Chapter 1

Lie Groups

Topics :

1.1 Lie Groups: Definition and Examples

Lie groups form an important class of smooth (in fact, *analytic*) manifolds. (Their prototype is any finite-dimensional group of linear transformations on a vector space.) The key idea of a Lie group is that it is a group in the usual sense, but with the additional property that it is also a smooth manifold, and in such a way that the group operations are smooth. A good example is the circle $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Lie groups (and their Lie algebras) play a central role in geometry, topology, and analysis, as well as in modern theoretical physics. The precise definition is given below.

1.1.1 DEFINITION. A (real) **Lie group** is a smooth manifold which is also a group such that the operations

$$G \times G \to G, \quad (g_1, g_2) \mapsto g_1 g_2 \qquad \text{and} \qquad G \to G, \quad g \mapsto g^{-1}$$

are smooth mapings.

1.1.2 EXAMPLE. The vector space $\mathbb{R}^m$, when equipped with its natural smooth structure (i.e. viewed as the Euclidean space $\mathbb{E}^m$), is an m-dimensional (Abelian) Lie group.

1.1.3 EXAMPLE. The *general linear group* $\mathsf{GL}(n, \mathbb{R})$ is evidently a Lie group. It is an open subset of (the vector space) $\mathbb{R}^{n \times n}$ (and hence a smooth submanifold of $\mathbb{E}^{n^2}$) and the group operations are given by rational functions of the coordinates.

NOTE : Let V be an n-dimensional vector space (over $\mathbb{R}$). Then the group $\mathsf{GL}(V)$ of all linear transformations on V is an n^2-manifold. Any choice of a basis in V

induces a linear isomorphism from $\mathsf{GL}(V)$ onto $\mathsf{GL}(n,\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ (an hence a global chart on $\mathsf{GL}(V)$). The coordinates of any product (composition) ST of elements in $\mathsf{GL}(V)$ are polynomial expressions of the coordinates of S and T, and the coordinates of S^{-1} are rational functions of the coordinates of S. It therefore follows that both group operations $(S,T) \mapsto ST$ and $S \mapsto S^{-1}$ are smooth (in fact, real analytic) mappings from $\mathsf{GL}(V) \times \mathsf{GL}(V)$ and $\mathsf{GL}(V)$, respectively, onto $\mathsf{GL}(V)$.

1.1.4 Example. The *special linear group* $\mathsf{SL}(n,\mathbb{R})$ and the *orthogonal group* $\mathsf{O}(n)$ are clearly Lie groups. Both subgroups $\mathsf{SL}(n,\mathbb{R})$ and $\mathsf{O}(n)$ are smooth submanifolds of (the Lie group) $\mathsf{GL}(n,\mathbb{R})$, hence smoothness of the group operations on $\mathsf{GL}(n,\mathbb{R})$ implies smoothness of their restrictions to $\mathsf{SL}(n,\mathbb{R})$ and $\mathsf{O}(n)$.

1.1.5 Example. The *complex general linear group* $\mathsf{GL}(n,\mathbb{C}) \subseteq \mathbb{R}^{2n^2}$ is a (real) Lie group. In particular, $\mathbb{C}^\times = \mathsf{GL}(1,\mathbb{C})$ is a Lie group. The unit circle $\mathbb{S}^1 \subseteq \mathbb{C}^\times$ is a subgroup and a (smoothly embedded) submanifold, hence also a Lie group.

1.1.6 Example. If G_1 and G_2 are Lie groups, then $G_1 \times G_2$ is a Lie group under the usual Cartesian group operations and the smooth product structure. In particular, the m-dimensional *torus*

$$\mathbb{T}^m = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$$

is a Lie group.

1.1.7 Example. Let $\mathbb{H}$ denote the division algebra of *quaternions*. The nonzero quaternions $\mathbb{H}^\times$ form a multiplicative group and a (smooth) manifold diffeomorphic to $\mathbb{R}^4 \setminus \{0\}$. It is clear that the group operations are smooth, so $\mathbb{H}^\times$ is a Lie group. The 3-*sphere* $\mathbb{S}^3 \subseteq \mathbb{H}^\times$ consists of the unit length quaternions, hence it is closed under multiplication and passing to inverses. This gives a Lie group structure on $\mathbb{S}^3$.

Usually, the *identity element* of a Lie group will be denoted by e. (For matrix groups, however, the customary symbol for the identity is I.)

NOTE : In most of the literature, Lie groups are defined to be *real analytic.* That is, G is a manifold with a C^ω (real analytic) atlas and the group operations are real analytic. In fact, no generality is lost by this more restrictive definition. *Smooth Lie groups always support an analytic group structure*, and something even stronger is true. HILBERT'S FIFTH PROBLEM was to show that if G is only assumed to be a topological manifold with continuous group operations, then it is, in fact, a real analytic Lie group. This was finally proven by the combined work of A. GLEASON, D. MONTGOMERY, and L. ZIPPIN (195?).

1.2 Invariant Vector Fields

One of the most important features of a Lie group is the existence of an associated Lie algebra that encodes many of the properties of the group. The crucial property of a Lie group that enables this to occur is the existence of the left and right translations on the group.

Let G be a Lie group. For any $g \in G$, the mappings

$$L_g : G \to G, \quad x \mapsto gx \qquad \text{and} \qquad R_g : G \to G, \quad x \mapsto xg$$

are called the **left** and **right translation** (by g), respectively. For each $g \in G$, both L_g and R_g are smooth mappings on G.

Exercise 1 Verify that (for every $g_1, g_2, g, h \in G$)

(a) $L_{g_1} \circ L_{g_2} = L_{g_1 g_2}$.

(b) $R_{g_1} \circ R_{g_2} = R_{g_2 g_1}$.

(c) $L_e = R_e = id_G$ ($e \in G$ denotes the identity element).

(d) $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$. (Hence L_g and R_g are diffeomorphisms.)

(e) $L_g \circ R_h = R_h \circ L_g$.

Note : Given any admissible chart on G, one can construct an entire atlas on the Lie group G by use of left (or right) translations. Suppose, for example, that (U, ϕ) is an admissible chart with $e \in U$. Define a chart (U_g, ϕ_g) with $g \in U_g$ by letting

$$U_g := L_g(U) = \{L_g(x) \mid x \in U\}$$

and defining

$$\phi_g := \phi \circ L_{g^{-1}} : U_g \to \phi(U), \quad x \mapsto \phi(g^{-1}x).$$

The collection of charts $\{(U_g, \phi_g)\}_{g \in G}$ forms a (smooth) atlas provided one can show that the transition mappings

$$\phi_{g_2} \circ \phi_{g_1}^{-1} = \phi \circ L_{g_2^{-1} g_1} \circ \phi^{-1} : \phi_{g_1}(U_{g_1} \cap U_{g_2}) \to \phi_{g_2}(U_{g_1} \cap U_{g_2})$$

is smooth. But this follows from the smoothness of group multiplication and passing to inverse.

By the chain rule,

$$\left(L_{g^{-1}}\right)_{*,gh} \circ (L_g)_{*,h} = \left(L_{g^{-1}} \circ L_g\right)_{*,h} = id_G.$$

Thus the tangent mapping $(L_g)_{*,h}$ is invertible and so, in particular,

$$(L_g)_* = (L_g)_{*,e} : T_e G \to T_g G$$

is a linear isomorphism. Likewise, $(R_g)_{*,h}$ is invertible.

1.2.1 Definition. A vector field X on G is called

- **left-invariant** if for every $g \in G$

$$(L_g)_* X(e) = X(g).$$

- **right-invariant** if for every $g \in G$

$$(R_g)_* X(e) = X(g).$$

It follows that a vector field (on G) that is either left- or right-invariant is determined by its value at the identity.

NOTE : Recall that smooth vector fields act as derivations on the space of smooth functions. (If X is a smooth vector field and f is a smooth function on M, then Xf denotes the (smooth) function $x \mapsto X(x)f$.) For any smooth vector fields X and Y, their *Lie bracket* $[X, Y]$ defined by

$$[X, Y]f = Y(Xf) - X(Yf)$$

is also a smooth vector field. The (vector) space $\mathfrak{X}(M)$ of all smooth vector space on M has the structure of a (real) Lie algebra, with the product given by the Lie bracket.

The set of all left-invariant (respectively, right-invariant) vector fields on a Lie group G is denoted $\mathfrak{X}_L(G)$ (respectively, $\mathfrak{X}_R(G)$). Clearly, both $\mathfrak{X}_L(G)$ and $\mathfrak{X}_R(G)$ are (real) vector spaces (under the pointwise vector addition and scalar multiplication).

NOTE : We defined the push forward $\Phi_{*,p} : T_pM \to T_{\Phi(p)}N$ induced by the (smooth) mapping $\Phi : M \to N$ (the so-called tangent mapping of Φ at $p \in M$). This is a linear mapping between the vector spaces T_pM and $T_{\Phi(p)}N$, and the question arises of whether it is similarly possible to define an induced mapping between the (vector) spaces of smooth vector fields $\mathfrak{X}(M)$ and $\mathfrak{X}(N)$. Given a vector field

$X \in \mathfrak{X}(M)$ and a smooth mapping $\Phi : M \to N$, a natural choice for an induced vector field $\Phi_* X \in \mathfrak{X}(N)$ might appear to be

$$\Phi_* X(\Phi(p)) = \Phi_{*,p}(X(p))$$

but this may fail to be well-defined for two reasons :

- If there are points $p_1, p_2 \in M$ such that $\Phi(p_1) = \Phi(p_2)$ (i.e. the mapping Φ is not one-to-one), then the "definition" above will be ambiguous when $\Phi_* X(p_1) \neq \Phi_* X(p_2)$.
- If Φ is not onto, then the defining equation does not specify the induced vector field outside the range of Φ.

Observe that if Φ is a diffeomorphism from M to N, then neither of these objections apply and an induced vector field $\Phi_* X$ *can* be defined via the above equation. However, it is possible that in certain cases the idea will work, even if Φ is not a diffeomeorphism, and this motivates the following definition : vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be Φ-*related* provided $\Phi_* X(p) = Y(\Phi(p))$ for all $p \in M$. We then write $\Phi_* X = Y$. It is not difficult to see that if $\Phi_* X_1 = Y_1$ and $\Phi_* X_2 = Y_2$, then $[X_1, X_2]$ is Φ-related to $[Y_1, Y_2]$ with

$$\Phi_*[X_1, X_2] = [\Phi_* X_1, \Phi_* X_2].$$

1.2.2 Proposition. *Let X and Y be any left-invariant (respectively, right-invariant) vector fields. Then $[X, Y]$ is a left-invariant (respectively, right-invariant) vector field.*

Proof : Let $X, Y \in \mathfrak{X}_L(G)$ and $g \in G$. Then (and only then) $(L_g)_* X = X$ and $(L_g)_* Y = Y$. Hence

$$(L_g)_*[X, Y] = [(L_g)_* X, (L_g)_* Y] = [X, Y]$$

and so $[X, Y] \in \mathfrak{X}_L(G)$. The case of right-invariant vector fields is similar. $\square$

Therefore, both $\mathfrak{X}_L(G)$ and $\mathfrak{X}_R(G)$ are Lie subalgebras of the (infinite dimensional) Lie algebra $\mathfrak{X}(G)$ of all smooth vector fields on G.

For each $A \in T_eG$, we define a (smooth) vector field X_A on G by letting

$$X_A(g) := (L_g)_{*,e}A.$$

Then

$$\begin{aligned}(L_g)_*X_A(e) &= (L_g)_*\left((L_e)_*A\right)\\ &= (L_g)_* \circ (L_e)_*A\\ &= (L_{ge})_{*,e}A\\ &= (L_g)_{*,e}A\\ &= X_A(g)\end{aligned}$$

which shows that X_A is left-invariant. Consider the mappings

$$\zeta_1 : \mathfrak{X}_L(G) \to T_eG, \qquad X \mapsto X(e)$$

and

$$\zeta_2 : T_eG \to \mathfrak{X}_L(G), \qquad A \mapsto X_A.$$

Exercise 2 Verify that ζ_1 and ζ_2 are linear mappings that satisfy

$$\zeta_1 \circ \zeta_2 = id_{T_e(G)} \qquad \text{and} \qquad \zeta_2 \circ \zeta_1 = id_{\mathfrak{X}_L(G)}.$$

(It is clear that ζ_2 is the inverse of ζ_1, and hence for a left-invariant vector field X

$$(L_g)_*X(e) = X(g) \qquad \text{and} \qquad (L_{g^{-1}})_*X_A(g) = A.)$$

Therefore, $\mathfrak{X}_L(G)$ *and* T_eG *are isomorphic (as vector spaces)*. It follows that the dimension of the vector space $\mathfrak{X}_L(G)$ is equal to $\dim T_eG = \dim G$.

Note : Since, by assumption, G is a (finite-dimensional) manifold it follows that $\mathfrak{X}_L(G)$ is a finite-dimensional, nontrivial subalgebra of the Lie algebra of all (smoth) vector fields on G.

For any $A, B \in T_eG$, we define their *Lie product* (bracket) $[A, B]$ by

$$[A, B] := [X_A, X_B](e)$$

where $[X_A, X_B]$ is the Lie bracket of vector fields. This makes T_eG into a Lie algebra. We say that this defines a Lie product in T_eG via *left extension.*

Note : By construction,

$$[X_A, X_B] = X_{[A,B]}$$

for all $A, B \in T_eG$.

1.2.3 Definition. The vector space T_eG with this Lie algebra structure is called the **Lie algebra** of G and is denoted by $\mathfrak{g}$.

Exercise 3 Let $\varphi : G \to H$ be a smooth homomorphism between the Lie groups G and H. Show that the induced mapping

$$d\varphi = \varphi_{*,e} : T_eG = \mathfrak{g} \to T_eH = \mathfrak{h}$$

is a homomorphism between the Lie algebras of the groups.

A similar construction to the above can be carried out with the Lie algebra $\mathfrak{X}_R(G)$ of right-invariant vector fields on G. In this case, for each $A \in T_eG$, the corresponding right-invariant vector field is defined by

$$Y_A(g) := (R_g)_{*,e}A.$$

We have (for $A, B \in T_eG$)

$$[Y_A, Y_B](e) = -[X_A, X_B](e).$$

Therefore, the Lie product $[\cdot,\cdot]^R$ in $\mathfrak{g}$ defined by *right extension* of elements of $\mathfrak{g}$:

$$[A,B]^R := [Y_A, Y_B](e)$$

is the *negative* of the one defined by left extension; that is,

$$[A,B]^R = -[A,B].$$

NOTE : There is a natural isomorphism between the (Lie algebras) $\mathfrak{X}_L(G)$ and $\mathfrak{X}_R(G)$. It is equal to the tangent mapping of $\Phi : G \to G, \quad x \mapsto x^{-1}$. In particular, we have (for $A \in \mathfrak{g} = T_e G$)

$$\Phi_* X_A = -Y_A.$$

Orbits of invariant vector fields

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1.3 The Exponential Mapping

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1.4 Matrix Groups as Lie Groups

We have seen that the matrix groups $\mathsf{GL}(n,\Bbbk)$, $\mathsf{SL}(n,\Bbbk)$, and $\mathsf{O}(n)$ are all Lie groups. These examples are typical of what happens for any matrix group that is a Lie subgroup of $\mathsf{GL}(n,\mathbb{R})$. The following important result holds.

1.4.1 THEOREM. *Let $G \leq \mathsf{GL}(n,\mathbb{R})$ be a matrix group. Then G is a Lie subgroup of $\mathsf{GL}(n,\mathbb{R})$.*

NOTE : In fact, a more general result also holds (but we will not give a proof) : *Every closed subgroup of a Lie group is a Lie subgroup.*

Our aim in this section is to prove THEOREM 4.5.1.

Let $G \leq \mathsf{GL}\,(n,\mathbb{R})$ be a matrix group, and let $\mathfrak{g} = T_I G$ denote its Lie algebra.

1.4.2 PROPOSITION. *Let*

$$\widetilde{\mathfrak{g}} := \{A \in \mathbb{R}^{n\times n} \mid \exp(tA) \in G \ \textit{ for all } \ t\}.$$

Then $\widetilde{\mathfrak{g}}$ *is a Lie subalgebra of* $\mathbb{R}^{n\times n}$.

PROOF : By definition, $\widetilde{\mathfrak{g}}$ is closed under (real) scalar multiplication. If $U, V \in \widetilde{\mathfrak{g}}$ and $r \geq 1$, then the following are in G :

$$\exp\left(\frac{1}{r}U\right)\exp\left(\frac{1}{r}V\right), \quad \left(\exp\left(\frac{1}{r}U\right)\exp\left(\frac{1}{r}V\right)\right)^r,$$
$$\exp\left(\frac{1}{r}U\right)\exp\left(\frac{1}{r}V\right)\exp\left(-\frac{1}{r}U\right)\exp\left(-\frac{1}{r}V\right),$$
$$\left(\exp\left(\frac{1}{r}U\right)\exp\left(\frac{1}{r}V\right)\exp\left(-\frac{1}{r}U\right)\exp\left(-\frac{1}{r}V\right)\right)^{r^2}.$$

For $t \in \mathbb{R}$, by the LIE-TROTTER PRODUCT FORMULA we have

$$\exp(tU + tV) = \lim_{r\to\infty}\left(\exp\left(\frac{1}{r}tU\right)\exp\left(\frac{1}{r}tV\right)\right)^r$$

and by the COMMUTATOR FORMULA

$$\begin{aligned}\exp(t[U,V]) &= \exp([tU,V]) \\ &= \lim_{r\to\infty}\left(\exp\left(\frac{1}{r}tU\right)\exp\left(\frac{1}{r}V\right)\exp\left(-\frac{1}{r}tU\right)\exp\left(-\frac{1}{r}V\right)\right)^{r^2}.\end{aligned}$$

As these are both limits of elements of the closed subgroup $G \leq \mathsf{GL}\,(n,\mathbb{R})$, they are also in G. This shows that $\widetilde{\mathfrak{g}}$ is a Lie subalgebra of $\mathfrak{gl}\,(n,\mathbb{R}) = \mathbb{R}^{n\times n}$.

$\square$

1.4.3 Corollary. *$\widetilde{\mathfrak{g}}$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof : Let $U \in \widetilde{\mathfrak{g}}$. Then the curve

$$\gamma : \mathbb{R} \to G, \quad t \mapsto \exp(tU)$$

has $\gamma(0) = I$ and $\dot{\gamma}(0) = U$, hence $U \in \mathfrak{g}$. □

Note : Eventually we will see that $\widetilde{\mathfrak{g}} = \mathfrak{g}$.

We will require a technical result.

1.4.4 Lemma. *Let $(A_r)_{r\geq 1}$ and $(\lambda_r)_{r\geq 1}$ be sequences in $\exp^{-1}(G)$ and $\mathbb{R}$, respectively. If $\|A_r\| \to 0$ and $\lambda_r A_r \to A \in \mathbb{R}^{n\times n}$ as $r \to \infty$, then $A \in \widetilde{\mathfrak{g}}$.*

Proof : Let $t \in \mathbb{R}$. For each r, choose an integer $m_r \in \mathbb{Z}$ so that $|t\lambda_r - m_r| \leq 1$. Then

$$\begin{aligned} \|m_r A_r - tA\| &\leq \|(m_r - t\lambda_r)A_r\| + \|t\lambda_r A_r - tA\| \\ &= |m_r - t\lambda_r|\|A_r\| + \|t\lambda_r A_r - tA\| \\ &\leq \|A_r\| + |t|\|\lambda_r A_r - A\| \to 0 \end{aligned}$$

as $r \to \infty$, showing that $m_r A_r \to tA$. Since $\exp(m_r A_r) = \exp(A_r)^{m_r} \in G$ and G is closed in $\mathsf{GL}\,(n, \mathbb{R})$, we have

$$\exp(tA) = \lim_{r\to\infty} \exp(m_r A_r) \in G.$$

Thus every scalar multiple tA is in $\exp^{-1}(G)$, showing that $A \in \widetilde{\mathfrak{g}}$. □

Proof of Theorem 4.5.1 : Choose a complementary $\mathbb{R}$-subspace $\mathfrak{w}$ to $\widetilde{\mathfrak{g}}$ in $\mathbb{R}^{n\times n}$; that is, any vector subspace such that

$$\begin{aligned} \widetilde{\mathfrak{g}} + \mathfrak{w} &= \mathbb{R}^{n\times n} \\ \dim \widetilde{\mathfrak{g}} + \dim \mathfrak{w} &= \dim \mathbb{R}^{n\times n} = n^2. \end{aligned}$$

(The second of these conditions is equivalent to $\widetilde{\mathfrak{g}} \cap \mathfrak{w} = 0$.) This gives a a *direct sum decomposition* of $\mathbb{R}^{n\times n}$, so every element $X \in \mathbb{R}^{n\times n}$ has a unique decomposition of the form

$$X = U + V \qquad (U \in \widetilde{\mathfrak{g}},\, V \in \mathfrak{w}).$$

Consider the mapping

$$\Phi : \mathbb{R}^{n\times n} \to \mathsf{GL}\,(n,\mathbb{R}), \quad U + V \mapsto \exp(U)\exp(V).$$

Φ is a smooth mapping which maps O to I. Observe that the factor $\exp(U)$ is in G. Consider the derivative (at O)

$$D\Phi(O) : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}.$$

To determine $D\Phi(O)\cdot(A+B)$, where $A \in \widetilde{\mathfrak{g}}$ and $B \in \mathfrak{h}$, we differentiate the curve $t \mapsto \Phi(t(A+B))$ at $t = 0$. Assuming that A and B small enough, for small $t \in \mathbb{R}$, there is a unique matrix $C(t)$ (depending on t) for which

$$\Phi(t(A+B)) = \exp(C(t)).$$

Then (by using the estimate in Proposition 3.5.6)

$$\|C(t) - tA - tB - \frac{t^2}{2}[A,B]\| \leq 65|t|^3\left(\|A\| + \|B\|\right)^3.$$

From this we obtain

$$\begin{aligned}\|C(t) - tA - tB\| &\leq \frac{t^2}{2}\|[A,B]\| + 65|t|^3\left(\|A\| + \|B\|\right)^3\\ &= \frac{t^2}{2}\left(\|[A,B]\| + 130|t|\left(\|A\| + \|B\|\right)^3\right)\end{aligned}$$

and so

$$\begin{aligned}D\Phi(O)\cdot(A+B) &= \left.\frac{d}{dt}\Phi(t(A+B))\right|_{t=0}\\ &= \left.\frac{d}{dt}\exp(C(t))\right|_{t=0}\\ &= A + B.\end{aligned}$$

Hence $D\Phi(O)$ is the identity mapping on $\mathbb{R}^{n\times n}$, and by the Inverse Mapping Theorem, there exists an open neighborhood (and we may take this to be an open ball) $\mathcal{B}_{\mathbb{R}^{n\times n}}(O,\delta)$ of O such that the restriction

$$\Phi_1 := \Phi|_{\mathcal{B}(O,\delta)} : \mathcal{B}(O,\delta) \to \Phi\left(\mathcal{B}(O,\delta)\right)$$

is a smooth diffeomeorphism.

Now we must show that Φ maps some open subset (which we may assume to be an open ball) of $\mathcal{B}_{\mathbb{R}^{n\times n}}(O,\delta)\cap\widetilde{\mathfrak{g}}$ onto an open neighborhood of I in G. Suppose not. Then there is a sequence of elements $(U_r)_{r\geq 1}$ in G with $U_r \to I$ as $r\to\infty$ but $U_r \notin \Phi(\widetilde{\mathfrak{g}})$. For large enough r, $U_r \in \Phi(\mathcal{B}(O,\delta))$, hence there are unique elements $A_r\in\widetilde{\mathfrak{g}}$ and $B_r \in \mathfrak{w}$ with $\Phi(A_r+B_r)=U_r$. Notice that $B_r\neq O$ since otherwise $U_r\in\Phi(\widetilde{\mathfrak{g}})$. As Φ_1 is a diffeomorphism, $A_r+B_r\to O$ and this implies that $A_r\to O$ and $B_r\to O$. By definition of Φ,

$$\exp(B_r) = \exp(A_r)^{-1}U_r \in G.$$

Hence $B_r\in\exp^{-1}(G)$. Consider the elements $\bar{B}_r = \frac{1}{\|B_r\|}B_r$ of unit norm. Each $\bar{B}_r$ is in the unit sphere in $\mathbb{R}^{n\times n}$, which is compact hence there is a convergent subsequence of $(\bar{B}_r)_{r\geq 1}$. By renumbering this subsequence, we can assume that $\bar{B}_r\to B$, where $\|B\|=1$. Applying Lemma 4.5.4 to the sequences $(B_r)_{r\geq 1}$ and $\left(\frac{1}{\|B_r\|}\right)_{r\geq 1}$, we find that $B\in\widetilde{g}$. But each B_r (and hence $\bar{B}_r$) is in $\mathfrak{w}$, so B must be too. Thus $B\in\widetilde{g}\cap\mathfrak{w}$, which contradicts the fact that $B\neq O$.

So there must be an open ball

$$\mathcal{B}_{\widetilde{\mathfrak{g}}}(O,\delta_1) = \mathcal{B}_{\mathbb{R}^{n\times n}}(O,\delta_1)\cap\widetilde{\mathfrak{g}}$$

which is mapped by Φ onto an open neighborhood of I in G. So the restriction of Φ to this open ball is a local diffeomorphism at O. The inverse mapping gives a local chart for G at I (and moreover $\mathcal{B}_{\widetilde{\mathfrak{g}}}(O, \delta_1)$ is then a smooth submanifold of $\mathbb{R}^{n\times n}$). We can use left translation to move this local chart to a new chart at any other point $U \in G$ (by considering $L_U \circ \Phi$).

So we have shown that $G \leq \mathsf{GL}\,(n, \mathbb{R})$ is a smooth submanifold. The matrix product $(A, B) \mapsto AB$ is clearly a smooth (in fact, analytic) function of the entries of A and B, and (in light of Cramer's rule) $A \mapsto A^{-1}$ is a smooth (in fact, analytic) function of the entries of A. Hence G is a Lie subgroup, proving Theorem 4.5.1.

$\square$

This is a fundamental result that can be usefully reformulated as follows : *A subgroup of* $\mathsf{GL}\,(n, \mathbb{R})$ *is a closed Lie subgroup if and only if it is a matrix subgroup.* (More generally, a subgroup of a Lie group G is a closed Lie subgroup if and only if is a closed subgroup.)

Note : Recall that the dimension of a matrix group G (as a manifold) is $\dim \widetilde{\mathfrak{g}}$. By Corollary 4.5.3, $\widetilde{\mathfrak{g}} \subseteq \mathfrak{g}$ and so $\dim \widetilde{\mathfrak{g}} \leq \dim \mathfrak{g}$. By definition of $\mathfrak{g} = T_I G$, these dimensions are in fact equal, giving

$$\widetilde{\mathfrak{g}} = \mathfrak{g}.$$

Combining with Proposition 3.3.3, this gives the following result : *For a matrix group* $G \leq \mathsf{GL}\,(n, \mathbb{R})$*, the exponential mapping*

$$\exp : \mathfrak{g} \to \mathbb{R}^{n\times n}$$

has image in G*. Moreover,* $\exp_G$ *is a local diffeomorphism at the origin (mapping some open neighborhood of* 0 *onto an open neighborhood of* I *in* G*).*

It is a remarkable fact that most of the important examples of Lie groups are (or can easily be represented as) matrix groups. However, *not all Lie groups are matrix groups.* For the sake of completeness, we shall describe the simplest example of a Lie group which is *not* a matrix group.

Consider the matrix group (of *unipotent* 3×3 matrices)

$$\mathsf{H}\,(1) = \left\{ \gamma(x,y,t) = \begin{bmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, t \in \mathbb{R} \right\} \leq \mathsf{GL}\,(3, \mathbb{R})$$

commonly referred to as the *Heisenberg group*. $\mathsf{H}\,(1)$ is a 3-dimensional Lie group.

NOTE : More generally, the *Heisenberg group* $\mathsf{H}\,(n)$ is defined by

$$\mathsf{H}\,(n) = \left\{ \gamma(x,y,t) = \begin{bmatrix} 1 & x^T & t \\ 0 & I_n & y \\ 0 & 0 & 1 \end{bmatrix} \mid (x,y) \in \mathbb{R}^{2n}, t \in \mathbb{R} \right\} \leq \mathsf{GL}\,(n+2, \mathbb{R}).$$

This (matrix) group is *isomorphic* to either one of the following groups :

- $\mathbb{R}^{2n+1}$ equipped with the group multiplication

$$(x,y,t) * (x',y',t') = (x+x', y+y', t+t'+x \bullet y').$$

- $\mathbb{R}^{2n+1}$ equipped with the group multiplication

$$(x,y,t)(x',y',t') = \left(x+x', y+y', t+t' + \frac{1}{2}(\Omega((x,y),(x',y'))) \right)$$

  where $\Omega((x,y),(x',y')) = x \bullet y' - x' \bullet y$ is the standard *symplectic form* on $\mathbb{R}^{2n}$.

The Lie algebra $\mathfrak{h}\,(n)$ of $\mathsf{H}\,(n)$ is given by

$$\mathfrak{h}\,(n) = \left\{ \Gamma(x,y,t) = \begin{bmatrix} 0 & x^T & t \\ 0 & O_n & y \\ 0 & 0 & 0 \end{bmatrix} \mid (x,y) \in \mathbb{R}^{2n}, t \in \mathbb{R} \right\}.$$

(The Lie algebra $\mathfrak{h}\,(1)$, which occurs throughout quantum physics, is essentially the same as the Lie algebra of operators on differentiable functions $f : \mathbb{R} \to \mathbb{R}$ spanned by the three operators $\mathbf{1}, \mathbf{p}, \mathbf{q}$ defined by

$$\mathbf{1}f(x) := f(x), \quad \mathbf{p}f(x) := \frac{d}{dx}f(x), \quad \mathbf{q}f(x) := xf(x).$$

The non-trivial commutator involving these three operators is given by the *canonical commutation relation* $[\mathbf{p}, \mathbf{q}] = \mathbf{pq} - \mathbf{qp} = \mathbf{1}$.)

Exercise 4 Determine the (group) commutator in $\mathsf{H}\,(1)$ (i.e. the product $\gamma\gamma'\gamma^{-1}\gamma'^{-1}$ for $\gamma, \gamma' \in \mathsf{H}\,(1)$) and hence deduce that the *centre* $Z(\mathsf{H}\,(1))$ of $\mathsf{H}\,(1)$ is

$$Z(\mathsf{H}\,(1)) = \{\gamma(0,0,t) \,|\, t \in \mathbb{R}\}\,.$$

Clearly, there is an isomorphism (of Lie groups) between $\mathbb{R}$ and $Z(\mathsf{H}(1))$, under which the subgroup $\mathbb{Z}$ of integers corresponds to the subgroup $\mathcal{Z}$ of $Z(\mathsf{H}(1))$. Thus

$$\mathcal{Z} = \{\gamma(0,0,t) \,|\, t \in \mathbb{Z}\}\,.$$

The subgroup $\mathcal{Z}$ is *discrete* and also *normal*.

NOTE : (1) By a *discrete* group Γ is meant a group with a countable number of elements and the discrete topology (every point is an open set). A discrete group is a 0-dimensional Lie group. Closed 0-dimensional Lie subgroups of a Lie group are usually called *discrete subgroups*. The following remarkable result holds : *If* Γ *is a discrete subgroup of a Lie group* G*, then the space of right (or left) cosets* G/Γ *is a smooth manifold* (and the natural projection $G \to G/\Gamma$ is a smooth mapping).

(2) A subgroup N of G is *normal* if for any $n \in N$ and $g \in G$ we have $gng^{-1} \in N$. A kernel of a homomorphism is normal. Conversely, if N is normal, we can define the *quotient group* G/N whose elements are equivalence classes $[g]$ of elements in G, and two elements g, h are equivalent if and only if $g = hn$ for some $n \in N$. The multiplication is given by $[g][h] = [gh]$ and the fact that N is normal says that this is well-defined. Thus *normal subgroups are exactly kernels of homomorphisms.*

Hence we can form the *quotient group*

$$\mathsf{H}\,(1)/\mathcal{Z}$$

which is in fact a (3-dimensional) Lie group. (Its Lie algebra is $\mathfrak{h}\,(1)$.)

The following result (which we will not prove) tells that *the Lie group* $\mathsf{H}\,(1)/\mathcal{Z}$ *cannot be realized as a matrix group.*

1.4.5 PROPOSITION. *There are no continuous homomorphisms* $\varphi : \mathsf{H}\,(1)/\mathcal{Z} \to \mathsf{GL}\,(n, \mathbb{C})$ *with trivial kernel.*

1.5 Hamiltonian Vector Fields

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1.6 Lie-Poisson Reduction

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Problems and Further Results

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