there is some

integrable mapping h such that

$$\xi(t) = \xi(a) + \int_a^t h(\tau)\, d\tau$$

for all $t \in J$). An absolutely continuous mapping is differentiable almost everywhere, and $\dot{\xi}(t) = h(t)$ holds for almost all t.

Solution curves can always be continued to the maximal interval of existence. Assuming that J is the maximal interval, then the solution curve (through x_0) is unique (up to a set of measure zero). We shall refer to it as an *integral curve* (or sometimes a *trajectory*) of the original control system, that corresponds to $u(\cdot)$.

For any integral curve $x(\cdot) : J \to M$, let $A(t)$ denote the matrix

$$A(t) := \left[\frac{\partial F_i}{\partial x_j}(x(t), u(t))\right] \in \mathbb{R}^{n\times n}.$$

Each entry $A_{ij} : J \to \mathbb{R}$ is an (essentially) bounded measurable function. The following linear system of ODEs is called the **variational system** along the trajectory $x(\cdot)$ (or, more precisely, along the *pair* $(x(\cdot), u(\cdot))$) :

$$\dot{v}_i = \sum_{j=1}^{n} A_{ij}(t) v_j, \qquad i = 1, 2, \ldots, n.$$

It follows from the theory of linear differential equations that for each $v_0 \in \mathbb{R}^n$, there exists an absolutely continuous curve $v(\cdot) : J \to M$ such that $v(t_0) = v_0$ and which satisfies the above conditions (linear ODEs) for almost all $t \in J$.

The **adjoint variational system** (along $x(\cdot)$) is given by

$$\dot{p}_i = -\sum_{j=1}^{n} p_j A_{ji}(t), \qquad i = 1, 2, \ldots, n.$$

The solution curves for the adjoint system are also defined on the entire interval J for each initial value $p_0 \in \mathbb{R}^n$.

Exercise 21 Verify that the solution curves $v(\cdot)$ and $p(\cdot)$ of the variational system and the adjoint variational system, respectively, satisfy (for almost $t \in J$)

$$p_1(t)v_1(t) + \cdots + p_n(t)v_n(t) = \text{ constant.}$$

NOTE : The pair of differential systems

$$\begin{aligned} \dot{x}_i &= F_i(x(t), u(t)) \\ \dot{p}_i &= -\sum_{j=1}^{n} p_j \frac{\partial F_j}{\partial x_i}(x(t), u(t)), \qquad i = 1, 2, \ldots, n \end{aligned}$$

can be expressed in terms of a single function H, given by

$$H(x, p, u) := p_1 F_1(x, u) + \cdots + p_n F_n(x, u)$$

by the formulas

$$\dot{x}_i = \frac{\partial H}{\partial p_i}(x, p, u) \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}(x, p, u) \qquad (i = 1, 2, \ldots, n)$$

(valid for any admissible control $u(\cdot)$).

Consider the optimal control problem

$$\dot{x} = F(x, u), \qquad x \in M \subseteq \mathbb{E}^n,\ u \in U$$

$$x(0) = x_0, \quad x(t_1) = x_1 \qquad (x_0, x_1 \text{ fixed})$$

$$\int_0^{t_1} L(x(t), u(t))\, dt \ \to \ \min.$$

The extended control system (on $M_{\text{ext}} = \mathbb{R} \times M$) defines its family of *Hamiltonians* $\mathcal{H}_u : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$ (parametrized by the control functions), given by

$$\mathcal{H}_u(x, p) := p_0 L(x, u) + p_1 F_1(x, u) + \cdots + p_n F_n(x, u).$$

NOTE : In a slight abuse of notation, we write (in the left-hand side) x instead of $x_{\text{ext}} = (\xi, x) \in \mathbb{R} \times M$ and also $p = \begin{bmatrix} p_0 & p_1 & \dots & p_n \end{bmatrix} \in \mathbb{R}^{1\times(n+1)}$ $\big(= (\mathbb{R}^{n+1})^*\big)$.

For each admissible control $u(\cdot)$, let $\vec{\mathcal{H}}_u$ denote the *Hamiltonian vector field* that corresponds to (the Hamiltonian function) $\mathcal{H}_u$. $\vec{\mathcal{H}}_u$ is the vector field (on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$) defined by

$$\vec{\mathcal{H}}_u(x,p) := \begin{bmatrix} \frac{\partial \mathcal{H}_u}{\partial p}(x,p) \\ \\ -\frac{\partial \mathcal{H}_u}{\partial x}(x,p) \end{bmatrix} \in \mathbb{R}^{2(n+1)\times 1}.$$

We have

$$\vec{\mathcal{H}}_u = \mathbb{J} \cdot \nabla \mathcal{H}_u$$

where $\mathbb{J} = \begin{bmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{bmatrix} \in \mathbb{R}^{2(n+1)\times 2(n+1)}$ and $\nabla \mathcal{H}_u$ is the naive gradient of $\mathcal{H}_u$; that is, the row matrix (the matrix of the derivative) $d\mathcal{H}_u$ written as a column matrix : $\nabla \mathcal{H}_u = \begin{bmatrix} \frac{\partial \mathcal{H}_u}{\partial x} & \frac{\partial \mathcal{H}_u}{\partial p} \end{bmatrix}^T$.

NOTE : Consider a vector space $E = V \times V^*$, where V is a (real) vector space and V^* is its dual. Define the *canonical symplectic form* Ω on E by

$$\Omega\left((v_1, \alpha_1), (v_2, \alpha_2)\right) := \alpha_2(v_1) - \alpha_1(v_2)$$

where $v_1, v_2 \in V$ and $\alpha_1, \alpha_2 \in V^*$. Then the induced linear mapping $\Omega^\flat : E \to E^*$, defined by

$$\Omega^\flat(v_1, \alpha_1)(v_2, \alpha_2) := \Omega\left((v_1, \alpha_1), (v_2, \alpha_2)\right),$$

is one-to-one. (If V is finite dimensional, then so is E and $\Omega^\flat$ is an isomorphism. In this case the matrix of $\Omega^\flat$ is $\mathbb{J}^T$.) A vector field $X : E \to E$ is called *Hamiltonian* if

$$\Omega^\flat(X(v, \alpha)) = dH(v, \alpha)$$

for all $(v, \alpha) \in E$, for some C^1 function $H : E \to \mathbb{R}$. (Here $dH = DH$ is alternative notation for the derivative of H.) If such an H exists, we write $X = X_H$ (or $X = \vec{H}$)

and call H a *Hamiltonian function* for X. In a number of important examples, H need not be defined on all of E. If (the vector space) E is finite dimensional, then the existence of X_H is guaranteed for any given (C^1) function H. Moreover, X_H is unique (since the mapping $\Omega^\flat$ is one-to-one).

All these considerations carry over to any *symplectic vector space* (E, Ω). Here E is a real Banach space, and $\Omega : E \times E \to \mathbb{R}$ is a non-degenerate skew-symmetric (continuous) bilinear form; non-degeneracy of Ω is equivalent to the injectivity of $\Omega^\flat$. If E is finite dimensional, then the induced mapping Ω is an isomorphism, and the dimension is *even*, since the determinant of a skew-symmetric matrix with an odd number of rows (and columns) is zero.

The integral curves of $\vec{\mathcal{H}}_u$ satisfy

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \vec{\mathcal{H}}_u(x,p), \qquad (x,p) \in \mathbb{R}^{n+1} \times \left(\mathbb{R}^{n+1}\right)^* ;$$

that is, they satisfy the following system of ODEs (for almost all t):

$$\begin{aligned}
\dot{\xi} &= \frac{\partial \mathcal{H}_u}{\partial p_0} = L(x,u) \\
\dot{x}_i &= \frac{\partial \mathcal{H}_u}{\partial p_i} = F_i(x,u), \qquad 1 = 1,2,\ldots,n \\
\dot{p}_0 &= -\frac{\partial \mathcal{H}_u}{\partial \xi} = p_0 \frac{\partial L}{\partial \xi}(x,u) + \sum_{j=1}^{n} p_j \frac{\partial F_j}{\partial \xi}(x,u) \\
\dot{p}_i &= -\frac{\partial \mathcal{H}_u}{\partial x_i} = p_0 \frac{\partial L}{\partial x_i}(x,u) + \sum_{j=1}^{n} p_j \frac{\partial F_j}{\partial x_i}(x,u), \qquad i = 1,2,\ldots,n.
\end{aligned}$$

The **maximal Hamiltonian** associated with each integral curve $(x(\cdot), p(\cdot))$ is defined by

$$\mathcal{H}(x,p) := \sup_{u \in U} \mathcal{H}_u(x,p).$$

The maximum principle consists of necessary conditions for optimality. We shall consider first OCPs with *free* final time.

3.2.1 THEOREM. *If* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *is an optimal solution of our OCP (with free final time* $t_1 > 0$*), then there exists a nonzero, absolutely continuous curve* $p = (p_0, p_1, \ldots, p_n) : [0, t_1] \to \left(\mathbb{R}^{n+1}\right)^*$ *such that :*

(MP1) $(\bar{x}(\cdot), p(\cdot))$ *is a solution curve of the differential system*

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \vec{\mathcal{H}}_{\bar{u}}(x, p), \qquad (x, p) \in \mathbb{R}^{n+1} \times \left(\mathbb{R}^{n+1}\right)^*.$$

(MP2) $\mathcal{H}_{\bar{u}}(\bar{x}, p) = \mathcal{H}(\bar{x}, p)$ *for almost all* $t \in [0, t_1]$.

(MP3) $p_0(t_1) \leq 0$ *and* $\mathcal{H}(\bar{x}(t_1), p(t_1)) = 0$.

(Furthermore, it can be shown that

$$\mathcal{H}_{\bar{u}}(\bar{x}, p) = \textit{const}, \qquad t \in [0, t_1]$$

and the coordinate $p_0(\cdot)$ *associated with the adjoint curve* $t \mapsto p(t)$ *is constant.)*

NOTE : (1) p_0 can always be normalized and so *we can assume that* $p_0 = -1$ *or* 0. It is then convenient to reduce the Hamiltonians to $M \times \mathbb{R}^n$ and regard p_0 as parameter.

(2) If we have a *maximization* problem instead of a minimization problem (OCP), then the inequality $p_0(t_1) \leq 0$ should be reversed.

A curve $t \mapsto (x(t), p(t), u(t))$ in $M \times (\mathbb{R}^n)^* \times U$ is called an **extremal triple** if there exists a constant $p_0 \leq 0$ such that $x(\cdot), p(\cdot), u(\cdot)$, and p_0 satisfy conditions (MP1)-(MP3) of the maximum principle and, in addition, satisfy $p(t) \neq 0$ whenever $p_0 = 0$. We shall also say that $(x(\cdot), p(\cdot))$ is the *extremal curve* generated by $u(\cdot)$. (Sometimes we may also refer to $u(\cdot)$ as the *extremal control.*)

NOTE : It is known that the maximal Hamiltonian $\mathcal{H} = \sup_{u \in U} \mathcal{H}_u$ is constant along each extremal curve $(x(\cdot), p(\cdot))$. Consequently, condition (MP3) of the maximum principle can be replaced by

$$\mathcal{H}(x, p) \equiv 0.$$

The passage to OCPs with *fixed* final time is as follows : Suppose that $x_0, x_1 \in M$ and (final time) $t_1 > 0$ are fixed in advance, and that $(x(\cdot), u(\cdot))$ is an optimal solution of our OCP; that is, $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a solution curve that minimizes the cost (functional)

$$\int_0^{t_1} L(x(t), u(t))\, dt$$

among all other solutions that transfer x_0 to x_1 in t_1 units of time.

Let $x_{n+1}(t) = t$ be another coordinate attached to the solution curve $t \mapsto x(t) = (x_1(t), \ldots x_n(t))$. Denote by $\tilde{x}_0 = (x_0, 0)$ and $\tilde{x}_1 = (x_1, t_1)$ the points in $M \times \mathbb{R} \subseteq \mathbb{E}^{n+1}$ defined by the boundary conditions x_0 and x_1. Then $\tilde{x}(\cdot) = (x_1(\cdot), \ldots, x_n(\cdot), x_{n+1}(\cdot))$, and $u(\cdot)$ is a solution curve for the OCP for the extended system (on $(\mathbb{R} \times M) \times \mathbb{R}$) :

$$\begin{aligned} \dot{\xi} &= L(x, u) \\ \dot{x}_i &= F_i(x, u), \quad i = 1, 2, \ldots, n \\ \dot{x}_{n+1} &= 1. \end{aligned}$$

An extended (controlled) trajectory $(\tilde{x}(\cdot), u(\cdot))$ of the foregoing system can transfer $\tilde{x}_0$ to $\tilde{x}_1$ *only* in t_1 units of time. Therefore, the adjoint curve defined by the maximum principle is defined on $[0, t_1]$. Let $p_{n+1}(\cdot)$ denote the component of the adjoint curve that corresponds to the optimal solution $(\tilde{x}(\cdot), u(\cdot))$ (of the extended system). Then $\dot{p}_{n+1} = 0$ (since the original

system is autonomous) and therefore $p_{n+1}(t) =$ constant. Hence condition (MP3) becomes

$$p_0 L(x,u) + \sum_{i=1}^{n} p_i(t) F_i(x(t),u(t)) + p_{n+1} = 0$$

for almost all $t \in [0,t_1]$. Thus (for almost all t)

$$\sum_{i=0}^{n} p_i(t) F_i(x(t),u(t)) = \mathcal{H}_{u(t)}(x(t),p(t)) = \text{ constant.}$$

The foregoing argument shows that *the necessary optimality conditions given by PMP, for fixed final time, differ from the conditions corresponding to a free (variable) final time only by the constant defined by* $\mathcal{H}(x,p)$.

NOTE : Nonautonomous (i.e. time-varying) problems can be reduced to autonomous ones by a similar trick, leading to necessary optimality conditions, except that the maximal Hamiltonian $\mathcal{H}(x(t),u(t))$ may no longer be constant along extremal curves.

Now consider the *time-optimal control problem* (T-OCP):

$$\dot{x} = F(x,u), \qquad x \in M \subseteq \mathbb{E}^n,\ u \in U$$

$$x(0) = x_0, \quad x(t_1) = x_1 \qquad (x_0, x_1 \text{ fixed})$$

$$t_1 = \int_0^{t_1} 1\, dt \ \to \ \min.$$

For this problem, Pontryagin's Maximum Principle (PMP) takes the following form:

3.2.2 COROLLARY. *If* $(\bar{x}(\cdot),\bar{u}(\cdot))$ *is an optimal solution of our T-OCP, then there exists a nonzero, absolutely continuous curve* $p = (p_1,\ldots,p_n) : [0,t_1] \to (\mathbb{R}^n)^*$ *such that :*

(1) $(\bar{x}(\cdot), p(\cdot))$ *is a solution curve of the differential system*

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \vec{\mathcal{H}}_{\bar{u}}(x,p), \qquad (x,p) \in \mathbb{R}^n \times (\mathbb{R}^n)^* .$$

(2) $\mathcal{H}_{\bar{u}}(\bar{x},p) = \mathcal{H}(\bar{x},p)$ *for almost all* $t \in [0,t_1]$.

(3) $\mathcal{H}(\bar{x},p) \geq 0$ *for almost all* $t \in [0,t_1]$.

Proof : Apply Theorem 6.2.1 (and the remark after it) by taking $L \equiv 1$. Then the Hamiltonian system (1) and the maximality condition (2) follow. Inequality (3) is equivalent to conditions

$$\mathcal{H}_{\bar{u}}(\bar{x},p) + p_0 = 0 \quad \text{and} \quad p_0 \leq 0.$$

The condition $p \neq 0$ is obtained as follows : if $p = 0$, then $\mathcal{H}_{\bar{u}}(\bar{x},p) = 0$ and hence $p_0 = 0$. But the pair $(p_0, p) \in (\mathbb{R})^* \times (\mathbb{R}^n)^*$ must be nontrivial. Consequently, $p \neq 0$.

□

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3.3 Simple Examples

We consider several optimal control problems (which can be solved by appying Pontryagin's Maximum Principle). We start with some simple concrete problems.

The fastest stop of a train at a station

Consider a train moving on a railway. The problem is *to drive the train to a station and stop it there in a minimal time.*

Describe position of the train by a coordinate z on the real line; the origin $0 \in \mathbb{R}$ corresponds to the station. Assume that the train moves without friction, and we can control acceleration of the train by applying a force bounded by absolute value. Using rescaling if necessary, we can assume that the absolute value of acceleration is bounded by 1. We write the equation of motion as

$$\ddot{z} = u, \qquad z \in \mathbb{R},\ |u| \leq 1$$

or, in the standard form (for $x_1 = z$ and $x_2 = \dot{z}$),

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \qquad x \in \mathbb{E}^2,\ |u| \leq 1. \end{aligned}$$

Our time-optimal control problem (T-OCP) is

$$\boxed{\begin{gathered} \dot{x} = \begin{bmatrix} x_2 \\ u \end{bmatrix}, \qquad x \in \mathbb{E}^2,\ |u| \leq 1 \\ x(0) = x_0, \quad x(t_1) = 0 \qquad (x_0 \text{ fixed}) \\ t_1 = \int_0^{t_1} 1\, dt \ \to \ \min. \end{gathered}}$$

First we verify existence of optimal controls by Filippov's theorem.

. . . . . .

Now we apply *Pontryagin's Maximum Principle* (PMP). Introduce canonical coordinates on the cotangent bundle :

$$\begin{aligned} T^*\mathbb{E}^2 &= \mathbb{R}^2 \times (\mathbb{R}^2)^* \\ &= \left\{ (x,p) \,|\, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ p = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \right\}. \end{aligned}$$

The control-dependent Hamiltonian function of PMP is

$$\mathcal{H}_u(x,p) = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \begin{bmatrix} x_2 \\ u \end{bmatrix} = p_1 x_2 + p_2 u$$

and the corresponding Hamiltonian system has the form

$$\begin{aligned} \dot{x} &= \frac{\partial \mathcal{H}_u}{\partial p} \\ \dot{p} &= -\frac{\partial \mathcal{H}_u}{\partial x} \cdot \end{aligned}$$

. . . . . .

So for any point of the Euclidean plane there exists exactly one extremal trajectory steering this point to the origin. Since optimal trajectories exist, then the solutions found are optimal.

Control of a linear oscillator

Consider a linear oscillator whose motion can be controlled by a force bounded in absolute value. The equation of motion (after appropriate rescaling) is

$$\ddot{z} + z = u, \qquad z \in \mathbb{R},\ |u| \leq 1$$

or, in the canonical form :

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u, \qquad x \in \mathbb{E}^2,\ |u| \leq 1. \end{aligned}$$

We consider the following time-optimal control problem (T-OCP) :

$$\dot{x} = \begin{bmatrix} x_2 \\ -x_1 + u \end{bmatrix}, \qquad x \in \mathbb{E}^2,\ |u| \leq 1$$

$$x(0) = x_0, \quad x(t_1) = 0 \qquad (x_0 \text{ fixed})$$

$$t_1 = \int_0^{t_1} 1\, dt \ \to \ \min.$$

By Filippov's theorem, optimal control exists.

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We apply *Pontryagin's Maximum Principle* (PMP) : the control-dependent Hamiltonian function is

$$\mathcal{H}_u(x,p) = p_1 x_2 - p_2 x_1 + p_2 u, \qquad (x,p) \in T^*\mathbb{E}^2 = \mathbb{R}^2 \times (\mathbb{R}^2)^*$$

and the Hamiltonian system reads

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u \\ \dot{p}_1 &= p_2 \\ \dot{p}_2 &= -p_1. \end{aligned}$$

. . . . . .

The time-optimal control problem is solved : in the part of the Euclidean plane over the switching curve the optimal control is $\bar{u} = -1$ and below this curve $\bar{u} = +1$. Through any point of the plane passes one optimal trajectory which corresponds to this optimal control rule. After finite number of switchings, any optimal trajectory comes to the origin.

The cheapest stop of a train

Again we control the motion of a train. Now the goal is *to stop the train at the fixed instant of time with a minimum expenditure of energy* (which is assumed to be proportional to the integral of squared acceleration).

So the T-OCP is as follows :

$$\dot{x} = \begin{bmatrix} x_2 \\ u \end{bmatrix}, \qquad x \in \mathbb{E}^2,\ |u| \le 1$$

$$x(0) = x_0, \quad x(t_1) = 0 \qquad (x_0, t_1 > 0 \ \text{ fixed})$$

$$\frac{1}{2}\int_0^{t_1} u^2(t)\,dt \ \to \ \min.$$

Filippov's theorem *cannot* be applied directly, since the rhs of the control system is not compact.

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In order to find (the) optimal control, we apply PMP. The Hamiltonian function is

$$\mathcal{H}_u(x,p) = \frac{p_0}{2}u^2 + p_1 x_2 + p_2 u, \qquad (x,p) \in \mathbb{R}^2 \times (\mathbb{R}^2)^*.$$

Along optimal trajectories

$$p_0 \le 0 \quad \text{and} \quad p_0 = \text{ constant.}$$

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So through the initial point (state) x_0 passes a unique extremal trajectory (ariving at the origin). It is a curve $t \mapsto (x_1(t), x_2(t)),\ t \in [0, t_1]$, where $x_1(\cdot)$ is a cubic polynomial that satisfies certain boundary conditions (see above) and $x_2(t) = \dot{x}_1(t)$. In view of existence, this is an optimal trajectory.

Control of a linear oscillator with cost

We control a linear oscillator (e.g. a pendulum with a small amplitude) by an unbounded force $u(\cdot)$, but take into account expenditure of energy measured by the inegral $\frac{1}{2}\int_0^{t_1} u^2(t)\,dt$. Our optimal control problem (OCP) is :

$$\dot{x} = \begin{bmatrix} x_2 \\ -x_1 + u \end{bmatrix}, \qquad x \in \mathbb{E}^2,\ u \in \mathbb{R}$$

$$x(0) = x_0, \quad x(t_1) = 0 \qquad (x_0, t_1 > 0 \ \text{ fixed})$$

$$\frac{1}{2}\int_0^{t_1} u^2(t)\,dt \ \to \ \min.$$

Existence of optimal control(s) can be proved by the same argument as in the previous example.

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The Hamiltonian function of PMP is

$$\mathcal{H}_u(x,p) = \frac{p_0}{2}u^2 + p_1x_2 - p_2x_1 + p_2u.$$

The corresponding Hamiltonian system yields

$$\begin{aligned} \dot{p}_1 &= p_2 \\ \dot{p}_2 &= -p_1. \end{aligned}$$

In the same way as the previous example, we show that there are no abnormal extremals. Hence we can assume that $p_0 = -1$.

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3.4 The Linear Time-Optimal problem

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be matrices, and let U be a compact convex *polytope* in $\mathbb{R}^m$. (The polytope U is the *convex hull* of a finite set of points $a_1, \ldots, a_k$ in $\mathbb{R}^m$: $U = \operatorname{conv}\{a_1, \ldots, a_k\}$. We assume that the points a_i do not belong to the convex hull of all the other points a_j, $j \neq i$ so that each a_i is a *vertex* of the polytope U.)

We consider the following time-optimal control problem (T-OCP) :

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in U$$

$$x(0) = x_0, \quad x(t_1) = x_1 \qquad (x_0, x_1 \text{ fixed})$$

$$t_1 = \int_0^{t_1} 1\, dt \ \to \ \min.$$

Such a problem is called a *linear time-optimal* control problem.

NOTE : T-OCPs constitute one of the basic concerns of optimal control theory. Minimal-time problems go back to the beginnings of the calculus of variations. JOHAN BERNOULLI's solution of the *brachistochrone problem* in 1697 was based on Fermat's principle of least time, which postulates that "light traverses any medium in the least possible time". Since then such problems have remained important sources of inspiration.

We shall assume that the following (general position) condition holds : For any edge $[a_i, a_j]$ of (the polytope) U, the vector $e_{ij} := a_j - a_i$ is such that

$$\operatorname{span}\{Be_{ij}, ABe_{ij}, \ldots, A^{n-1}Be_{ij}\} = \mathbb{R}^n.$$

(This condition is equivalent to the controllability of the linear control system $\dot{x} = Ax + Bu$ with the set of control parameters $u \in \mathbb{R}e_{ij}$. The condition can

be achieved by a small perturbation of the matrices A, B.)

Existence of optimal control for any points x_0, x_1 such that $x_1 \in \mathbf{A}(x_0)$ is guaranteed by Filippov's Theorem.

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NOTE : For the analogous problem with an unbounded set of control parameters, optimal control may not exist.

Optimal control in the linear time-optimal control problem is "bang-bang" : it is piecewiwe constant and takes values in the vertices of the polytope U.

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The next three examples are special cases of linear time-optimal control problems.

Time-optimal control of linear mechanical systems

Consider the problem of controlling a *linear mechanical system*

$$\ddot{z} + k\dot{z} + \beta z = u(t)$$

by an external force $u(\cdot)$ restricted in magnitude : $|u| \leq \varepsilon$. Here k and β are some non-negative constants (see EXAMPLE 5.1.1). The equivalent first-order system (induced by the state variables $x_1 := z$ and $x_2 := \dot{z}$) is given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\beta x_1 - k x_2 + u$$

or, in vector notation,

$$\dot{x} = Ax + bu, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\beta & -k \end{bmatrix} \quad \text{and} \quad b = e_2.$$

Exercise 22 Show that the foregoing linear control system is *controllable* if and only if $k = 0$.

We shall consider only the cases when $k = 0$.

Case I : $\beta = 0$. In this case, ...

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Case II : $\beta \neq 0$. This case corresponds to the control of a linear harmonic oscillator through an external force $u(\cdot)$.

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The next example represents a class of OCPs very "popular" in applications.

3.5 The Linear-Quadratic Problem

Minimizing the integral of a quadratic form over the trajectories of a linear control system, known as the *linear quadratic problem*, was one of the earliest OCPs (Kalman, 1960).

We consider linear control systems with quadratic cost :

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

$$x(0) = x_0, \quad x(t_1) = x_1 \qquad (x_0, x_1, t_1 > 0 \ \text{ fixed})$$

$$\frac{1}{2}\int_0^{t_1} \langle Pu, u\rangle + \langle Qx, u\rangle + \langle Rx, x\rangle \, dt \ \to \ \min.$$

Here A, B, P, Q, R are matrices of appropriate dimensions, P and R are *symmetric* (i.e. $P^T = P$ and $R^T = R$), and the angle brackets $\langle \cdot, \cdot \rangle$ denote the standard inner product in $\mathbb{R}^n$ and $\mathbb{R}^m$.

NOTE : One can write the "Lagrangian" L (i.e. the integrand of the cost functional) in the (matrix) form

$$L(x(t), u(t)) = \frac{1}{2}\left[u^T(t)Pu(t) + u^T(t)Qx(t) + x^T(t)Rx(t)\right].$$

One can show that the condition $P \geq 0$ (i.e. the matrix P is positive semi-definite) is necessary for the existence of optimal control. We do not discuss here the case of degenerate P and *assume that* $P > 0$ *(i.e. the matrix* P *is positive definite).*

Exercise 23 Verify that the substitution (of variables) $u \mapsto v = P^{1/2}u$ transforms the cost functional $\mathcal{J}(u) = \int_0^{t_1} L(x(t), u(t))\, dt$ into a similar one with the identity matrix I instead of P.

We assume that $P = I$. Another change of variables "kills" the matrix Q.

Exercise 24 Find a change of variables that would transform the cost functional into a similar one with the zero matrix O instead of Q.

Hence we can write the cost functional as follows :

$$\mathcal{J}(u) = \frac{1}{2}\int_0^{t_1} \|u(t)\|^2 + \langle Qx(t), u(t)\rangle\, dt.$$

We further assume that the linear control system $\dot{x} = Ax + Bu$ is controllable :

$$\operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n.$$

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3.6 Optimal Control on Matrix Lie Groups

In approaching variational problems on Lie groups, optimal control theory finds itself on the same ground as Hamiltonian mechanics : facing an already developed theory of Hamiltonian systems that it needs to understand and absorb in order to arrive at the proper solutions of its own problems. This theory of Hamiltonian systems on Lie groups is based on a particular *realization* of the cotangent bundle of a Lie group G as the product of G and the dual of its Lie algebra $\mathfrak{g}$. (For vector spaces V, which are also commutative Lie groups, that realization of the cotangent bundle coincides with the usual representation : $T^*V = V \times V^*$.)

Let G be a (real) Lie group with Lie algebra $\mathfrak{g}$, and let e denote the group identity of G. Recall that $\mathfrak{g}$ is (isomorphic to) the tangent space to G at e : $\mathfrak{g} = T_eG$. Let T^*G denote the *cotangent bundle* of G.

The symplectic structure of T^*G

For each element $g \in G$, let L_g denote the left-translation by g (i.e. $x \mapsto L_g(x) := gx$). The *tangent mapping* $dL_g = (L_g)_*$ maps (the tangent space) T_xG onto $T_{gx}G$ for each x in G. Let dL_g^* denote the *dual mapping* of dL_g. Then (for each $x \in G$)

$$dL_g^* : T_{gx}^*G \to T_x^*G.$$

For each (tangent covector) $p \in T_{gx}^*G$ we have

$$dL_g^*(p) = p \circ dL_g.$$

In particular, at $x = g$, $dL_{g^{-1}}^*$ maps T_e^*G onto T_g^*G.

The correspondence

$$(g,p) \longleftrightarrow dL^*_{g^{-1}}(p)$$

realizes T^*G *as* $G \times \mathfrak{g}^*$.

NOTE : In this representation of T^*G, the Hamiltonians of left-invariant vector fields are linear functionals on $\mathfrak{g}^*$, and the Hamiltonians of right-invariant vector fields are functions that depend on both factors G and $\mathfrak{g}^*$. The explicit expressions are as follows :

The Hamiltonian H_X of a left-invariant vector field X is given by

$$H_X(g,p) := p(X(e))$$

and the Hamiltonian of a right-invariant vector field X is given by

$$H'_X(g,p) := p(dL_{g^{-1}}X(g)) = p(dL_{g^{-1}}dR_g(X(e))).$$

T^*G could also have been realized as $G \times \mathfrak{g}^*$ in terms of the right multiplications $x \mapsto R_g(x) = xg$. Then the correspondence would be given by

$$(g,f) \longleftrightarrow dR^*_{g^{-1}}f$$

and therefore the Hamiltonians of right-invariant vector fields would become linear functionals on $\mathfrak{g}^*$. These representations are equally suitable for applications. (The left-invariant realization is better for the applications that follow.)

The *tangent bundle* of $G \times \mathfrak{g}^*$ is naturally identified with $TG \times T\mathfrak{g}^*$. We shall further identify TG with $G \times \mathfrak{g}$ via the correspondence

$$(g,X) \longleftrightarrow dL_g(X)$$

for each $g \in G$ and $X \in \mathfrak{g}$. Since $T\mathfrak{g}^* = \mathfrak{g}^* \times \mathfrak{g}^*$, we get

$$\begin{aligned} T(T^*G) &= T(G \times \mathfrak{g}^*) = TG \times T\mathfrak{g}^* \\ &= (G \times \mathfrak{g}) \times (\mathfrak{g}^* \times \mathfrak{g}^*). \end{aligned}$$

In this realization, each element $((g, X), (p, Y^*))$ is a tangent vector (X, Y^*) based at (g, p) in T^*G. With these conventions, vector fields on T^*G will be represented by pairs (X, Y^*), with X taking values in $\mathfrak{g}$, and Y^* taking values in $\mathfrak{g}^*$.

NOTE : Having identified T^*G with $G \times \mathfrak{g}^*$, functions on T^*G become functions on $G \times \mathfrak{g}^*$.

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Left-invariant control systems and co-adjoint orbits

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Casimir functions and the conservation laws

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Lie-Poisson reduction and the Maximum Principle

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Chapter 4

Applications

Topics :

1. The Brachistochrone Problem
2. The Elastic Problem
3. Dubins' Problem
4. Other Problems

4.1 The Brachistochrone Problem

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Two very interesting classic problems will be considered next. If the curvature function $\kappa(\cdot)$ (of a curve in the Euclidean plane $\mathbb{E}^2$) is regarded as a control function, many classic variational problems in geometry become optimal control problems (OCPs).

4.2 The Elastic Problem

Consider the following problem : Given points $x_0, x_1 \in \mathbb{E}^2$ and unit tangent vectors $v_0, v_1 \in T_0\mathbb{E}^2 = \mathbb{R}^2$, find a (differentiable) curve $\gamma : [0, T] \to \mathbb{E}^2$ such that :

- γ is parametrized by arc length.
- γ has *curvature* $\kappa(\cdot)$ (almost everywhere).
- γ satisfies the *boundary conditions* :

$$\gamma(0) = x_0, \quad \dot{\gamma} = v_0, \quad \gamma(T) = x_1, \quad \dot{\gamma}(T) = v_1.$$

- γ *minimizes* the (cost) functional

$$\mathcal{J} = \frac{1}{2} \int_0^T \kappa^2(t)\, dt.$$

This (variational) problem, known as the **elastic problem**, goes back to Leonhard Euler (1707-1783), and the solution curves are called the *elastica*.

Note : The *elastic problem* has a rich classical heritage inspired by the following physical situation : a thin elastic rod, when subjected to bending only, assumes the

shape of an elastica in its equilibrium position. In this context, EULER made the initial study of the (planar) elastica in 1744. Much of the development in the theory of the elastic rods is based on a discovery of GUSTAV R. KIRCHHOFF (1824-1887) (known as *the kinetic analogue of the elastic problem*) that the equations for the equilibrium configurations of an elastic rod are the same as the equations for the (Lagrange's) *spinning top*.

The geometric significance of minimizing $\frac{1}{2}\int \kappa^2\, dt$ (or, more generally, *any* functional of κ) was recognized by WILHELM BLASCHKE (1885-1962) under the name of *Radon's problem* (after the name of the mathematician JOHANN RADON (1887-1956)).

Investigations of motion of the rigid body (the kinetic analogue of the elastic problem) in *non-Euclidean spaces* were done by WILLIAM K. CLIFFORD (1845-1879) as early as 1874.

Euler's elastic problem admits a natural formulation (as a OCP) on the matrix Lie group $\mathsf{SE}(2)$ (of proper rigid motions on $\mathbb{E}^2$). Recall that $\mathsf{SE}(2)$ is the semidirect product of $\mathbb{R}^2$ with $\mathsf{SO}(2)$ which can also be regarded as the subgroup of $\mathsf{GL}(3,\mathbb{R})$ consisting of 3×3 matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ x_1 & \alpha & -\beta \\ x_2 & \beta & \alpha \end{bmatrix}$$

with $(x_1, x_2) \in \mathbb{R}^2$ and $\alpha^2 + \beta^2 = 1$. This group can also be viewed as the set of all pairs $(x, \underline{b})$, with x a point in $\mathbb{E}^2$ and $\underline{b}$ a positively-oriented (orthonormal) frame at x. Also recall that the Lie algebra $\mathfrak{se}(2)$ of $\mathsf{SE}(2)$ consists of 3×3 matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & -a_3 \\ a_2 & a_3 & 0 \end{bmatrix}$$

with $(a_1, a_2, a_3) \in \mathbb{R}^3$. Let

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

denote the standard basis (so that any element in the Lie algebra is writen $a_1A_1 + a_2A_2 + a_3A_3$). Corresponding to each element A in the Lie algebra, $\vec{A}$ denotes the left-invariant vector field $\vec{A} : g \mapsto gA$. We shall consider the following (left-invariant) control system on $\mathsf{SE}\,(2)$:

$$\dot{g} = \vec{A}_1(g) + u(t)\vec{A}_3(g)$$

with

$$g(t) = \begin{bmatrix} 1 & 0 \\ x(t) & R(t) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \text{and} \quad R(t) \in \mathsf{SO}\,(2).$$

NOTE : This control system is the classic *Serret-Frenet control system*, associated with a curve $x(\cdot)$ parametrized by its arc length. The state equation of the system can also be written as

$$\dot{x} = R(t)e_1, \qquad \dot{R} = R(t) \begin{bmatrix} 0 & -u(t) \\ u(t) & 0 \end{bmatrix}.$$

Observe that the rotation matrix $R(t)$, when parametrized by the angle θ, yields the following differential system in $\mathbb{E}^3$:

$$\dot{x}_1 = \cos\theta, \quad \dot{x}_2 = \sin\theta, \quad \dot{\theta} = u.$$

It follows that $\ddot{x}(t) = \dot{R}(t)e_1 = u(t)R(t)e_2$ and therefore

$$\|\ddot{x}(t)\| = |u(t)|$$

and the control $u(\cdot)$ is equal to the *geodesic curvature*.

Our OCP is the following :

$$\dot{g} = g\,(A_1 + uA_3)\,, \qquad g \in \mathsf{SE}\,(2),\ u \in \mathbb{R}$$

$$g(0) = (x_0, R(0))\,, \quad g(t_1) = (x_1, R(t_1)) \qquad (x_0, x_1, R(0), R(t_1)\ \text{fixed})$$

$$\frac{1}{2}\int_0^{t_1} u^2(t)\,dt \ \to\ \min.$$

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We shall determine the extremals for Euler's elastic problem. Denote by H_1, H_2, H_3 the Hamiltonians of the left-invariant vector fields $\vec{A}_1, \vec{A}_2, \vec{A}_3$. respectively. (Because $[\vec{A}_1, \vec{A}_2] = 0$, $[\vec{A}_1, \vec{A}_3] = \vec{A}_2$, and $[\vec{A}_2, \vec{A}_3] = -\vec{A}_1$, it follows that the Poisson brackets of H_1, H_2, and H_3 satisfy the same relations.) It follows that the *regular extremals* are the integral curves of the Hamiltonian vector field $\vec{H}$, defined by

$$H := \frac{1}{2}H_3^2 + H_1$$

and that along each extremal curve $\xi(\cdot)$ the corresponding control $u(\cdot)$ is equal to $H_3(\xi(\cdot))$.

For *abnormal extremals*,

$$H_3(\xi(t)) = 0 \quad \text{and} \quad \{H_3, H_1\}\,(\xi(t)) = 0.$$

Because $\{H_3, H_1\} = -H_2$, it follows that $H_2(\xi(t)) = 0$.

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4.3 Dubins' Problem

In 1957 L.E. Dubins considered (and solved) the problem of finding the (parametrized) curves of *minimal length* that would connect two given *config-*

urations (x_0, v_0) and (x_1, v_1) (in the *tangent bundle* of $\mathbb{E}^2$) and would satisfy the additional constraint that $|\kappa(t)| \leq k_0$ (almost everywhere).

NOTE : DUBINS proved that *optimal arcs are concatenations of circular arcs (with constant curvature* k_0 *) and straight line segments.* Moreover, he proved that optimal arcs consists of at most three pieces and that the line segment – if there is any – has to be in the middle. This reduces finding the optimal arcs to a finite problem. There are at most six candidates for optimal arcs. So all one has to do is to determine these arcs and compare their lengths.

One of the well-known interpretations of this problem is to think of a car moving with constant speed in the plane subject to the constraint that it cannot make arbitrarily sharp turns (see also the *unicycle*).

Indeed, consider a car moving in the plane. The car can move forward with a fixed *linear velocity* and simultaneously rotate with a bounded *angular velocity.* Given unitial and terminal positions, and orientation of the car in the plane, *the problem is to drive the car from the initial configuration to the terminal one in minimal time.*

Admissible paths of the car are (geometric) curves with bounded curvature. Suppose that curves are parametrized by arc length; then our problem can be stated geometrically : Given two points in the plane and two unit velocity vectors attached respectively at these points, one has to find a (parametrized) curve in the plane that starts at the first point with the first velocity vector and comes to the second point with second velocity vector, has curvature bounded by a given constant, and has the minimal length among all such curves.

NOTE : If curvature is unbounded, then the problem, in general, has no solution. Indeed, the infimum of lengths of all curves that satisfy the boundary conditions without bound on curvature is the distance between the initial and terminal points :

the segment of the straight line through these points can be approximated by smooth curves with the required boundary conditions. But this infimum is not attained when the boundary velocity vectors do not lie on the line through the boundary points and are not collinear one to another.

After rescaling, we obtain a time-optimal control problem (T-OCP) :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ u \end{bmatrix}, \qquad (x,\theta) = (x_1, x_2, \theta) \in \mathbb{E}^2 \times \mathbb{S}^1, \ |u| \leq 1$$

$$x(0) = x_0, \theta(0) = \theta_0, x(t_1) = x_1, \theta(t_1) = \theta_1 \qquad (x_0, \theta_0, x_1, \theta_1 \ \text{fixed})$$

$$t_1 = \int_0^{t_1} 1\, dt \ \to \ \min.$$

NOTE : The problem of Dubins also admits a natural formulation (as a T-OCP) on the special Euclidean group $\mathsf{SE}(2)$, associated with the Serret-Frenet control system.

Existence of solutions is guaranted by the Filippov's Theorem.

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We apply *Pontryagin's Maximum Priciple* (PMP). We have $(x_1, x_2, \theta) \in M = \mathbb{E}^2 \times \mathbb{S}^1$ and let (ξ_1, ξ_2, μ) be the corresponding coordinates of the adjoint vector. Then

$$\lambda = (x, \theta, \xi, \mu) \in T^*M$$

and the control-dependent Hamiltonian is

$$\mathcal{H}_u(\lambda) = \xi_1 \cos\theta + \xi_2 \sin\theta + \mu u.$$

The Hamiltonian system of PMP yields

$$\begin{aligned} \dot{\xi} &= 0 \\ \dot{\mu} &= \xi_1 \sin\theta - \xi_2 \cos\theta \end{aligned}$$

and the maximality condition reads

$$\mu(t)u(t) = \max_{|u|\leq 1} \mu(t)u.$$

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NOTE : T-OCPs constitute one of the basic concerns of optimal control theory. Minimal-time problems go back to the beginnings of the calculus of variations. JOHAN BERNOULLI's solution of the *brachistochrone problem*in 1697 was based on Fermat's principle of least time, which postulates that "light traverses any medium in the least possible time". Since then such problems have remained important sources of inspiration.

4.4 Other Problems

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An OCP on $\mathsf{SO}(3)$

Let $\mathsf{SO}(3)$ be the rotation group. (Recall that $\mathsf{SO}(3)$ is a compact and connected matrix Lie group, of dimension 3, whose associated Lie algebra $\mathfrak{so}(3)$ consists of all 3×3 skew-symmetric matrices). A *driftless, left-invariant control system* on $\mathsf{SO}(3)$ can be written in the following form :

$$\dot{g} = g\left(u_1 A_1 + u_2 A_2 + u_3 A_3\right), \qquad g \in \mathsf{SO}(3)$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

form the *standard basis* of $\mathfrak{so}(3)$ (see **Exercise 162**). The Lie algebra structure of $\mathfrak{so}(3)$ is given by the following table for the Lie bracket (commutator) :

$[\cdot,\cdot]$	A_1	A_2	A_3
A_1	0	A_3	$-A_2$
A_2	$-A_3$	0	A_1
A_3	A_2	$-A_1$	0

NOTE : The *minus Lie-Poison structure* on $\mathfrak{so}(3)^*$ is given by

$$\Pi = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}.$$

Exercise 25 Show that there are *only* four different driftless, left-invariant control systems on $\mathsf{SO}(3)$, and these are :

(1) $\dot{g} = g\,(u_1 A_1 + u_2 A_2)$.

(2) $\dot{g} = g\,(u_1 A_1 + u_3 A_3)$.

(3) $\dot{g} = g\,(u_2 A_2 + u_3 A_3)$

(4) $\dot{g} = g\,(u_1 A_1 + u_2 A_2 + u_3 A_3)$.

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