## Chapter 2

## Control Systems

## Topics :

1. Control Systems: Definition and Examples
2. Invariant Systems on Matrix Lie Groups
3. Examples
4. Controllability
5. Linear Control Systems
6. Serret-Frenet Control Systems

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### 2.1 Control Systems: Definition and Examples

There are many important problems (notably in engineering and physical sciences), envolving the study of various control systems, which cannot be treated satisfactory by "classical" (i.e. linear) control theory. This is the case essentially because the state space (of the control system under consideration) is not a vector space, but is, in a natural way, a much more sofisticated "nonlinear" space, namely a manifold. Linearization often destroys the essence of the problem and new and different methods are needed (especially for treating global questions). It appears that differential-geometric methods, introduced in the 1970s, provide a very useful language and, at the same time, a powerful machinery for tackling most of these problems.

In what follows we shall restrict ourselves to the special, but very interesting case, when the state space is a matrix Lie group.

Matrix Lie groups arise naturally as the models for the configuration space of mechanical systems. For instance, the position and orientation of a rigid body in Euclidean 3-space can be completely characterized by the special Euclidean group SE (3). Control systems on matrix Lie groups thus find application in modeling and motion control of mechanical systems such as robotic manipulators, wheeled robots, underwater vehicles, and spacecraft.

Next to mechanical applications, matrix Lie groups also arise from physical conservation principles such as conservation of energy. For instance, electrical networks used for power conversion can be modeled as control systems evolving on the special orthogonal group $\mathrm{SO}(3)$, and so-called multilevel systems used to model molecular bonds in the context of coherent control of quantum dynamics can naturally be represented as control systems on the unitary group $\mathrm{U}(n)$.

Furthermore, matrix Lie groups arise in the study of the state transition matrix of a time-varying linear control system (on some Euclidean space).

From a theoretical point of view, control systems on matrix Lie grups are also an interesting subject of study since they form an important sub-class of nonlinear control systems. Their structure leads to simplifications which allows us to study the essence of various nonlinear control questions of more general formulations.

Control systems on matrix Lie groups were first introduced in 1972 by Roger Brockett who expressed notions such as (nonlinear) controllability, observability, and realization theory for (right-invariant) control systems evolving on matrix Lie groups. Velimir Jurdjevic and Héctor Sussmann further investigated the controllability properties of control systems on $a b-$ stract Lie groups. One of the most important insights derived from this work was the recognition that questions about these kind of control systems on Lie groups can be reduced to questions about their associated Lie algebras. Since Lie algebras are (finite-dimensional) vector spaces, whereas Lie groups are manifolds, this reduction greatly simplifies the problem.

Constructive questions for control systems on matrix Lie groups such as deriving optimal controls for certain lower-dimensional control systems on matrix Lie groups were taken up by P.S. Krishnaprasad and Naomi Leonard in the early 1990s.

The study of (invariant) control systems on abstract Lie groups has been a subject of active research in mathematical control theory in the last three decades or so. The study is motivated both by important applications (in engineering and physical sciences) and by essential links with various branches of mathematics outside control theory (e.g. Lie groups and Lie algebras, dif-
ferential geometry, Lie semigroups, dynamical systems).

## Control Systems

Roughly speaking, a control system (on a smooth manifold) is any system of ordinary differential equations in which control functions appear as parameters.

Note : A control system can be viewed as a (deterministic, smooth, finite dimensional) dynamical system whose dynamical laws are not entirely fixed but depend on parameters, called controls, that can vary and with which one can control the behaviour of the system.

From a geometric viewpoint, each control determines a vector field, and therefore a control system can be viewed as a family $\mathcal{F}=\left(F_{u}\right)_{u \in U}$ of vector fields. A trajectory of such a system is a (continuous) curve made up of finitely many segments of integral curves of vector fields in the family.

Note : More generally, let $\mathcal{F}$ be an arbitrary family of vector fields (on the smooth manifold $M$ ). For the sake of simplicity we shall assume that all the elements of $\mathcal{F}$ are complete vector fields. Then each element $X \in \mathcal{F}$ generates a one-parameter group of diffeomorphisms of $M(\exp t X)_{t \in \mathbb{R}}$. Let $G(\mathcal{F})$ denote the group of diffeomorphisms generated by $\bigcup_{X \in \mathcal{F}}(\exp t X)_{t \in \mathbb{R}}$. (The elements of $G(\mathcal{F})$ are precisely the diffeomorphisms $\Phi$ of $M$ of the form

$$
\Phi=\left(\exp t_{k} X_{k}\right) \circ\left(\exp t_{k-1} X_{k-1}\right) \circ \cdots \circ\left(\exp t_{1} X_{1}\right)
$$

for some $t_{1}, \ldots, t_{k} \in \mathbb{R}$ and $X_{1}, \ldots, X_{k} \in \mathcal{F}$.) $G(\mathcal{F})$ acts on $M$ in the obvious way and partitions $M$ into its orbits:

$$
M=\bigcup_{p \in M} \mathcal{O}(p) .
$$

(The $G(\mathcal{F})$-orbit through the point $p \in M$ is $\mathcal{O}(p)=\{\Phi(p) \mid \Phi \in G(\mathcal{F})\}$.) The $G(\mathcal{F})$-orbits are referred to as the orbits of $\mathcal{F}$ and their structure is described in the following fundamental result :
(Orbit Theorem) Every orbit $\mathcal{O}(p)$ of $\mathcal{F}$ is a connected, immersed submanifold of M. Moreover, the tangent space to $\mathcal{O}(p)$ at $q \in \mathcal{O}(p)$ is

$$
T_{q} \mathcal{O}(p)=\operatorname{span}\left\{\Phi_{*} X(q) \mid \Phi \in G(\mathcal{F}), X \in \mathcal{F}\right\}
$$

This result has a remarkable significance in geometric control theory.
Let Lie $(\mathcal{F})$ denote the Lie algebra of vector fields generated by the family $\mathcal{F}$. (Lie $(\mathcal{F})$ can be described as

$$
\operatorname{Lie}(\mathcal{F})=\operatorname{span}\left\{\operatorname{ad} X_{1} \circ \operatorname{ad} X_{2} \circ \cdots \circ \operatorname{ad} X_{k-1}\left(X_{k}\right) \mid X_{1}, \ldots, X_{k} \in \mathcal{F}\right\}
$$

where $\operatorname{ad} X: \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ is the mapping $Y \mapsto \operatorname{ad} X(Y):=[X, Y]$.) For each $q \in M$, the evaluation of $\operatorname{Lie}_{q}(\mathcal{F})$ at $q$ is (the vector space)

$$
\operatorname{Lie}_{q}(\mathcal{F})=\{X(q) \mid X \in \mathcal{F}\} \subseteq T_{q} M
$$

The following relation holds for every $q \in M$ :

$$
\operatorname{Lie}_{q}(\mathcal{F}) \subseteq T_{q} \mathcal{O}(p)
$$

In many important cases (for instance, when $M$ is a Lie group), this inclusion turns out to be an equality.

We make the following definition.
2.1.1 Definition. A control system is (given by) a mapping

$$
F: M \times U \rightarrow T M, \quad(x, u) \mapsto F_{u}(x)
$$

where

- $M$ is a smooth $m$-dimensional manifold, called the state space;
- $U$ is an arbitrary subset of (the Cartesian $\ell$-space) $\mathbb{R}^{\ell}$, called the control set;
- $T M$ is the tangent bundle of $M$ (a smooth $2 m$-dimensional manifold).

It is assumed that

- the mapping $F$ is continuous;
- for each $u \in U$, the mapping

$$
F_{u}=F(\cdot, u): M \rightarrow T M
$$

is smooth. ( $F_{u}$ is a smooth vector field on M.)

Such a control system is usually written (in classical notation) as follows

$$
\dot{x}=F(x, u), \quad x \in M, u \in U \subseteq \mathbb{R}^{\ell} .
$$

The variable $x$ is the state and represents the "memory" of the system. The variable $u$ is the control (or the input) and represents the external influence on the system. We define a control to be a $U$-valued mapping defined on some (compact) interval:

$$
u(\cdot):[a, b] \rightarrow U, \quad t \mapsto u(t)=\left(u_{1}(t), \ldots, u_{\ell}(t)\right) \in U .
$$

Generally, a control must satisfy certain regularity conditions, in which case it is referrred to as an admissible control. For all geometric considerations it is sufficient to consider only piecewise constant controls.

Note: (1) The control functions, when regarded as an $\ell$-tuple $u=\left(u_{1}, \ldots, u_{\ell}\right)$, are constrained to take value in a fixed subset $U$ of $\mathbb{R}^{\ell}$, called the control set. Generally, $U$ is assumed to be a closed subset of $\mathbb{R}^{\ell}$ (sometimes a compact or even compact
convex subset) with nonempty interior. Whenever $U=\mathbb{R}^{\ell}$, we may refer to the control system as an unrestricted control system.
(2) Although convenient for geometric considerations, piecewise constant controls are not particularly suitable for problems of optimal control. For such problems, the class of admissible controls $\mathcal{U}$ needs to be enlarged to accomodate more general controls (like piecewise continuous ones).
(3) Formally, a (nonlinear) control system is a 4-tuple

$$
\Sigma=(M, U, \mathcal{U}, F)
$$

where the manifold $M$ is the state space, $U \subseteq \mathbb{R}^{\ell}$ is the control set, $\mathcal{U}$ is the class of admissible controls, and the mapping $F$ is the dynamics. It is the dynamics, or the associated family of vector fields $\mathcal{F}=\left(F_{u}\right)_{u \in U}$, which provides a local in time description (i.e. the state equation) of $\Sigma$ :

$$
\dot{x}=F_{u}(x), \quad x \in M, u=\left(u_{1}, \ldots, u_{\ell}\right) \in U
$$

The case when $F_{u}$ is of the form

$$
F_{u}=X_{0}+u_{1} X_{1}+\cdots+u_{\ell} X_{\ell}, \quad u=\left(u_{1}, \ldots, u_{\ell}\right) \in U \subseteq \mathbb{R}^{\ell}
$$

(i.e. each vector field $F_{u}$ is an affine combination of some fixed vector fields $\left.X_{0}, X_{1}, \ldots, X_{\ell}\right)$ is of particular importance for applications. Such a controlaffine system is usually written as follows

$$
\dot{x}=X_{0}(x)+u_{1} X_{1}(x)+\cdots+u_{\ell} X_{\ell}(x)
$$

with piecewise constant control functions $u_{1}(\cdot), u_{2}(\cdot), \ldots, u_{\ell}(\cdot)$. The vector field $X_{0}$ is called the drift, and the remaining vector fields $X_{1}, \ldots, X_{\ell}$ are called the controlled vector fields. If $X_{0}=0$ and $0 \in \operatorname{int} U$, then we say that the system is driftless (or homogeneous).

The class of control-affine systems serves as a kinematic model for a wide range of problems relevant to mechanics, geometry, and control.
2.1.2 Example. (The Liénard control system) A general nonlinear oscillator with an external force $u(\cdot)$ is described by the (second-order differential) equation

$$
\ddot{z}+a(z) \dot{z}+b(z)=u(t)
$$

(known as the Liénard equation). This equation can be expressed as an equivalent first-order system (of diferential equations) in the phase plane by introducing the new variables $x_{1}:=z$ and $x_{2}:=\dot{z}$. Then

$$
\dot{x}_{1}=x_{2}, \quad \dot{x}_{2}=-a\left(x_{1}\right) x_{2}-b\left(x_{1}\right)+u
$$

If we set

$$
X:=\left[\begin{array}{c}
x_{2} \\
-a\left(x_{1}\right) x_{2}-b\left(x_{1}\right)
\end{array}\right] \quad \text { and } \quad Y:=\left[\begin{array}{l}
0 \\
1
\end{array}\right]
$$

then we get (the state equation describing) a control-affine system (on $\mathbb{E}^{2}$ )

$$
\dot{x}=X(x)+u Y(x), \quad x \in \mathbb{E}^{2}, u \in U \subseteq \mathbb{R}
$$

The external force $u(\cdot)$ plays the role of a (scalar) control.

### 2.1.3 EXAMPLE. (MEChanical System with damping controls) Consider

 the problem of controlling a mechanical system$$
\ddot{z}+u \dot{z}+f(z)=0
$$

by a damping control function $u(\cdot)$. The equivalent first-order system is given by

$$
\dot{x}_{1}=x_{2}, \quad \dot{x}_{2}=-f\left(x_{1}\right)-u x_{2} .
$$

For the sake of simplicity, assume that $f(z)=k z$ for same constant $k$. Then the forgoing system can be rewritten as

$$
\begin{aligned}
\dot{x} & =A x+u B x \\
& =(A+u B) x, \quad x \in \mathbb{E}^{2}, u \in U \subseteq \mathbb{R}
\end{aligned}
$$

where

$$
A=\left[\begin{array}{cc}
0 & 1 \\
-k & 0
\end{array}\right] \quad \text { and } \quad B=\left[\begin{array}{cc}
0 & 0 \\
0 & -1
\end{array}\right] .
$$

We have obtained (the state equation of) a control-affine system (on $\mathbb{E}^{2}$ ).

Note : A control-affine system of the form

$$
\begin{aligned}
\dot{x} & =A x+u_{1} B_{1} x+\cdots+u_{\ell} B_{\ell} x \\
& =\left(A+u_{1} B_{1}+\cdots+u_{\ell} B_{\ell}\right) x, \quad x \in \mathbb{E}^{m}, u=\left(u_{1}, \ldots, u_{\ell}\right) \in U \subseteq \mathbb{R}^{\ell}
\end{aligned}
$$

where $A, B_{i} \in \mathbb{R}^{m \times m}$, is called a bilinear system (on the Euclidean $m$-space $\mathbb{E}^{m}$ ).

### 2.1.4 Example. (Linear control systems) A linear control system

 is a control-affine system on a Euclidean space $M=\mathbb{E}^{m}$ with a linear drift $X_{0}$ and each controlled vector field $X_{i}$ constant.Denoting the constant values of $X_{1}, \ldots, X_{\ell}$ by $b_{1}, \ldots, b_{\ell}$, and the drift term by a linear vector field $A, x \mapsto A x$, the corresponding linear control system is given by

$$
\begin{aligned}
\dot{x} & =A x+u_{1} b_{1}+\cdots+u_{\ell} b_{\ell} \\
& =A x+B u, \quad x \in \mathbb{E}^{m}, u=\left(u_{1}, \ldots u_{\ell}\right) \in U \subseteq \mathbb{R}^{\ell}
\end{aligned}
$$

where $B=\left[\begin{array}{lll}b_{1} & \ldots & b_{\ell}\end{array}\right] \in \mathbb{R}^{m \times \ell}$. The case $\ell=1$ is called the single-input case. Single-input linear control systems are intricately connected with $m^{\text {th }}-$ order ODEs with constant coefficients.

Exercise 5 Verify that the $m^{\text {th }}$-order ODE

$$
z^{(m)}+a_{1} z^{(m-1)}+\cdots+a_{m} z=u(t)
$$

can be converted into its single-input linear control system

$$
\dot{x}=\left[\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_{m} & -a_{m-1} & -a_{m-2} & \cdots & -a_{1}
\end{array}\right] x+\left[\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
1
\end{array}\right] u, \quad x=\left[\begin{array}{c}
x_{1} \\
\vdots \\
x_{m}
\end{array}\right] \in \mathbb{E}^{m} .
$$

NOTE : It is somewhat remarkable that "almost all" single-input linear control systems are higher-order ODEs in disguise. More precisly, if

$$
\dot{x}=A x+b u, \quad x \in \mathbb{E}^{m}, u \in \mathbb{R}
$$

is a single-input linear control system such that rank $\left[\begin{array}{llll}b & A b & \cdots & A^{m-1} b\end{array}\right]=m$, than there exists a linear transformation (change of coordinates) $\widetilde{x}=T x$ such that

$$
\begin{aligned}
\widetilde{A} & =T A T^{-1}=\left[\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-a_{m} & -a_{m-1} & -a_{m-2} & \cdots & -a_{1}
\end{array}\right] \\
\widetilde{b} & =T b=\left[\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
1
\end{array}\right] .
\end{aligned}
$$

Abstract Lie groups (in particular, matrix Lie groups) form an important class of smooth (in fact, analytic) manifolds. Henceforth, in this chapter, we shall consider only control-affine systems on matrix Lie groups.

### 2.2 Invariant Control Systems on Matrix Lie Groups

## Invariant vector felds

Let $G \leq G L(n, \mathbb{R})$ be an $m$-dimensional matrix Lie group with identity $e=$ $I_{n} \in G$. Let $\mathfrak{g} \subseteq \mathfrak{g l}(n, \mathbb{R})=\mathbb{R}^{n \times n}$ denote the Lie algebra of $G$ (i.e. the tangent space $T_{e} G$ at the identity).

Note : Given an arbitrary element (matrix) $A \in \mathbb{R}^{n \times n}$, the Cauchy problem (i.e. ODE + initial condition), on the general linear group $\mathrm{GL}(n, \mathbb{R})$,

$$
\dot{g}=g A, \quad g(0)=g_{0} \quad(g \in \mathrm{GL}(n, \mathbb{R}))
$$

has a (unique) solution of the form $g(t)=g_{0} \exp (t A)$ (see Exercise 163).
A similar problem, on the matrix Lie group $G$, fails to be well-defined, unless $A \in \mathfrak{g}$. (This is the case because $\exp (t A) \in G$ for all $t \Longleftrightarrow A \in \mathfrak{g}$.)

Recall that

$$
T_{g} G=\{\dot{\gamma}(0) \mid \gamma(t) \in G, \gamma(0)=g\}
$$

Exercise 6 Show that (for $g \in G$ )

$$
T_{g} G=g T_{e} G=\{g A \mid A \in \mathfrak{g}\} .
$$

(The left translation $L_{g}$ moves the tangent space at the identity to the tangent space at $g$.)

Thus for any element $A \in \mathfrak{g}$, the correspondence

$$
g \in G \mapsto g A \in T_{g} G
$$

defines a (smooth) vector field on (the matrix Lie group) $G$.

A vector field $X$ on $G$ is left-invariant if $X(g)=g A$ for some fixed (matrix) $A \in \mathfrak{g}$.

Note : (1) Recall that a vector field $X$ on an abstract Lie group $G$ is leftinvariant if (and only if) for every $g \in G$

$$
\left(L_{g}\right)_{*} X(e)=X(g)
$$

The left translation $L_{g}: \mathrm{GL}(n, \mathbb{R}) \rightarrow \mathrm{GL}(n, \mathbb{R})$ is a linear transformation, hence $\left(L_{g}\right)_{*}=L_{g}: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}\left(T_{e} \mathrm{GL}(n, \mathbb{R})=\mathbb{R}^{n \times n}\right.$; see also Exercise 202). When $G \leq \mathrm{GL}(n, \mathbb{R})$, it follows that (for $g \in G$ )

$$
\begin{aligned}
X(g) & =\left(L_{g}\right)_{*} X(e) \\
& =L_{g}(A) \\
& =g A
\end{aligned}
$$

where $A=X(e) \in \mathfrak{g}$.
(2) The set $\mathfrak{X}_{L}(G)$ of all left-invariant vector fields on $G$ has the structure of a vector space (in fact, a Lie algebra). The correspondence

$$
X \in \mathfrak{X}_{L}(G) \mapsto X(e) \in T_{e} G=\mathfrak{g}
$$

is an isomorphism (of Lie algebras) and so we can identify any left-invariant vector field on the matrix Lie group $G$ with its value at the identity.
(3) Similarly, a vector field $Y$ on $G$ is right-invariant if $Y(g)=B g$ for some fixed $B \in \mathfrak{g}$. Again, the space (Lie algebra) $\mathfrak{X}_{R}(G)$ of all right-invariant vector fields on $G$ is isomorphic to the Lie algebra $\mathfrak{g}$ of $G$ (and thus to $\mathfrak{X}_{L}(G)$ ).

Henceforth we shall not distinguish - notationwise - between an element (matrix) $A \in \mathfrak{g}$ and its corresponding left-invariant vector field $g \mapsto g A$.

It follows that the ODE, on the matrix Lie group $G$,

$$
\dot{g}=g A \quad(g \in G)
$$

is well-defined and has solutions $g(t)=g(0) \exp (t A)$. Equivalently, in geometric language, the integral curve of the left-invariant vector field $A=\mathfrak{X}_{L}(G)=$
$\mathfrak{g}$ through $g_{0} \in G$ is

$$
t \mapsto g_{0} \exp (t A)
$$

Note : We can write

$$
(\exp t A)(g)=g \exp (t A)
$$

Caution : The left-hand side represents the flow - in "exponential notation" - of the vector field $A, g \mapsto g A$, whereas the right-hand side represents the product (matrix multiplication) of g with the matrix exponential of $A \in \mathfrak{g}$.

It follows that any left-invariant vector field on $G$ is complete.
Note : More generally, let $\mathrm{Fl}^{X}$ be the flow corresponding to the left-invariant vector field $X$ on the abstract Lie group $G$, and let $\gamma_{e}: t \mapsto \mathrm{Fl}^{X}(t, e)$ denote the integral curve through the group identity $e \in G$. Then the curve

$$
\gamma_{g}: J_{g} \rightarrow G, \quad t \mapsto g \gamma_{e}(t)\left(=L_{g}\left(\gamma_{e}(t)\right)\right)
$$

is the integral curve of $X$ through $g$, which furthermore satisfies (for each $g \in G$ and each $t \in \mathbb{R}$ )

$$
\mathrm{Fl}^{X}(t, g)=g \mathrm{Fl}^{X}(t, e)
$$

This equality has several implications :
(i) The integral curve $\gamma_{e}$ is defined for all $t \in \mathbb{R}$. One can easily verify that the set $\left\{\gamma_{e}(t) \mid t \in \mathbb{R}\right\}$ is an Abelian subgroup of $G$. Now, if the curve $\gamma_{e}$ is defined for a particular value of $t$, then $\gamma_{e}$ must be defined for $t+\epsilon$ (where $\epsilon$ is independent of $t$ ) since $\gamma_{e}(t+\epsilon)=\gamma_{e}(t) \gamma_{e}(\epsilon)$. Therefore $\gamma_{e}$ is defined for all $t \in \mathbb{R}$.
(ii) $X$ is a complete vector field, because $\mathrm{Fl}^{X}(t, g)=g \gamma_{e}(t)$, and therefore $t \mapsto$ $\mathrm{Fl}^{X}(t, g)$ is defined for all $t \in \mathbb{R}$.
(iii) If $X$ were right-invariant, then its flow $\mathrm{Fl}^{X}$ would satisfy $\mathrm{Fl}^{X}(t, g)=\mathrm{Fl}^{X}(t, e) g$. It therefore follows that implications $(i)$ and (ii) are also true for the rightinvariant vector fields.
(iv) In particular, if a left-invariant vector field and a right-invariant vector field are equal to each other at the identity, then their integral curves through the identity are the same.

We further observe the following important fact : the left translation $L_{h}$ maps an integral curve into an integral curve. Indeed, if $t \mapsto g(0) \exp (t A)$ is an integral curve of $A \in \mathfrak{X}_{L}(G)=\mathfrak{g}$, then

$$
\begin{aligned}
h(t) & =h g(t) \\
& =h g(0) \exp (t A) \\
& =h(0) \exp (t A) .
\end{aligned}
$$

Hence its left-translation is also an integral curve of $A$. (This explains the title "left-invariant" for vector fields of the form $g \mapsto g A$.)

The Lie bracket of left-invariant vector fields is also a left-invariant vector field (see also Proposition 4.4.9). More precisely, the following result holds.
2.2.1 Proposition. Let $A, g \mapsto g A$ and $B, g \mapsto g B$ be left-invariant vector fields on the matrix Lie group $G$. Then (for $g \in G$ )

$$
[A, B](g)=g(A B-B A)
$$

Proof : We shall give a "direct" proof based on the following characterization of the Lie bracket of two (arbitrary) vector fields :

$$
[X, Y](p)=\left.\frac{d}{d t} \gamma(\sqrt{t})\right|_{t=0}
$$

where the curve $t \mapsto \gamma(t)$ is defined by

$$
\gamma(t)=(\exp -t Y) \circ(\exp -t X) \circ(\exp t Y) \circ(\exp t X)(p)
$$

The flows (in "exponential notation") of $A, B \in \mathfrak{X}_{L}(G)=\mathfrak{g}$ are given by

$$
(\exp t A)(g)=g \exp (t A) \quad \text { and } \quad(\exp t B)=g \exp (t B)
$$

Then (by computing the low-order terms of the curve $\gamma$ )

$$
\begin{aligned}
\gamma(t)= & (\exp -t B) \circ(\exp -t A) \circ(\exp t B) \circ(\exp t A)(g) \\
= & g \exp (t A) \exp (t B) \exp (-t A) \exp (-t B) \\
= & g\left(I+t A+\frac{t^{2}}{2} A^{2}+\cdots\right)\left(I+t B+\frac{t^{2}}{2} B^{2}+\cdots\right) \\
& \left(I-t A+\frac{t^{2}}{2} A^{2}+\cdots\right)\left(I-t B+\frac{t^{2}}{2} B^{2}+\cdots\right) \\
& \left(I+t(A+B)+\frac{t^{2}}{2}\left(A^{2}+2 A B+B^{2}\right)+\cdots\right) \\
= & g\left(I-t(A+B)+\frac{t^{2}}{2}\left(A^{2}+2 A B+B^{2}\right)+\cdots\right) \\
= & g\left(I+t^{2}(A B-B A)+\cdots\right)
\end{aligned}
$$

hence

$$
\gamma(\sqrt{t})=g(I+t(A B-B A)+\cdots)
$$

Thus

$$
\begin{aligned}
{[A, B](g) } & =\left.\frac{d}{d t} \gamma(\sqrt{t})\right|_{t=0} \\
& =g(A B-B A)
\end{aligned}
$$

Note : For right-invariant vector fields $A, g \mapsto A g$ and $B, g \mapsto B g$ the following "less convenient" formula holds (for $g \in G$ )

$$
[A, B](g)=(B A-A B) g
$$

Let $\mathcal{A} \subseteq \mathbb{R}^{n \times n}$ be an arbitrary family of matrices. Each element $A \in \mathcal{A}$ may be viewed as a left-invariant vector field on $G L(n, \mathbb{R})$. By the Orbit

Theorem, the orbit $\mathcal{O}(e)$ of $\mathcal{A}$ through the identity is a connected, immersed submanifold of $G L(n, \mathbb{R})$. Moreover,

$$
\begin{aligned}
\mathcal{O}(e) & =\left\{\left(\exp t_{k} A_{k}\right) \circ \cdots \circ\left(\exp t_{1} A_{1}\right)(e) \mid t_{i} \in \mathbb{R}, A_{i} \in \mathcal{A}, k \in \mathbb{Z}^{+}\right\} \\
& =\left\{\exp \left(t_{1} A_{1}\right) \cdots \exp \left(t_{k} A_{k}\right) \mid t_{i} \in \mathbb{R}, A_{i} \in \mathcal{A}, k \in \mathbb{Z}^{+}\right\}
\end{aligned}
$$

Consequently, the orbit $\mathcal{O}(e)$ is a subgroup of $G L(n, \mathbb{R})$. This subgroup is in fact a Lie subgroup of $\operatorname{GL}(n, \mathbb{R})$. We also have

$$
\begin{aligned}
T_{e} \mathcal{O}(e) & =\operatorname{Lie}(\mathcal{A}) \\
T_{g} \mathcal{O}(e) & =g \operatorname{Lie}(\mathcal{A}) \\
\mathcal{O}(g) & =\left\{g \exp \left(t_{1} A_{1}\right) \cdots \exp \left(t_{k} A_{k}\right) \mid t_{i} \in \mathbb{R}, A_{i} \in \mathcal{A}, k \in \mathbb{Z}^{+}\right\} \\
& =g \mathcal{O}(e)
\end{aligned}
$$

In particular, by restricting to Lie subalgebras $\mathcal{A}=\operatorname{Lie}(\mathcal{A}) \subseteq \mathbb{R}^{n \times n}$, we get the following result : to any Lie subalgebra $\mathcal{A} \subseteq \mathbb{R}^{n \times n}$ there corresponds a connected Lie subgroup $G$ of $G L(n, \mathbb{R})$ such that $T_{e} G=\mathcal{A}$. (Here $G=$ $\mathcal{O}(e)$.$) The converse is also true. (This can be proved by using arguments$ based, again, on the Orbit Theorem.) Hence we get the following classical result (due to Sophus LIE): there exists a one-to-one correspondence between Lie subalgebras $\mathcal{A} \subseteq \mathbb{R}^{n \times n}$ and connected Lie subgroups $G$ of $G L(n, \mathbb{R})$ such that $T_{e} G=\mathcal{A}$.

Note : A remarkable and very deep result, due to Igor Ado, states that every finite-dimensional Lie algebra is (isomorphic to) a Lie algebra of matrices. This is in contrast to the situation for Lie groups, where most but not all Lie groups are matrix Lie groups.

## Invariant control systems

Let $G \leq G L(n, \mathbb{R})$ be an $m$-dimensional matrix Lie group with its Lie algebra $\mathfrak{g}$. A control-affine system (on $G$ ) determined by left-invariant vector fields is said to be left-invariant. We make the following definition.
2.2.2 Definition. A left-invariant control system on the matrix Lie group $G$ is (given by) a collection $\Gamma$ of elements in $\mathfrak{g}$ of the form

$$
\Gamma=\left\{A_{u}=A_{0}+u_{1} A_{1}+\cdots+u_{\ell} A_{\ell} \mid u=\left(u_{1}, \ldots, u_{\ell}\right) \in U \subseteq \mathbb{R}^{\ell}\right\}
$$

for some fixed $A_{0}, A_{1}, \ldots, A_{\ell} \in \mathfrak{g}=\mathfrak{X}_{L}(G)$.

NOTE : $\quad \Gamma \subseteq \mathfrak{g}$ is in fact a collection of matrices (in $\mathbb{R}^{n \times n}$ ).

In classical notation, a left-invariant control system on $G$ is written as

$$
\begin{aligned}
\dot{g} & =g\left(A_{0}+u_{1} A_{1}+\cdots+u_{\ell} A_{\ell}\right) \\
& =g A_{u}(t), \quad g \in G, u=\left(u_{1}, \ldots, u_{\ell}\right) \in U \subseteq \mathbb{R}^{\ell}
\end{aligned}
$$

where $g(\cdot)$ is a curve in the matrix Lie group $G$ and $A_{u}(\cdot)$ is a curve in the associated Lie algebra $\mathfrak{g}=\mathfrak{X}_{L}(G)$.

Note : We may assume that $\ell \leq m$ and also that $A_{1}, \ldots, A_{\ell}$ are linearly independent elements (matrices) of $\mathfrak{g}$ which can be completed such that $\left\{A_{1}, \ldots, A_{\ell}, A_{\ell+1}, \ldots, A_{m}\right\}$ is a basis for $\mathfrak{g}$.

A trajectory of a left-invariant control system (given by) $\Gamma$ on $G$ is a continuous curve $t \mapsto g(t)$ in $G$, defined on an interval $[0, T] \subset \mathbb{R}$ so that there exists a partition $0=t_{0}<t_{1}<\cdots<t_{N}=T$ and elements (leftinvariant vector fields) $X_{1}, \ldots, X_{N} \in \Gamma$ such that the restriction of $g(\cdot)$ to each open interval $\left(t_{i-1}, t_{i}\right)$ is smooth and (for $\left.t \in\left(t_{i-1}, t_{i}\right)\right)$

$$
\dot{g}(t)=X_{i}(g(t)), \quad i=1,2, \ldots, N
$$

Note : Because the elements of $\Gamma$ are parametrized by controls, it follows that each left-invariant vector field $X_{i}$ is equal to $A_{u_{i}}$ for some $u_{i}$. Hence $g(\cdot)$ is the integral curve of the time-varying vector field $(t, g) \mapsto A(g, u(t)):=g A_{u}(t)$, with $u(\cdot)$ equal to the piecewise constant control, which takes constant value $u_{i}$ in each interval $\left[t_{i-1}, t_{i}\right]$, and $t \mapsto g(t)$ can be visualized as a "broken" continuous curve consisting of pieces of integral curves of vector fields corresponding to different choices of control values.

Similarly, a right-invariant control system on $G$ can be written as

$$
\begin{aligned}
\dot{g} & =\left(A_{0}+u_{1} A_{1}+\cdots+u_{\ell} A_{\ell}\right) g \\
& =A_{u}(t) g, \quad g \in G, u=\left(u_{1}, \ldots, u_{\ell}\right) \in U \subseteq \mathbb{R}^{\ell}
\end{aligned}
$$

where $g(\cdot)$ is a curve in the matrix Lie group $G$ and $t \mapsto A_{u}(t):=A_{0}+$ $u_{1}(t) A_{1}+\cdots+u_{\ell}(t) A_{\ell}$ is a curve in the associated Lie algebra $\mathfrak{g}=\mathfrak{X}_{R}(G)$.

We focus on left-invariant control systems on matrix Lie graoups, but analogue results can be derived for right-invariant control systems. In fact, given a right-invariant control system written as

$$
\dot{g}=A_{u}(t) g, \quad g \in G, u=\left(u_{1}, \ldots, u_{\ell}\right) \in U \subseteq \mathbb{R}^{\ell}
$$

we can always convert it into a left-invariant control system by considering $t \mapsto g^{-1}(t)$ as our state trajectory.

Exercise 7 Show that if the curve $t \mapsto g(t)$ in $G$ satisfies the condition

$$
\dot{g}=A_{u}(t) g
$$

then the curve $t \mapsto h(t):=g^{-1}(t)$ satisfies

$$
\dot{h}=-h A_{u}(t)
$$

Thus, there is no loss of generality in specializing to left-invariant control systems.

Consider the general affine group

$$
\mathrm{GA}(n, \mathbb{R})=\left\{\left.g=\left[\begin{array}{ll}
1 & 0 \\
c & A
\end{array}\right] \right\rvert\, c \in \mathbb{R}^{n}, X \in \mathrm{GL}(n, \mathbb{R})\right\} .
$$

Embedding $\mathbb{E}^{n}$ into $\mathbb{E}^{n+1}$ as the hyperplane

$$
\{1\} \times \mathbb{E}^{n}=\left\{(1, p) \mid p \in \mathbb{E}^{n}\right\} \subset \mathbb{E}^{n+1}
$$

we obtain the affine transformation on $\mathbb{E}^{n}$ defined by an element $g \in \mathrm{GA}(n, \mathbb{R})$

$$
x=\left[\begin{array}{l}
1 \\
x
\end{array}\right] \mapsto g x=\left[\begin{array}{ll}
1 & 0 \\
c & A
\end{array}\right]\left[\begin{array}{l}
1 \\
x
\end{array}\right]=\left[\begin{array}{c}
1 \\
A x+c
\end{array}\right]=A x+c .
$$

That is, the group $\mathrm{GA}(n, \mathbb{R})$ acts on (the Euclidean space) $\mathbb{E}^{n}$ as follows :

$$
(g, x) \mapsto g x:=A x+c .
$$

The Lie algebra of $G A(n, \mathbb{R})$ is

$$
\mathfrak{g a}(n, \mathbb{R})=\left\{\left.\bar{A}=\left[\begin{array}{ll}
0 & 0 \\
a & A
\end{array}\right] \right\rvert\, a \in \mathbb{R}^{n}, A \in \mathbb{R}^{n \times n}\right\} .
$$

Every element (matrix) $\bar{A} \in \mathfrak{g a}(n, \mathbb{R})$ induces a vector field on $\mathbb{R}^{n}$ :

$$
x \mapsto A x+a .
$$

Now let $G \leq \mathrm{GA}(n, \mathbb{R})$ be a connected matrix subgroup of $\mathrm{GA}(n, \mathbb{R})$ (that acts transitively on $\left.\mathbb{R}^{n}\right)$; for instance, $\mathrm{GA}^{+}(n, \mathbb{R})$ or $\mathrm{SE}(n)$.

A right-invariant control system on the matrix Lie group $G$ written as

$$
\dot{g}=\left(\bar{A}+u_{1} \bar{B}_{1}+\cdots+u_{\ell} \bar{B}_{\ell}\right) g, \quad g \in G
$$

with

$$
\bar{A}=\left[\begin{array}{ll}
0 & 0 \\
a & A
\end{array}\right], \bar{B}_{i}=\left[\begin{array}{cc}
0 & 0 \\
b_{i} & B_{i}
\end{array}\right] \in \mathfrak{g a}(n, \mathbb{R}), \quad i=1,2, \ldots, \ell
$$

induces the following affine control system on $\mathbb{E}^{n}$ :

$$
\dot{x}=A x+a+u_{1}\left(B_{1} x+b_{1}\right)+\cdots+u_{\ell}\left(B_{\ell} x+b_{\ell}\right), \quad x \in \mathbb{E}^{n}
$$

Particular cases of such control systems (on $\mathbb{E}^{n}$ ) are

- the bilinear control systems

$$
\begin{aligned}
\dot{x} & =A x+u_{1} B_{1} x+\cdots+u_{\ell} B_{\ell} x \\
& =\left(A+u_{1} B_{1}+\cdots+u_{\ell} B_{\ell}\right) x, \quad x \in \mathbb{E}^{n}
\end{aligned}
$$

(obtained for $a=b_{1}=\cdots=b_{\ell}=0$ )

- the linear control systems

$$
\begin{aligned}
\dot{x} & =A x+u_{1} b_{1}+\cdots+u_{\ell} b_{\ell} \\
& =A x+B u, \quad x \in \mathbb{E}^{n}
\end{aligned}
$$

(obtained for $\left.a=0, B_{1}=\cdots=B_{\ell}=0\right)$.

Note : An abstract Lie group $G$ is said to act (from the left) on the (analytic) manifold $M$ if there exists an (analytic) mapping $\theta: G \times M \rightarrow M, \quad(g, x) \mapsto$ $\theta(g, x)=g x$ that satisfies (for $g_{1}, g_{2} \in G$ and $x \in M$ )

$$
\left(g_{2} g_{1}\right) x=g_{2}\left(g_{1} x\right) \quad \text { and } \quad e x=x .
$$

For each $g \in G$, consider the (analytic) diffeomorphism $\theta_{g}: M \rightarrow M, \quad x \mapsto \theta_{g}(x)=$ $g x$ (the inverse of $\theta_{g}$ is $\theta_{g^{-1}}$ ). The mapping $g \mapsto \theta_{g}$ is called the (left) action of $G$ on $M$. Any action is a homomorphism from the group $G$ to the group of (analytic)
diffeomorhisms of $M$. For any element $A \in \mathfrak{g}, \theta_{\exp (t A)}$ is a one-parameter group of diffeomorphisms of $M$ with the generator $\theta_{*}(A)$ - an (analytic) vector field on $M$ :

$$
\theta_{*}(A)(x):=\left.\frac{d}{d t} \theta_{\exp (t A)}(x)\right|_{t=0}, \quad x \in M, A \in \mathfrak{g}
$$

Such vector fields $\theta_{*}(A), A \in \mathfrak{g}$ are called subordinated to the action $\theta$. They form a (finite-dimensional) Lie algebra $\theta_{*}(\mathfrak{g})$. A collection of vector fields $\mathcal{F}$ on $M$ is called subordinated to the action $\theta$ if $\mathcal{F} \subseteq \theta_{*}(\mathfrak{g})$. If $\mathcal{F}=\theta_{*}(\Gamma)$ for some right-invariant control system (determined by) $\Gamma \subseteq \mathfrak{g}$, then $\mathcal{F}$ is called induced by $\Gamma$.

A Lie group $G$ is said to act transitively on (the manifold) $M$ if for any $x \in M$ the orbit $\left\{\theta_{g}(x) \mid g \in G\right\}$ coincides with the whole $M$. A manifold that admits a transitive action of a Lie group is called a homogeneous space (of this Lie group). Homogeneous spaces are exactly manifolds that can be represented as quotients of Lie groups.

Given a right-invariant control system on a Lie group that acts on (the manifold) $M$, one can construct the control system (on $M$ ) induced by $\Gamma$. In particular, for $G$ either $\mathrm{GA}^{+}(n, \mathbb{R})$ or $\operatorname{SE}(n)$ (or, more generally, any connected matrix subgroup of the general affine group $G A(n, \mathbb{R})$ that acts transitively on $\left.\mathbb{E}^{n}\right)$, one obtain bilinear and affine control systems on $\mathbb{E}^{n}$.

Control systems on homogeneous spaces subordinated to a group action (in particular, bilinear and affine control systems) were among the most important motivations for the study of (righ-)invariant control systems on (matrix) Lie groups.

### 2.3 Examples

We will give some interesting examples of invariant control systems on matrix Lie groups.

The so-called Brockett system is a simple (nonlinear) control system on $\mathbb{E}^{3}$ defined (after a change of variables) as

$$
\begin{aligned}
\dot{x}_{1} & =u_{1} \\
\dot{x}_{2} & =u_{2} \\
\dot{x}_{3} & =x_{2} u_{1} .
\end{aligned}
$$

Exercise 8 Verify that the Brockett system is a driftless, control-affine system on $\mathbb{E}^{3}$.

Note : In the literature, the Brockett system is also referred to as the Brockett integrator (or the nonholonomic integrator or even the Heisenberg system); it usually appears in the following form

$$
\begin{aligned}
& \dot{x}_{1}=u_{1} \\
& \dot{x}_{2}=u_{2} \\
& \dot{x}_{3}=x_{2} u_{1}-x_{1} u_{2} .
\end{aligned}
$$

Since its appearance in the early 80 's, the Brockett integrator has attracted the interest of several researchers. It is the simplest control system with nonholonomic constraint as well as the first example of a (globally) controllable nonlinear system which is not (smoothly) stabilizable. Despite its simplicity, the Brockett integrator presents challenging problems, many of them not yet solved. It arises in numerous applications and moreover has an educational relevance (it is a useful example to approach and understand difficult mathematical and control theoretic issues).

The Brockett system can be expressed as a driftless, left-invariant control system on the Heisenberg group (consisting of unipotent $3 \times 3$ matrices)

$$
\mathrm{H}(1)=\left\{\left.g=\left[\begin{array}{ccc}
1 & x_{2} & x_{3} \\
0 & 1 & x_{1} \\
0 & 0 & 1
\end{array}\right] \right\rvert\, x_{1}, x_{2}, x_{3} \in \mathbb{R}\right\} \leq \mathrm{GL}(3, \mathbb{R}) .
$$

Consider the associated Lie algebra (consisting of all $3 \times 3$ strictly upper triangular matrices)

$$
\mathfrak{h}(1)=\left\{\left.A=\left[\begin{array}{ccc}
0 & a_{2} & a_{3} \\
0 & 0 & a_{1} \\
0 & 0 & 0
\end{array}\right] \right\rvert\, a_{1}, a_{2}, a_{3} \in \mathbb{R}\right\}
$$

Define a basis $\left\{A_{1}, A_{2}, A_{3}\right\}$ for this Lie algebra, where

$$
A_{1}:=\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right], A_{2}:=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right], A_{3}:=\left[\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

with the following table for the Lie bracket (commutator) :

| $[\cdot, \cdot]$ | $A_{1}$ | $A_{2}$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | 0 | $-A_{3}$ | 0 |
| $A_{2}$ | $A_{3}$ | 0 | 0 |
| $A_{3}$ | 0 | 0 | 0 |

(This means, for instance, that $\left[A_{1}, A_{2}\right]=-\left[A_{2}, A_{1}\right]=-A_{3}$.)
A simple computation shows that the Brockett's system can be written as

$$
\dot{g}=g\left(u_{1} A_{1}+u_{2} A_{2}\right), \quad g \in H(1), u=\left(u_{1}, u_{2}\right) \in \mathbb{E}^{2} .
$$

Indeed,

$$
\begin{aligned}
\dot{g} & =\left[\begin{array}{ccc}
0 & \dot{x}_{2} & \dot{x}_{3} \\
0 & 0 & \dot{x}_{1} \\
0 & 0 & 0
\end{array}\right] \\
& =\left[\begin{array}{lll}
0 & u_{2} & u_{1} x_{2} \\
0 & 0 & u_{1} \\
0 & 0 & 0
\end{array}\right] \\
& =\left[\begin{array}{ccc}
1 & x_{2} & x_{3} \\
0 & 1 & x_{1} \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{lll}
0 & u_{2} & 0 \\
0 & 0 & u_{1} \\
0 & 0 & 0
\end{array}\right] \\
& =g\left(u_{1} A_{1}+u_{2} A_{2}\right) .
\end{aligned}
$$

## Unicycle

Let us consider a simplified model of a unicycle, where we just model the wheel which is assumed to roll without slipping on a plane with the wheel axis always parallel to the plane. The configuration space is

$$
\mathbb{R}^{2} \times \mathbb{S}^{1}=\left\{\left(x_{1}, x_{2}, \theta\right) \mid x_{1}, x_{2} \in \mathbb{R}, \theta \in \mathbb{S}^{1}\right\}
$$

where $\left(x_{1}, x_{2}\right)$ describes the position of the unicycle on the plane (relative to an orthonormal inertial frame $\left(r_{1}, r_{2}\right)$ ) and $\theta$ describes the orientation of the unicycle (specifically, the angle between the tangent to the wheel and the $r_{1}$-axis). Further we assume that we have control over the forward velocity as well as the steering velocity, which describes the angular velocity of the wheel. So, with $u_{1}=\dot{\theta}$ (steering speed) and $u_{2}=v$ (rolling speed) as controls, the control system (i.e. the motion of the unicycle) can be described by (the scalar
state equations)

$$
\begin{aligned}
\dot{x}_{1} & =u_{2} \cos \theta \\
\dot{x}_{2} & =u_{2} \sin \theta \\
\dot{\theta} & =u_{1} .
\end{aligned}
$$

This control-affine system (on the manifold $\mathbb{R}^{2} \times \mathbb{S}^{1}$ ) can be viewed as a driftless, left-invariant control system on the special Euclidean group SE (2). Indeed, let

$$
\operatorname{SE}(2)=\left\{\left.g=\left[\begin{array}{ccc}
1 & 0 & 0 \\
x_{1} & \cos \theta & -\sin \theta \\
x_{2} & \sin \theta & \cos \theta
\end{array}\right] \right\rvert\, x_{1}, x_{2} \in \mathbb{R}, \theta \in[0,2 \pi)\right\} .
$$

Then its Lie algebra

$$
\mathfrak{s e}(2)=\left\{\left.A=\left[\begin{array}{ccc}
0 & 0 & 0 \\
a_{2} & 0 & -a_{1} \\
a_{3} & a_{1} & 0
\end{array}\right] \right\rvert\, a_{1}, a_{2}, a_{3} \in \mathbb{R}\right\}
$$

is generated by the elements (matrices)

$$
A_{1}:=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right], A_{2}:=\left[\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right], A_{3}:=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{array}\right]
$$

with the following table for the Lie bracket (commutator) :

| $[\cdot, \cdot]$ | $A_{1}$ | $A_{2}$ | $A_{3}$ |
| :---: | :---: | :---: | :---: |
| $A_{1}$ | 0 | $A_{3}$ | $-A_{2}$ |
| $A_{2}$ | $-A_{3}$ | 0 | 0 |
| $A_{3}$ | $A_{2}$ | 0 | 0 |

Again, a simple computation shows that the unicycle control system can be written as

$$
\dot{g}=g\left(u_{1} A_{1}+u_{2} A_{2}\right), \quad g \in \operatorname{SE}(2), u=\left(u_{1}, u_{2}\right) \in \mathbb{R}^{2} .
$$

## Spacecraft

Let us consider a spacecraft free to move in the Euclidean 3 -space $\mathbb{E}^{3}$. In the attitude control problem we restrict our attention to the orientation of the spacecraft (satellite) with respect to a reference frame $\left(r_{1}, r_{2}, r_{3}\right)$. Let $\underline{b}=\left(b_{1}, b_{2}, b_{3}\right)$ be an orthonormal frame fixed on the body and assume that the origins of the two frames coincide. We assume that the actuators of the satellite (i.e. thrusters or momentum wheels) are fixed to the body such that resulting angular velocity vectors are alingned with the body frame $\underline{b}$. Further we make the idealizing assumption that we have direct control over the angular velocities (resulting from the actuators).

We define $g(t) \in \mathbf{S O}(3)$ such that

$$
r_{i}=g(t) b_{i}, \quad i=1,2,3
$$

(i.e. $g(t)$ determines the attitude of the spacecraft at time $t$ ). Hence the configuration space is the special orthogonal group SO (3). Its associated Lie algebra

$$
\mathfrak{s o}(3)=\left\{A \in \mathbb{R}^{3 \times 3} \mid A^{T}+A=0\right\}
$$

(consisting of all $3 \times 3$ skew-symmetric matrices) is customarily identified with the Lie algebra $\mathbb{R}^{3}$, via the canonical mapping

$$
x=\left(x_{1}, x_{2}, x_{3}\right) \mapsto \widehat{x}:=\left[\begin{array}{ccc}
0 & -x_{3} & x_{2} \\
x_{3} & 0 & -x_{1} \\
-x_{2} & x_{1} & 0
\end{array}\right] .
$$

Define

$$
A_{i}:=\widehat{e}_{i}, \quad i=1,2,3
$$

where $e_{1}, e_{2}, e_{3}$ are the standard vectors in $\mathbb{R}^{3}$. That is,

$$
A_{1}=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right], A_{2}=\left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{array}\right], A_{3}=\left[\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

Exercise 9 Compute the corresponding table for the Lie bracket (commutator).

Then $\left\{A_{1}, A_{2}, A_{3}\right\}$ is a (standard) basis for $\mathfrak{s o}(3)$ and $g(t)$ satisfies :

$$
\begin{aligned}
\dot{g} & =g \widehat{\omega} \\
& =g\left(\omega_{1} A_{1}+\omega_{2} A_{2}+\omega_{3} A_{3}\right)
\end{aligned}
$$

where

$$
\omega=\left[\begin{array}{l}
\omega_{1} \\
\omega_{2} \\
\omega_{3}
\end{array}\right] \in \mathbb{R}^{3 \times 1}
$$

is the angular velocity of the spacecraft in the body-fixed coordinates. If we let

$$
u_{i}:=\omega_{i}, \quad i=1,2,3
$$

(i.e. if we interpret the components of the angular velocity as controls), then the kinematics of the spacecraft can be described by (the state equation) :

$$
\dot{g}=g\left(u_{1} A_{1}+u_{2} A_{2}+u_{3} A_{3}\right), \quad g \in \mathrm{SO}(3), u=\left(u_{1}, u_{2}, u_{3}\right) \in \mathbb{R}^{3}
$$

This is a driftless, left-invariant control system on the special orthogonal group SO (3).

An interesting particular case is when only two components of the angular velocity can be controlled (due, for instance, to a failure). Without loss of generality we may assume that $u_{3}=0$ and then $g(t)$ satisfies

$$
\dot{g}=g\left(u_{1} A_{1}+u_{2} A_{2}\right), \quad g \in \mathrm{SO}(3), u=\left(u_{1}, u_{2}\right) \in \mathbb{R}^{2} .
$$

Note : Any control configuration can be represented by choosing the appropriate basis for $\mathfrak{s o}$ (3). For example, suppose there are only two independent control inputs defined by

$$
u_{1}:=\omega_{1}+\omega_{2} \quad \text { and } \quad u_{2}:=\omega_{2}+\omega_{3} .
$$

Then the (left-invariant) control system is described by

$$
\dot{g}=g\left(u_{1} A_{1}^{\prime}+u_{2} A_{2}^{\prime}\right), \quad g \in \operatorname{SO}(3), u=\left(u_{1}, u_{2}\right) \in \mathbb{R}^{2}
$$

where

$$
A_{1}^{\prime}:=A_{1}+A_{2}, \quad A_{2}^{\prime}:=A_{2}+A_{3}, \quad A_{3}^{\prime}:=A_{3} .
$$

## Underwater vehicle

Consider an autonomous underwater vehicle (AUV) and let $\underline{b}=\left(b_{1}, b_{2}, b_{3}\right)$ be an orthonormal frame fixed on the vehicle. The configuration of the vehicle is modeled as the position and orientation of the body-fixed frame $\underline{b}$ with respect to an inertial frame $\left(r_{1}, r_{2}, r_{3}\right)$. We assume that the individual actuators are configured such that the resulting angular and translational velocities are aligned with the body frame $\underline{b}$. We define

$$
g(t)=\left[\begin{array}{cc}
1 & 0 \\
x(t) & R(t)
\end{array}\right] \in \mathrm{SE}(3)
$$

such that

$$
\left[\begin{array}{c}
1 \\
r_{i}
\end{array}\right]=g(t)\left[\begin{array}{c}
1 \\
b_{i}
\end{array}\right], \quad i=1,2,3 .
$$

Note : This is essentially the same condition as

$$
r_{i}=x(t)+R(t) b_{i}, \quad i=1,2,3 .
$$

Thus $g(t)$ describes the position and orientation of the AUV at time $t$. Let

$$
A_{i}:=\left[\begin{array}{cc}
0 & 0 \\
0 & \widehat{e}_{i}
\end{array}\right], \quad A_{i+3}:=\left[\begin{array}{cc}
0 & 0 \\
e_{i} & 0
\end{array}\right], \quad i=1,2,3 .
$$

Then $\left\{A_{1}, A_{2}, A_{3}, A_{4}, A_{5}, A_{6}\right\}$ defines a basis for the Lie algebra $\mathfrak{s e}(3)$ associated with (the configuration space) SE (3).

Exercise 10 Compute the corresponding table for the Lie bracket (commutator).

Now let

$$
\omega=\left[\begin{array}{l}
\omega_{1} \\
\omega_{2} \\
\omega_{3}
\end{array}\right], v=\left[\begin{array}{l}
v_{1} \\
v_{2} \\
v_{3}
\end{array}\right] \in \mathbb{R}^{3 \times 1}
$$

define the angular velocity and the translational velocity of the vehicle (in the body-fixed coordinates), respectively. Then $g(t)$ satisfies

$$
\dot{g}=g\left(\omega_{1} A_{1}+\omega_{2} A_{2}+\omega_{3} A_{3}+v_{1} A_{4}+v_{2} A_{5}+v_{3} A_{6}\right) .
$$

If we let

$$
u_{i}:=\omega_{i}, u_{i+3}:=v_{i}, \quad i=1,2,3
$$

(i.e. if we interpret the components of the angular and translational velocities as controls), then the kinematics of the AUV can be described by (the state equation):
$\dot{g}=g\left(u_{1} A_{2}+u_{2} A_{2}+\cdots+u_{6} A_{6}\right), \quad g \in \operatorname{SE}(3), u=\left(u_{1}, \ldots, u_{6}\right) \in \mathbb{R}^{6}$.

This is a driftless, left-invariant control system on the special Euclidean group SE (3).

As in the spacecraft attitude control problem, we are interested in the case when fewer than $m(=6)$ control components are available (i.e. $\ell<6$ ). For example, suppose that we can control angular velocity about $b_{1}, b_{2}, b_{3}$ and translational velocity along $b_{1}$. Then $g(t)$ satisfies
$\dot{g}=g\left(u_{1} A_{1}+u_{2} A_{2}+u_{3} A_{3}+u_{4} A_{4}\right), \quad g \in \operatorname{SE}(3), u=\left(u_{1}, u_{2}, u_{3}, u_{4}\right) \in \mathbb{R}^{4}$.

Note : The AUV is controllable with as few as two controls.

## Kinematic car

Let us consider a simple kinematic model for a front-wheel drive car of length $l$. The front-wheel pair and the rear-wheel pair are each modelled as a single wheel located at the midpoint of each axle. We assume that only the front wheels are allowed to turn. The car, like the unicycle, is a nonholonomic system if we assume that the wheels do not slip.

Note : Holonomic systems are mechanical systems that are subject to constraints that limit their possible configurations. The word holonomic is comprised of the Greek words holos and nomos meaning "integral" (or "whole") and "law", respectively, and refers to the fact that such constraints, given as constraints on the velocity, may be integrated and reexpressed as constraints on the configuration variables. Examples of holonomic constraints are length constraints for simple pendula and rigidity constraints for rigid body motion.

Nonholonomic mechanics describes the motion of systems constrained by nonintegrable constraints (i.e. constraints on the system velocities that do not arise from constraints on the configurations alone). Classic examples are rolling and skating motion.

Nonholonomic mechanics fits uneasily into the classical mechanics, since it is not variational in nature : it is neither Lagrangian nor Hamiltonian in the strict sense of the word. It is important however, for the theory of optimal control. (There is a close link between nonholonomic constraints and controllability of nonlinear systems. Nonholonomic constraints are given by nonintegrable distributions - that is, taking the Lie bracket of two vector fields in such a distribution may give rise to a vector field not contained in this distribution. It is precisely this property that one wants in nonlinear control systems so that we can drive the system to as large a part of the state space as possible.)

The configuration space is

$$
\mathbb{R}^{2} \times \mathbb{S}^{1} \times \mathbb{S}^{1}=\left\{(x, y, \theta, \varphi) \mid x_{1}, x_{2} \in \mathbb{R}, \theta, \varphi \in \mathbb{S}^{1}\right\}
$$

where $(x, y)$ describes the car's position on a plane (relative to an inertial frame $\left.\left(r_{1}, r_{2}\right)\right)$. On the other hand, $\left(b_{1}, b_{2}\right)$ is an orthonormal frame fixed on the car. $\theta$ denotes the orientation of the car (i.e. the angle between the $b_{1}$-axis of the car and the $r_{1}$-axis), and $\varphi$ denotes the steering angle (i.e. the angle betweeen the $b_{1}$-axis of the car and the front wheels). Assuming that we can control $u_{1}=\dot{\varphi}$ (steering speed) and $u_{2}=v$ (rolling speed), then the kinematic state equations are :

$$
\begin{aligned}
\dot{x} & =u_{2} \cos \theta \\
\dot{y} & =u_{2} \sin \theta \\
\dot{\varphi} & =u_{1} \\
\dot{\theta} & =u_{2} \frac{1}{l} \tan \varphi .
\end{aligned}
$$

Note : This control affine system (on the manifold $\mathbb{R}^{2} \times \mathbb{S}^{1} \times \mathbb{S}^{1}$ ) can be viewed as a nonlinear control system on the matrix Lie group $\mathrm{SE}(2) \times \mathrm{SO}(2)$. Indeed,
the configuration of the car is more naturally described by the matrix Lie group SE (2) $\times$ SO (2). SE (2) describes the position and orientation of the car (as in the unicycle case) and $\operatorname{SO}(2)=\mathbb{S}^{1}$ describes the angular position of the front wheel. Let

$$
g(t)=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
x(t) & \cos \theta & -\sin \theta & 0 & 0 \\
y(t) & \sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 0 & \cos \varphi & -\sin \varphi \\
0 & 0 & 0 & \sin \varphi & \cos \varphi
\end{array}\right] \in \mathrm{SE}(2) \times \mathrm{SO}(2)
$$

describe the configuration of the car at time $t$. Define

$$
A_{1}:=\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0
\end{array}\right], A_{2}:=\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right], A_{3}:=\left[\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\right] .
$$

Then $\left\{A_{1}, A_{2}, A_{3},\left[A_{3}, A_{2}\right]\right\}$ is a basis for the associated Lie algebra $\mathfrak{s e}(2) \times \mathfrak{s o}(2)$, and $g(t)$ satisfies

$$
\dot{g}=g\left(u_{1} A_{1}+u_{2} A_{2}+\tan \left(\tilde{u}_{1}\right) u_{2} A_{3}\right)
$$

where $\tilde{u}_{1}:=\int_{0}^{t} u_{1}(\tau) d \tau$. This is a "left-invariant" control system on SE (2) $\times \mathrm{SO}(2)$, nonlinear in the controls $u_{1}, u_{2}$.

An alternative way of describing the kinematics of the car is to convert the state equations into chained form.

Exercise 11 Verify that, by making the change of variables

$$
v_{1}=u_{2} \cos \theta, \quad v_{2}=u_{1}, \quad \alpha=\sin \theta
$$

the kinematic state equations of the car become

$$
\begin{aligned}
\dot{x} & =v_{1} \\
\dot{\varphi} & =v_{2} \\
\dot{\alpha} & =v_{1} \frac{1}{l} \tan \varphi \\
\dot{y} & =v_{1} \frac{\alpha}{\sqrt{1-\alpha^{2}}} .
\end{aligned}
$$

If we take (for the sake of simplicity) $l=1$ and make the following approximations

$$
\tan \varphi \approx \varphi, \quad \frac{\alpha}{\sqrt{1-\alpha^{2}}} \approx \alpha
$$

the equations take the form :

$$
\begin{aligned}
\dot{x} & =v_{1} \\
\dot{\varphi} & =v_{2} \\
\dot{\alpha} & =v_{1} \varphi \\
\dot{y} & =v_{1} \alpha .
\end{aligned}
$$

This system (of equations) is in chained form and we shall write it as follows (for $x_{1}:=x, x_{2}:=\varphi, x_{3}:=\alpha, x_{4}:=y$ ) :

$$
\begin{aligned}
\dot{x}_{1} & =v_{1} \\
\dot{x}_{2} & =v_{2} \\
\dot{x}_{3} & =v_{1} x_{2} \\
\dot{x}_{4} & =v_{1} x_{3} .
\end{aligned}
$$

This chained form control system can be expressed as a driftless, leftinvariant control system on a matrix Lie group $\mathrm{G}_{4}$ of unipotent $4 \times 4$ matrices
(see Exercise 127). Indeed, let

$$
\mathbf{G}_{4}=\left\{\left.g=\left[\begin{array}{cccc}
1 & x_{2} & x_{3} & x_{4} \\
0 & 1 & x_{1} & \frac{x_{1}^{2}}{2} \\
0 & 0 & 1 & x_{1} \\
0 & 0 & 0 & 1
\end{array}\right] \right\rvert\, x_{1}, x_{2}, x_{3}, x_{4} \in \mathbb{R}\right\} .
$$

Then its (nilpotent) Lie algebra $\mathfrak{g}_{4}$ is generated by the elements (matrices)

$$
B_{1}:=\left[\begin{array}{llll}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{array}\right], \quad B_{2}:=\left[\begin{array}{llll}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right] .
$$

Exercise 12 Check that $\left\{B_{1}, B_{2}, B_{3}, B_{4}\right\}=\left\{B_{1}, B_{2},\left[B_{2}, B_{1}\right],\left[\left[B_{2}, B_{1}\right], B_{1}\right]\right\}$ is a basis for $\mathfrak{g}_{4}$.

A simple computation shows that the kinematic car control system (under simplifying conditions) can be written as

$$
\dot{g}=g\left(v_{1} B_{1}+v_{2} B_{2}\right), \quad g \in \mathrm{G}_{4}, v=\left(v_{1}, v_{2}\right) \in \mathbb{R}^{2} .
$$

Note : Other more general chained form control systems can equivalently be described as driftless, left-invariant control systems on some matrix Lie groups of unipotent matrices. For example, consider the (two-input) chained form system

$$
\begin{aligned}
\dot{x}_{1} & =u_{1} \\
\dot{x}_{2} & =u_{2} \\
\dot{x}_{3} & =u_{1} x_{2} \\
\dot{x}_{4} & =u_{1} x_{3} \\
& \vdots \\
\dot{x}_{k} & =u_{1} x_{k-1} .
\end{aligned}
$$

It can be shown that the kinematic state equations of a car with $k-3$ trailers can be converted into this form. Such a control system can be expressed as a driftless, left-invariant control system on a matrix Lie group (of unipotent $k \times k$ matrices) $\mathrm{G}_{k}$.

### 2.4 Controllability

Let $G$ be an $m$-dimensional matrix Lie group with associated Lie algebra $\mathfrak{g}=T_{e} G=\mathfrak{X}_{L}(G)$. Consider a left-invariant control system on $G$ written as

$$
\dot{g}=g\left(A_{0}+u_{1} A_{1}+\cdots+u_{\ell} A_{\ell}\right), \quad g \in G, u=\left(u_{1}, \ldots, u_{\ell}\right) \in \mathbb{R}^{\ell}
$$

where $A_{0}, A_{1}, \ldots, A_{\ell} \in \mathfrak{g}$ and $\ell \leq m . A_{1}, \ldots, A_{\ell}$ are assumed to be linearly independent. (For simplicity, the control set $U$ coincides with $\mathbb{R}^{\ell}$.) Henceforth, in this chapter, any such (left-invariant) control system on $G$ will be identified with the corresponding collection

$$
\Gamma=\left\{A_{0}+u_{1} A_{1}+\cdots+u_{\ell} A_{\ell} \mid u=\left(u_{1}, \ldots, u_{\ell}\right) \in \mathbb{R}^{\ell}\right\}
$$

of elements (matrices) in $\mathfrak{g}$.

NOTE : $\quad \Gamma \subseteq \mathfrak{g}$ is an affine subspace (i.e. translation of a vector subspace) in $\mathfrak{g}$.

## Reachable sets and orbits

Let $\Gamma$ be a left-invariant control system on $G$ and let $\operatorname{Traj}(\Gamma)$ denote the set of all trajectories of $\Gamma$.

For any $T \geq 0$ and any point $g \in G$, the time $T$ reachable set from $g$ is the set

$$
\mathbf{A}(g, T):=\{g(T) \mid g(\cdot) \in \operatorname{Traj}(\Gamma), \quad g(0)=g\} .
$$

That is, $\boldsymbol{A}(g, T)$ is the set of all points (in $G$ ) that can be reached from (the initial point) $g$ in exactly $T$ units of time. We also define

$$
\mathbf{A}(g, \leq T):=\bigcup_{0 \leq t \leq T} \mathbf{A}(g, t) .
$$

The reachable (or attainable) set from $g$ is the set $\boldsymbol{A}(g)$ of all terminal points $g(T), T \geq 0$ of all trajectories $g(\cdot)$ starting at (the initial point) $g$. That is,

$$
\mathbf{A}(g):=\bigcup_{T \geq 0} \mathbf{A}(g, T) .
$$

2.4.1 Definition. The left-invariant control system $\Gamma$ is called (completely) controllable if, for any $g \in G$,

$$
\mathbf{A}(g)=G .
$$

In other words, $\Gamma$ is (completely) controllable if, given any pair of points $g_{0}, g_{1} \in G$, the point $g_{1}$ can be reached from $g_{0}$ (along a trajectory of $\Gamma$ ) for a nonnegative time $T: g_{1} \in \boldsymbol{A}\left(g_{0}, T\right)$.

Note: (1) The weaker property of accessibility is essential for the description of reachable sets : $\Gamma$ is called accessible at a point $g \in G$ if the reachable set $\boldsymbol{A}(g)$ has nonempty interior (in $G$ ).
(2) There are various controllability concepts, all of which involve reachable sets being "very large" in some sense (e.g. complete controllability, controllability from a point, local controllability, or small-time local controllability). In general, controllability theory is the study of the structure of reachable sets. One major concern is to determine "reasonable" (and, if possible, "effectively computable") conditions for the various controllability (and accesibility) conditions.
(3) All these considerations and concepts can be extended to the more general case of control-affine systems on manifolds. In particular, they are valid for linear control
systems. For the linear control system (with unrestricted controls)

$$
\dot{x}=A x+B u, \quad x \in \mathbb{E}^{m},
$$

the reachable set from the origin is

$$
\boldsymbol{A}(0, T)=\left\{\int_{0}^{T} \exp ((T-\tau) A) B u(\tau) d \tau \mid u(\cdot) \in \mathcal{U}\right\} .
$$

This reachable set is a linear subspace of $\mathbb{E}^{m}$. The control system is controllable from the origin if (for every $T>0$ ) $\boldsymbol{A}(0, T)=\mathbb{E}^{m}$. This immediately implies that also (for any $T>0$ and $x \in \mathbb{E}^{m}$ )

$$
\boldsymbol{A}(x, T)=\mathbb{E}^{m}
$$

Given $A \in \mathfrak{g}=\mathfrak{X}_{L}(G)$, its integral curve through $g \in G$ is $t \mapsto g \exp (t A)$. One can use this simple fact to obtain a (very useful) description of an endpoint of a trajectory.
2.4.2 Lemma. Let $g(\cdot) \in \operatorname{Traj}(\Gamma)$ with $g(0)=g_{0}$. Then there exist $t_{1}, \ldots, t_{k}>0$ and $X_{1}, \ldots, X_{k} \in \Gamma$ such that

$$
g(T)=g_{0} \exp \left(t_{1} X_{1}\right) \cdots \exp \left(t_{k} X_{k}\right), \quad t_{1}+\cdots+t_{k}=T
$$

Proof : Let $g(\cdot):[0, T] \rightarrow G$ be a trajectory of $\Gamma$ with initial point $g_{0}$. Then there exist a partition $0=\tau_{0}<\tau_{1}<\cdots<\tau_{k}=T$ and elements $X_{1}, \ldots, X_{k} \in \Gamma$ such that

$$
t \in\left(\tau_{i-1}, \tau_{i}\right) \quad \Rightarrow \quad \dot{g}=g(t) X_{i} \quad(i=1,2, \ldots k)
$$

For $i=1$ :

$$
t \in\left(0, \tau_{1}\right) \quad \Rightarrow \quad \dot{g}=g(t) X_{1}, \quad g(0)=g_{0}
$$

It follows that

$$
g(t)=g_{0} \exp \left(t X_{1}\right)
$$

and also (by continuity)

$$
g\left(\tau_{1}\right)=g_{0} \exp \left(\tau_{1} X_{1}\right)
$$

For $i=2$ :

$$
t \in\left(\tau_{1}, \tau_{2}\right) \quad \Rightarrow \quad \dot{g}=g(t) X_{2}, \quad g\left(\tau_{1}\right)=g \exp \left(\tau_{1} X_{1}\right)
$$

It follows that

$$
\begin{aligned}
g(t) & =g_{0} \exp \left(\tau_{1} X_{1}\right) \exp \left(\left(t-\tau_{1}\right) X_{2}\right) \\
g\left(\tau_{2}\right) & =g_{0} \exp \left(\tau_{1} X_{1}\right) \exp \left(\left(\tau_{2}-\tau_{1}\right) X_{2}\right), \quad t_{1}=\tau_{1}, t_{2}=\tau_{2}-\tau_{1}
\end{aligned}
$$

and so on. Finally, we get (for $i=k$ ):

$$
g(T)=g\left(\tau_{k}\right)=g_{0} \exp \left(t_{1} X_{1}\right) \cdots \exp \left(t_{k} X_{k}\right)
$$

where $t_{k}=\tau_{k}-\tau_{k-1}, \ldots, t_{2}=\tau_{2}-\tau_{1}, t_{1}=\tau_{1} \quad\left(t_{1}+\cdots+t_{k}=T\right)$.

Now we can derive a description, as well as some elementary properties, of reachable sets.
2.4.3 Proposition. Let $\Gamma$ be a left-invariant control system on $G$ and let $g \in G$ be an arbitrary point. Then
(RS1) $\quad \mathbf{A}(g)=\left\{g \exp \left(t_{1} X_{1}\right) \cdots \exp \left(t_{k} X_{k}\right) \mid X_{i} \in \Gamma, t_{i} \geq 0, k \in \mathbb{N}\right\}$.
$(\mathrm{RS} 2) \quad \boldsymbol{A}(g)=g \mathbf{A}(e)$.
(RS3) $\boldsymbol{A}(e)$ is a subsemigroup of $G$.
(RS4) $\boldsymbol{A}(g)$ is a path-connected subset of $G$.

Proof : (RS1) and (RS2) follow immediately from Lemma 5.4.2.
(RS3) Since

$$
\mathbf{A}(e)=\left\{\exp \left(t_{1} X_{1}\right) \cdots \exp \left(t_{k} X_{k}\right) \mid X_{i} \in \Gamma, t_{i} \geq 0, k \in \mathbb{N}\right\}
$$

it follows that for any $g_{1}, g_{2} \in \mathbf{A}(e), g_{1} g_{2} \in \mathbf{A}(e)$.
(RS4) Any point in $\boldsymbol{A}(e)$ is connected with the initial point $g$ by a path $g(\cdot) \in \operatorname{Traj}(\Gamma)$.
2.4.4 Corollary. The left-invariant control system $\Gamma \subseteq \mathfrak{g}$ is controllable if and only if $\mathbf{A}(e)=G$.

Proof : By definition, $\Gamma$ is controllable if (and only if) $\boldsymbol{A}(g)=G$ for every $g \in G$. Since $\boldsymbol{A}(g)=g \boldsymbol{A}(e)$, it follows that controllability is equivalent to the condition $\boldsymbol{A}(e)=G$.

The orbit through the point $g \in G$ is denoted by $\mathcal{O}(g)$ and is defined as the set

$$
\mathcal{O}(g):=\{g(T) \mid g(\cdot) \in \operatorname{Traj}(\Gamma), g(0)=g, T \in \mathbb{R}\}
$$

This set is defined analogously to the reachable set $\boldsymbol{A}(g)$ but the terminal time $T$ may take both positive and negative values. The structure of orbits is simpler than that of reachable sets. Clearly (for $g \in G$ ),

$$
\boldsymbol{A}(g) \subseteq \mathcal{O}(g) .
$$

2.4.5 Proposition. Let $\Gamma$ be a left-invariant control system on $G$ and let $g \in G$ be an arbitrary point. Then
(O1) $\mathcal{O}(g)=\left\{g \exp \left(t_{1} X_{1}\right) \cdots \exp \left(t_{k} X_{k}\right) \mid X_{i} \in \Gamma, t_{i} \in \mathbb{R}, k \in \mathbb{N}\right\}$.
(O2) $\quad \mathcal{O}(g)=g \mathcal{O}(e)$.
(O3) $\mathcal{O}(e)$ is the connected Lie subgroup of $G$ with the Lie algebra Lie ( $\Gamma$ ).

Proof : (O1) and (O2) follow immediately from (RS1), (RS2) and the definition of an orbit.
(O3) The orbit $\mathcal{O}(e)$ is a subgroup of $G$. Indeed, if $g_{1}, g_{2} \in \mathcal{O}(e)$, then $g_{1} g_{2}^{-1} \in \mathcal{O}(e)$. For the Lie subalgebra Lie $(\Gamma) \subseteq \mathfrak{g}$, by the Orbit Theorem, the orbit $\mathcal{O}(e) \subseteq G$ is a connected, immersed submanifold such that $T_{e} \mathcal{O}(e)=$ Lie $(\Gamma)$. Then $\mathcal{O}(e)$ is a connected Lie subgroup of $G$ with the Lie algebra Lie ( $\Gamma$ ) (see the Lie correspondence).

Since all essential properties of reachable sets (including controllability) are expressed in terms of the reachable set from the identity $\boldsymbol{A}(e)$, in the sequel we restrict ourselves to this set and denote it by $\boldsymbol{A}$. Likewise, we denote the orbit (through the identity) $\mathcal{O}(e)$ simply by $\mathcal{O}$.

## Basic controllability conditions

Let $\Gamma \subseteq \mathfrak{g}$ be a left-invariant control system on the matrix Lie group $G$. We can see that a necessary condition for $\Gamma$ to be controllable is that $G$ be connected. Henceforth, all matrix Lie groups are assumed to be connected, unless otherwise stated.

We denote by Lie ( $\Gamma$ ) the Lie algebra generated by $\Gamma \subseteq \mathfrak{g}$ (i.e. the smallest subalgebra of $\mathfrak{g}$ containing $\Gamma$ ). It follows that $\operatorname{Lie}(\Gamma)$ is the smallest vector subspace $S$ of $\mathfrak{g}$ that also satisfies (for any $X \in \mathfrak{g}$ )

$$
[X, S] \subseteq S
$$

Lie $(\Gamma)$ can also be described in terms of the following notation : for each $X \in \mathfrak{g}$, let ad $X: \mathfrak{g} \rightarrow \mathfrak{g}$ denote the mapping $\operatorname{ad} X(Y):=[X, Y]$ for $Y \in \mathfrak{g}$ (ad : $X \mapsto \operatorname{ad} X$ is the adjoint representation of the Lie algebra $\mathfrak{g}$ ). Then Lie $(\Gamma)$ is equal to the smallest vector subspace $S$ of $\mathfrak{g}$ for which ad $X_{1} \circ$ ad $X_{2} \circ \cdots \circ$ ad $X_{k-1}\left(X_{k}\right) \in S$ for any finite set of elements $X_{1}, \ldots, X_{k} \in \Gamma$. That is,

$$
\operatorname{Lie}(\Gamma)=\operatorname{span}\left\{\operatorname{ad} X_{1} \circ \operatorname{ad} X_{2} \circ \cdots \circ \operatorname{ad} X_{k-1}\left(X_{k}\right) \mid X_{1}, \ldots, X_{k} \in \Gamma\right\}
$$

2.4.6 Proposition. If $\Gamma \subseteq \mathfrak{g}$ is controllable, then $\operatorname{Lie}(\Gamma)=\mathfrak{g}$.

Proof : If $\boldsymbol{A}=G$, then $\mathcal{O}=G$ and hence (by Proposition 5.4.5)

$$
\operatorname{Lie}(\Gamma)=\mathfrak{g}
$$

The condition that $\Gamma$ generates $\mathfrak{g}$ as a Lie algebra (i.e. Lie $(\Gamma)=\mathfrak{g}$ ) is referred to as the Lie algebra rank condition (LARC). A left-invariant control system $\Gamma$ satisfying (LARC) is said to have full rank.

Note : If a point $h \in G$ is reachable (or accessible) from a point $g \in G$, then there exist elements $X_{1}, \ldots, X_{k} \in \Gamma$ and $t_{1}, \ldots, t_{k} \in \mathbb{R}$ (with $t_{i}>0$ ) such that

$$
h=g \exp \left(t_{1} X_{1}\right) \cdots \exp \left(t_{k} X_{k}\right) .
$$

The following stronger concept turns out to be important in the study of topological properties of reachable sets (and hence of controllability). A point $h \in G$ is said to be normally accessible from a point $g \in G$ if there exists elements $X_{1}, \ldots, X_{k} \in \Gamma$ and $t_{1}, \ldots, t_{k} \in \mathbb{R}$ (with $t_{i}>0$ ) such that the mapping

$$
\Psi: \mathbb{R}^{k} \rightarrow G, \quad\left(s_{1}, \ldots, s_{k}\right) \mapsto g \exp \left(s_{1} X_{1}\right) \cdots \exp \left(s_{k} X_{k}\right)
$$

satisfies the following conditions :
(i) $\Psi\left(\left(t_{1}, \ldots, t_{k}\right)\right)=h$.
(ii) The rank of $\Psi$ at $t=\left(t_{1}, \ldots, t_{k}\right)$ is equal to $m$ (the dimension of $G$ ).
(The rank of $\Psi$ at $t \in \mathbb{R}^{k}$ is the rank of the differential $D \Psi(t)$.) We say that the point $h$ is normally accessible from the point $g$ by $X_{1}, \ldots, X_{k}$. It can be proved that if $\Gamma \subseteq \mathfrak{g}$ has full rank, then in any neighborhood $N$ of the identity $e \in G$ there are points normally accessible from $e$.

Exercise 13 Show that if $\Gamma \subseteq \mathfrak{g}$ has full rank (i.e. Lie $(\Gamma)=\mathfrak{g}$ ), then
(a) for any neighborhood $N$ of $e$, the set $\operatorname{int}(\boldsymbol{A}) \cap N$ is nonempty.
(b) the reachable set $\boldsymbol{A}$ has nonempty interior (i.e. $\Gamma$ is accessible at the identity).

In general, the Lie algebra rank condition (LARC) is not sufficient for controllability, but is equivalent to accessibility.
2.4.7 Proposition. The left-invariant control system $\Gamma \subseteq \mathfrak{g}$ is accessible at the identity (and thus at any point $g \in G$ ) if and only if $\operatorname{Lie}(\Gamma)=\mathfrak{g}$.

Proof : $\quad(\Leftarrow)$ If Lie $(\Gamma)=\mathfrak{g}$, then (by Exercise 252) int $(\boldsymbol{A})$ is nonempty (in $G$ ); that is, $\Gamma$ is accessible at the identity. Since the left translation $L_{g}$ is a homeomorphism, by Proposition 5.4.3, it follows that

$$
\operatorname{int}(\mathbf{A}(g))=\operatorname{int}(g \mathbf{A}) \neq \emptyset
$$

Thus $\Gamma$ is accessible at $g \in G$.
$(\Rightarrow) \quad$ Let $\operatorname{Lie}(\Gamma) \neq \mathfrak{g}$. Then

$$
\operatorname{dim} \mathcal{O}=\operatorname{dim} \operatorname{Lie}(\Gamma)<\operatorname{dim} \mathfrak{g}=\operatorname{dim} G
$$

Thus $\operatorname{int}(\mathcal{O})=\emptyset$ and so

$$
\operatorname{int}(\boldsymbol{A})=\emptyset
$$

### 2.4.8 Theorem. (Group Test) The left-invariant control system $\Gamma \subseteq \mathfrak{g}$

 is controllable if and only if(i) The reachable set $\boldsymbol{A}$ is a subgroup of $G$.
(ii) $\operatorname{Lie}(\Gamma)=\mathfrak{g}$.

Proof : $(\Rightarrow)$ Condition ( $i$ ) is obvious, and condition (ii) follows from the Lie algebra rank condition (LARC).
$(\Leftarrow)$ If $\boldsymbol{A} \subseteq G$ is a subgroup, then for any $g \in \boldsymbol{A}$, its inverse $g^{-1}$ also belongs to $\boldsymbol{A}$. For any exponential $\exp (t X) \in \boldsymbol{A}$, its inverse

$$
(\exp (t X))^{-1}=\exp (-t X) \in \mathbf{A} .
$$

Thus the reachable set $\boldsymbol{A}$ coincides with the orbit $\mathcal{O}$. But $\mathcal{O} \subseteq G$ is a connected Lie subgroup with Lie algebra $\operatorname{Lie}(\Gamma)=\mathfrak{g}$. Then (by the Lie correspondence) $\mathcal{O}=G$ and hence

$$
\boldsymbol{A}=\mathcal{O}=G .
$$

Note : A control system (on manifold $M$ ) is called locally controllable at a point $p \in M$ if $p \in \operatorname{int}(\boldsymbol{A}(p))$. For such general control systems, the local controllability property is weaker than the global controllability property. However, for left-invariant control systems on matrix Lie groups, these two notions coincide. Hence the following result holds :
(Local Controllability Test) The left-invariant control system $\Gamma \subseteq \mathfrak{g}$ is controllable if and only if the group identity $e$ is contained in the interior of $\mathbf{A}$.

This particular result can be used to derive another interesting test :
(Closure Test) The left-invariant control system $\Gamma \subseteq \mathfrak{g}$ is controllable if and only if the (topological) closure of the reachable set $\boldsymbol{A}$ is the entire group $G$ : $\operatorname{cl}(\boldsymbol{A})=G$.

This means that in the study of controllability one can replace the reachable set $\boldsymbol{A}$ by its closure $\operatorname{cl}(\boldsymbol{A})$. This fact has far-reaching consequences.

## Other controllability criteria

Let $\Gamma \subseteq \mathfrak{g}$ be a driftless (or homogeneous) left-invariant control system on $G$; that is,

$$
\begin{aligned}
\Gamma & =\left\{u_{1} A_{1}+u_{2} A_{2}+\cdots+u_{\ell} A_{\ell} \mid u=\left(u_{1}, \ldots, u_{\ell}\right) \in \mathbb{R}^{\ell}\right\} \\
& =\operatorname{span}\left\{A_{1}, \ldots, A_{\ell}\right\} \subseteq \mathfrak{g}
\end{aligned}
$$

where $A_{1}, \ldots, A_{\ell}$ are assumed to be linearly independent. (Again, for the sake of simplicity, the system is assumed to be unconstained : $U=\mathbb{R}^{\ell}$.)

Note : If $\Gamma \subseteq \mathfrak{g}$ is driftless, then together with any element $X$, it contains also the negative $-X$ :

$$
X \in \Gamma \Rightarrow-X \in \Gamma
$$

(This fact can also be expressed by saying that the "symmetry condition" : $\Gamma=-\Gamma$ is satisfied.)

Exercise 14 Show that if $\Gamma \subseteq \mathfrak{g}$ is a driftless left-invariant control system on $G$, then its reachable set $\boldsymbol{A}$ is a subgroup of $G$ and coincides with the orbit $\mathcal{O}$.

Thus deciding controllability for a driftless left-invariant control system $\Gamma \subseteq \mathfrak{g}$ reduces to verifying the algebraic condition of coincidence of the (connected) matrix Lie groups $\mathcal{O}$ and $G$.

Exercise 15 Show that a driftless left-invariant control system $\Gamma \subseteq \mathfrak{g}$ is controllable if and only if $\operatorname{Lie}(\Gamma)=\mathfrak{g}$.
2.4.9 THEOREM. Consider a driftless left-invariant control system

$$
\Gamma=\left\{u_{1} A_{1}+\cdots+u_{\ell} A_{\ell} \mid u=\left(u_{1}, \ldots, u_{\ell}\right) \in \mathbb{R}^{\ell}\right\} \subset \mathfrak{g}
$$

on a (not necessarily connected) matrix Lie group G. Then :
(i) The reachable set $\boldsymbol{A}$ coincides with the orbit $\mathcal{O}$ (i.e. the connected matrix Lie subgroup of $G$ with associated Lie algebra Lie $(\Gamma))$.
(ii) Any point of $\boldsymbol{A}$ can be reached from the identity $e \in G$ in an arbitrary time :

$$
\mathbf{A}(e, T)=\mathbf{A}=\mathcal{O} \quad \text { for any } T>0
$$

(iii) If $G$ is connected, then $\Gamma$ is controllable if and only if

$$
\operatorname{Lie}\left(\left\{A_{1}, \ldots, A_{\ell}\right\}\right)=\mathfrak{g}
$$

Proof : (i) and (iii) follow immediately from Exercise 253 and Exercise 254, respectively.

To prove (ii), choose any $T>0$. Let a point $g \in G$ be reachable from $e$ in some time $T_{1}>0:$

$$
g=\exp \left(t_{1} X_{1}\right) \cdots \exp \left(t_{k} X_{k}\right), \quad t_{1}+\cdots+t_{k}=T_{1}
$$

where $t_{1}, \ldots, t_{k}>0$ and $X_{1}, \ldots, X_{k} \in \Gamma$. The elements (vector fields)

$$
\widehat{X}_{i}:=\alpha X_{i}, \quad i=1,2, \ldots, k
$$

belong to $\Gamma$ for $\alpha=\frac{T_{1}}{T}$. Thus $g$ can be reached from the identity $e$ in time $T$ :

$$
g=\exp \left(s_{1} \widehat{X}_{1}\right) \cdots \exp \left(s_{k} \widehat{X}_{k}\right), \quad s_{1}+\cdots+s_{k}=T
$$

where $s_{i}=\frac{1}{\alpha} t_{i}, \quad i=1,2, \ldots, k$.

Note : For compact (connected) matrix Lie groups, the following result holds : The left-invariant control system $\Gamma \subseteq \mathfrak{g}$ is controllable if and only if $\operatorname{Lie}(\Gamma)=\mathfrak{g}$. (Moreover, if $\Gamma$ is controllable, then there exists $T>0$ such that, for every two points $g_{0}, g_{1} \in G$, there is a control $u(\cdot)$ that steers $g_{0}$ into $g_{1}$ in no more than $T$ units of time.)

Exercise 16 Let $A_{0}, A_{1}$ be any two linearly independent $3 \times 3$ real skew-symmetric matrices (i.e. two linearly independent elements of the Lie algebra $\mathfrak{s o}$ (3)). Show that the left-invariant control system

$$
\Gamma=A_{0}+\operatorname{span}\left\{A_{1}\right\} \subseteq \mathfrak{s o}(3)
$$

(or, in classical notation,

$$
\left.\dot{g}=g\left(A_{0}+u A_{1}\right), \quad g \in \mathrm{SO}(3), u \in \mathbb{R}\right)
$$

is controllable.

Exercise 17 Investigate for controllability each of the following driftless left-invariant control systems on a specific (connected) matrix Lie group $G$ :
(a) The Brockett system on $G=\mathrm{H}$ (1).
(b) The unicycle on $G=\mathrm{SE}(2)$.
(c) The spacecraft on $G=\mathrm{SO}$ (3).
(d) The autonomous underwater vehicle (AUV) on $G=\mathrm{SE}(3)$.
(e) The kinematic car on $G=\mathrm{G}_{4}$.

### 2.5 Linear Control Systems

$\qquad$

### 2.6 Serret-Frenet Control Systems

The arc length parametrization of a (geometric) curve describing the path of (the center of mass of) a rigid body in Euclidean 3-space can be used to express the state equation of the "motion" of this (left-invariant) control system (on the special Euclidean group SE (3)).

Consider a unit-speed curve $x(\cdot)$ in $\mathbb{E}^{3}$.

Note : The map $t \mapsto x(t) \in \mathbb{E}^{3}$ is assumed to be smooth. For the sake of convenience, we use the variable $t$ (instead of $s$ ) for the arc length parameter of the curve.

The Serret-Frenet frame $(T, N, B)$ along the curve $x(\cdot)$ is described by the (unipotent orthogonal) matrix

$$
R(t):=\left[\begin{array}{lll}
T(t) & N(t) & B(t)
\end{array}\right] \in \mathrm{SO}(3)
$$

that relates this frame to the natural frame $\left(e_{1}, e_{2}, e_{3}\right)$ in $\mathbb{E}^{3}$ (we have omitted any notational distinctions between tangent vectors and parallel vector fields) and that further satisfies the following differential equation (in matrices) :

$$
\dot{R}=R\left[\begin{array}{ccc}
0 & -\kappa & 0 \\
\kappa & 0 & -\tau \\
0 & \tau & 0
\end{array}\right]
$$

where $\kappa(\cdot)$ and $\tau(\cdot)$ represent the curvature and torsion function, respectively.

Note : $\quad R(\cdot)$ is the attitude matrix of the frame field $(T, N, B)$ and the differential equation satisfied by $R(\cdot)$ represents the Serret-Frenet formulas. Clearly,

$$
R(t) e_{1}=T, \quad R(t) e_{2}=N, \quad R(t) e_{3}=B
$$

The curve $x(\cdot)$ (i.e.

$$
\left.t \mapsto x(t)=\left[\begin{array}{l}
x_{1}(t) \\
x_{2}(t) \\
x_{3}(t)
\end{array}\right] \in \mathbb{R}^{3 \times 1}\right)
$$

and the rotation matrix $R(t) \in \mathrm{SO}(3)$ can be expressed as (the curve)

$$
g(t)=\left[\begin{array}{cc}
1 & 0 \\
x(t) & R(t)
\end{array}\right] \in \mathrm{SE}(3)
$$

in the (matrix Lie) group of proper rigid motions on the Euclidean 3 -space $\mathbb{E}^{3}$. Since

$$
\dot{x}=T=R(t) e_{1},
$$

we get

$$
\begin{aligned}
\dot{g} & =g\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 0 & -\kappa & 0 \\
0 & \kappa & 0 & -\tau \\
0 & 0 & \tau & 0
\end{array}\right] \\
& =g\left(X_{0}+\kappa X_{1}+\tau X_{2}\right), \quad g \in \operatorname{SE}(3)
\end{aligned}
$$

where

$$
X_{0}=\left[\begin{array}{llll}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right], \quad X_{1}=\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right], \quad X_{2}=\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{array}\right] .
$$

Note : Recall that the matrices $A_{1}, A_{2}, \ldots, A_{6}$ defined by

$$
A_{i}:=\left[\begin{array}{cc}
0 & 0 \\
0 & \widehat{e}_{i}
\end{array}\right], \quad A_{i+3}:=\left[\begin{array}{cc}
0 & 0 \\
e_{i} & 0
\end{array}\right], \quad i=1,2,3
$$

form a basis for the Lie algebra $\mathfrak{s e}(3)$ associated with the special Euclidean group SE (3). We can see that

$$
X_{0}=\left[\begin{array}{cc}
0 & 0 \\
e_{1} & 0
\end{array}\right]=A_{4}, \quad X_{1}=\left[\begin{array}{cc}
0 & 0 \\
0 & \widehat{e}_{3}
\end{array}\right]=A_{3}, \quad X_{2}=\left[\begin{array}{cc}
0 & 0 \\
0 & \widehat{e}_{1}
\end{array}\right]=A_{1}
$$

2.6.1 Definition. The left-invariant control system on the special Euclidean group SE (3), written as

$$
\dot{g}=g\left(X_{0}+\kappa X_{1}+\tau X_{2}\right), \quad g \in \mathrm{SE}(3)
$$

with the curvature and torsion functions playing the role of controls, is called the Serret-Frenet control system (on SE (3)).

For $\tau(\cdot)=0$, we obtain a left-invariant control system (on SE (2) ), described by (the state equation)

$$
\begin{aligned}
\dot{g} & =g\left[\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & -\kappa \\
0 & \kappa & 0
\end{array}\right] \\
& =g\left(X_{0}^{\prime}+\kappa X_{1}^{\prime}\right), \quad g \in \operatorname{SE}(2), \kappa \geq 0 .
\end{aligned}
$$

Consider now the special case where the torsion function $\tau(\cdot)$ is constant. This assumption reduces the number of controls to a single control ( $u:=\kappa$ ) and introduces a drift term in the rotational part of the equation (corresponding to the constant torsion).

Under this assumption, the differential equation satisfied by the rotation matrix $R(\cdot)$ can be written

$$
\dot{R}=R(A+u B), \quad R \in \mathrm{SO}(3), u \geq 0
$$

where

$$
A=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -\tau \\
0 & \tau & 0
\end{array}\right], \quad B=\left[\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

Note : We have

$$
A=\tau \widehat{e}_{1}, \quad B=\widehat{e}_{2}
$$

where $\left\{\widehat{e}_{1}, \widehat{e}_{2}, \widehat{e}_{3}\right\}$ is the standard basis for the Lie algebra $\mathfrak{s o}(3)$.
We call the forgoing (left-invariant) control system the stiff Serret-Frenet control system (on SO (3)).

Note : Writing $h(t)$ for the matrix $R(t)^{-1}$, turns this left-invariant control system into a right-invariant control system

$$
\dot{h}=-(A+u B) h, \quad h \in \mathrm{SO}(3), u \geq 0
$$

The matrix Lie group $\mathrm{SO}(3)$ is compact and connected, and hence the stiff Serret-Frenet control system is controllable if and only if the set $\Gamma=$ $\{A+u B \mid u \geq 0\}$ generates $\mathfrak{s o}(3)$ as a Lie algebra; that is,

$$
\operatorname{Lie}(\{A, B\})=\mathfrak{s o}(3)
$$

Exercise 18 Show that if the fixed torsion $\tau$ is nonzero in the expression for $A$, then the Lie algebra generated by $A$ and $B$ equals the Lie algebra $\mathfrak{s o}(3)$.

Consider the (left-invariant) Serret-Frenet control system on SE (2)

$$
\dot{g}=g(X+u Y), \quad g \in \mathrm{SE}(2), u \in \mathbb{R}
$$

where

$$
X=\left[\begin{array}{ccc}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] \quad \text { and } \quad Y=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right]
$$

Exercise 19 Calculate the Lie algebra generated by $\{X, Y\}$.

Note : For a left-invariant control system on a connected (but not compact) matrix Lie group, the Lie algebra rank condition (LARC) is only a necessary condition. The Serret-Frenet control system on (the noncompact) matrix Lie group $\operatorname{SE}(2)$ is, in fact, controllable.

Consider now the matrix Lie group $\operatorname{SE}(n)$. Recall that an arbitrary element $g \in \operatorname{SE}(n)$ can be expressed as a matrix $\left[\begin{array}{ll}1 & 0 \\ c & R\end{array}\right] \in \operatorname{GL}(n+1, \mathbb{R})$ with $c \in \mathbb{R}^{n \times 1}$ and $R \in \operatorname{SO}(n)$. We may denote such an element by $(c, R) \in \mathbb{R}^{n} \times \mathrm{SO}(n)$.

Note : The group product in $\mathbb{R}^{n} \times \mathrm{SO}(n)$ is defined by

$$
\left(c_{1}, R_{1}\right) \cdot\left(c_{2}, R_{2}\right):=\left(c_{1}+R_{1} c_{2}, R_{1} R_{2}\right)
$$

We say that (the group) $\mathrm{SE}(n)$ is the semidirect product of (the vector space) $\mathbb{R}^{n}$ and (the group) $\mathrm{SO}(n)$ and write $\mathrm{SE}(n)=\mathbb{R}^{n} \ltimes \mathrm{SO}(n)$.

Likewise, the Lie algebra $\mathfrak{s e}(n)$ of the special Euclidean group $\operatorname{SE}(n)$ is the semidirect sum $\mathbb{R}^{n} \lambda \mathfrak{s o}(n)$; that is, the vector space $\mathfrak{s e}(n)$ is the direct sum of the vector spaces $\mathbb{R}^{n}$ and $\mathfrak{s o}(n)$, and the Lie bracket is as follows :

$$
[(a, A),(b, B)]:=(A b-B a,[A, B]) .
$$

Exercise 20 Verify that the commutator of the matrices

$$
M_{1}=\left[\begin{array}{cc}
0 & 0 \\
a_{1} & A_{1}
\end{array}\right] \quad \text { and } \quad M_{2}=\left[\begin{array}{cc}
0 & 0 \\
a_{2} & A_{2}
\end{array}\right]
$$

is

$$
\left[M_{1}, M_{2}\right]=\left[\begin{array}{cc}
0 & 0 \\
A_{1} a_{2}-A_{2} a_{1} & A_{1} A_{2}-A_{2} A_{1}
\end{array}\right]
$$

(Observe that for $a \in \mathbb{R}^{n \times 1}$ and $A \in \mathfrak{s o}(n)$, the matrix $\left[\begin{array}{ll}0 & 0 \\ a & A\end{array}\right]$ is an element of $\mathfrak{s e}(n)$.)

Let

$$
\pi: \mathrm{SE}(n) \rightarrow \mathrm{SO}(n), \quad(c, R) \mapsto R
$$

denote the projection (on the second factor). Projection $\pi$ is a Lie homomorphism, and hence the derivative

$$
d \pi=\pi_{*}: \mathfrak{s e}(n) \rightarrow \mathfrak{s o}(n)
$$

is a Lie algebra homomorphism (see Theorem 3.4.17).
2.6.2 Theorem. A left-invariant control system $\Gamma \subseteq \mathfrak{s e}(n)$ on the special Euclidean group $\mathrm{SE}(n)$ is controllable if and only if

$$
\operatorname{Lie}(\Gamma)=\mathfrak{s e}(n) .
$$

Proof : ( $\Rightarrow$ ) : The Lie algebra rank condition Lie $(\Gamma)=\mathfrak{s e}(n)$ is necessary for controllability (see Proposition 5.4.6).
$(\Leftarrow): \quad$ Assume that Lie $(\Gamma)=\mathfrak{s e}(n)$. Then the left-invariant control system $\Gamma_{\mathrm{SO}_{(n)}}:=\pi_{*}(\Gamma) \subset \mathfrak{s o}(n)$ on $\mathrm{SO}(n)$ is controllable since $\mathrm{SO}(n)$ is compact and connected (see the note after Theorem 5.4.9). That is,

$$
\pi(\boldsymbol{A})=\mathrm{SO}(n) .
$$

It follows (see Corollary 5.4.4) that it is sufficient to show that the group identity $e=(0, I) \in \mathrm{SE}(n)=\mathbb{R}^{n} \ltimes \mathrm{SO}(n)$ is contained in the interior of $\boldsymbol{A}$.

Let $(x, g) \in \operatorname{int}(\boldsymbol{A}) \neq \emptyset$. There exists $y \in \mathbb{R}^{n}$ such that $\left(y, g^{-1}\right) \in \boldsymbol{A}$. Then $(x, g) \cdot\left(y, g^{-1}\right)=(x+g y, I)$, and this product is in the interior of $(\boldsymbol{A})$.

Denote $x+g y$ by $v$. Let $\Omega$ be a neighborhood of $I$ in $\mathrm{SO}(n)$ such that $(v, \Omega) \subset \operatorname{int}(\boldsymbol{A})$. For any $h \in \Omega$ and $r \in \mathbb{N}$, the element

$$
(v, h)^{r}=\left(v+h v+\cdots+h^{r-1} v, h^{r}\right)
$$

is contained in $\operatorname{int}(\boldsymbol{A})$. If $h^{r}=I$, and if $v=h w-w$ for some $w \in \mathbb{R}^{n}$, then $v+h v+\cdots+h^{r-1} v=0$ and $(0, I) \in \operatorname{int}(\boldsymbol{A})$.

To finish the proof, we need to show that for any $v \in \mathbb{R}^{n}$ and any neighborhood $\Omega$ of $I$ in $\mathrm{SO}(n)$, there exists an element $h \in \Omega$ such that

- $v=h w-w$ for some $w \in \mathbb{R}^{n}$
- $h^{s}=I$ for some $s \in \mathbb{N}$.
(We outline a proof. Let $P$ denote a plane in $\mathbb{R}^{n}(n \geq 2)$ that contains a given point $v \in \mathbb{R}^{n}$. Then for any neighborhood $\Omega$ of $I$ in $\mathrm{SO}(2, P)$, there exists a rotation $R \in \Omega$ such that $R-I$ is nonsingular and $R^{s}=I$ for some $s \in \mathbb{N}$. Then $R$ can be extended to $\mathbb{R}^{n}$ by defining it equal to the identity on the orthogonal complement $P^{\perp}$ of $P$ in $\mathbb{R}^{n}$. Hence

$$
\left.v \in \operatorname{im}(R-I) \quad \text { and } \quad R^{s}=I .\right)
$$

Note : A more general result holds : Let $K$ be a compact connected Lie group which acts linearly on a (real) vector space $V$, and suppose that $V$ admits no nonzero fixed points (with respect to $K$ ). Then a left-invariant control system $\Gamma \subseteq \mathfrak{g}$ on the Lie group $G=V \ltimes K$ is controllable if and only if $\operatorname{Lie}(\Gamma)=\mathfrak{g}$.

Besides the case $G=\mathbb{R}^{n} \ltimes \mathrm{SO}(n)$, another interesting case (in applications) is $G=\mathbb{R}^{2 n} \ltimes \mathrm{U}(2 n)$.

Theorem 5.5.2 has far-reaching implications (in the theory of curves), as the following examples illustrate.
2.6.3 Example. The Serret-Frenet system associated with a curve $x(\cdot)$ in $\mathbb{E}^{3}$ is given by

$$
\dot{x}=R(t) e_{1} \quad \text { and } \quad \dot{R}=R\left[\begin{array}{ccc}
0 & -\kappa & 0 \\
\kappa & 0 & -\tau \\
0 & \tau & 0
\end{array}\right]
$$

If both the curvature $\kappa$ and the torsion $\tau$ are constant, then

$$
\omega=\left[\begin{array}{l}
\tau \\
0 \\
\kappa
\end{array}\right]
$$

is the axis of rotation for

$$
A=\left[\begin{array}{ccc}
0 & -\kappa & 0 \\
\kappa & 0 & -\tau \\
0 & \tau & 0
\end{array}\right]
$$

Then $\exp (t A)$ is the rotation about $\omega$ through the angle $t \sqrt{\tau^{2}+\kappa^{2}}$, and $x(\cdot)$ is a helix (along $\omega$ ).
2.6.4 Example. Suppose now that we consider curves whose curvature $\kappa=$ constant $(\neq 0)$ and whose torsion can take only two distinct values : $\tau_{1}$ and $\tau_{2}$. Such curves are concatenations of helices along

$$
\omega_{1}=\left[\begin{array}{c}
\tau_{1} \\
0 \\
\kappa
\end{array}\right] \quad \text { and } \quad \omega_{2}=\left[\begin{array}{c}
\tau_{2} \\
0 \\
\kappa
\end{array}\right]
$$

The corresponding family of left-invariant vector fields on the special Euclidean group $\operatorname{SE}(3)=\mathbb{R}^{3} \ltimes \mathrm{SO}(3)$ is

$$
\Gamma=\left\{\left(e_{1}, A\right),\left(e_{1}, B\right)\right\} \subset \mathfrak{s e}(3)=\mathbb{R}^{3} \lambda \mathfrak{s o}(3)
$$

with

$$
A=\left[\begin{array}{ccc}
0 & -\kappa & 0 \\
\kappa & 0 & -\tau_{1} \\
0 & \tau_{1} & 0
\end{array}\right] \quad \text { and } \quad B=\left[\begin{array}{ccc}
0 & -\kappa & 0 \\
\kappa & 0 & -\tau_{2} \\
0 & \tau-2 & 0
\end{array}\right]
$$

It follows that $\operatorname{Lie}(\Gamma)=\mathbb{R}^{3} \lambda \mathfrak{s o}$ (3) because of the following calculations:

$$
\left(e_{1}, A\right)-\left(e_{1}, B\right)=\left(\tau_{1}-\tau_{2}\right)\left(0, A_{1}\right)
$$

and

$$
\left[\left(e_{1}, A\right),\left(e_{1}, B\right)\right]=\left(\tau_{1}-\tau_{2}\right)\left(0, A_{2}\right)
$$

where we denote

$$
A_{1}:=E_{23}, \quad A_{2}:=E_{13}, \quad \text { and } \quad A_{3}:=E_{12} .
$$

(see Proposition 3.4.9). Then $\left[\left(0, A_{1}\right),\left(0, A_{2}\right)\right]=\left(0, A_{3}\right)$, and therefore $(0, \mathfrak{s o}(3)) \subset \operatorname{Lie}(\Gamma)$. Hence $\left(e_{1}, 0\right) \in \operatorname{Lie}(\Gamma)$, and then $\left[\left(e_{1}, 0\right),(0, \mathfrak{s o}(3))\right]=$ $\left(\mathbb{R}^{3}, 0\right) \subset \operatorname{Lie}(\Gamma)$. Thus

$$
\operatorname{Lie}(\Gamma)=\mathbb{R}^{3} \lambda \mathfrak{s o}(3)=\mathfrak{s e}(3) .
$$

According to Theorem 5.5.2, any initial point $x_{0} \in \mathbb{E}^{3}$ and any initial frame at $x_{0}$ can be connected to any terminal point $x_{1} \in \mathbb{E}^{3}$ and any terminal frame at $x_{1}$ along the integral curves of the left-invariant family $\Gamma=\left\{X_{A}, X_{B}\right\}$ in $\operatorname{SE}(3)=\mathbb{R}^{3} \ltimes \mathrm{SO}$ (3) (with $X_{A}$ and $X_{B}$ equal to the left-invariant vector fields that coincide with $\left(e_{1}, A\right)$ and $\left(e_{1}, B\right)$ at the group identity, respectively).

## Problems and Further Results

