## Chapter 3

## Optimal Control

## Topics :

1. Optimal Control Problems
2. Pontryagin's Maximum Principle
3. Simple Examples
4. The Linear Time-Optimal Problem
5. The Linear-Quadratic Problem
6. Optimal Control on Matrix Lie Groups

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### 3.1 Optimal Control Problems

Optimal control theory, recognized initially as an engineering subject, reveals a distinct relationship to classic forebears : the calculus of variations, differential geometry, and mechanics. This distinctive character of optimal control theory can be traced back to the mathematical problems of the subject in the mid 1950s dealing with inequality constraints. Faced with the practical, time-optimal control problems of that period, mathematicians (and engineers) looked to the calculus of variations for answers, but soon discovered that the answers to their problems were outside the scope of the classic theory (and would require different mathematical tools). That realization initiated a search for new necessary conditions for optimality suitable for control problems. That search, further intensified by the space programme and the race to the moon, eventually led to the "maximum principle" (1959), due to the Russian mathematician Lev S. Pontryagin (1908-1988) and his co-workers.

Note : Optimal control is significantly richer and broader than the calculus of variations, from which it differs in some fundamental ways. The calculus of variations deals mainly with optimization problems of the following "standard" form :

$$
\mathcal{J}=\int_{t_{0}}^{t_{1}} L(x(t), \dot{x}(t), t) d t \rightarrow \min
$$

subject to

$$
x\left(t_{0}\right)=x_{0} \quad \text { and } \quad x\left(t_{1}\right)=x_{1}
$$

or, equivalently, of the form

$$
\mathcal{J}=\int_{t_{0}}^{t_{1}} L(x(t), u(t), t) d t \rightarrow \min
$$

subject to

$$
x\left(t_{0}\right)=x_{0}, \quad x\left(t_{1}\right)=x_{1}, \quad \text { and } \quad \dot{x}(t)=u(t) \text { for } t_{0} \leq t \leq t_{1} .
$$

The distinct feature of these problems is that the minimization takes place in the space of "all" curves, so nothing interesting happens on the level of the set of curves under consideration, and all the nontrivial features of the problem arise because of the Lagrangian L. Optimal control problems, by contrast, involve a minimization over a set $\mathcal{C}$ of curves which is itself determined by some dynamical constraints. For example, $\mathcal{C}$ might be the set of all curves $t \mapsto x(t)$ that satisfy a differential equation

$$
\dot{x}=F(x, u, t)
$$

for some choice of the "control function" $t \mapsto u(t)$. (Even more precisely, since it may happen that a member of $\mathcal{C}$ does not uniquely determine the control $u(\cdot)$ that generates it, we should be talking about trajectory-control pairs $(x(\cdot), u(\cdot)))$. So in an optimal control problem there are at least two objects that give the situation interesting structure, namely, the dynamics $F$ and the functional $\mathcal{J}$ to be minimized. In particular, optimal control theory contains at the opposite extreme from the calculus of variations, problems where the "Lagrangian" $L$ is (identically) 1 (i.e. completely trivial), and therefore the interesting action occurs because of the dynamics $F$. Such problems, in which it is desired to minimize time (i.e. the integral of $\mathcal{J}$ with $L \equiv 1$ ) among all curves $t \mapsto x(t)$ that satisfy endpoint constraints and are solutions of (time-dependent) differential equations for some control $t \mapsto u(t)$, are called timeoptimal problems. It is in these problems that the difference between optimal control and the calculus of variatons is most clearly seen, and it is no accident that these were the problems that propelled the development of optimal control in the early 1960s, and that time-optimal control is prominently represented in today's research.

## Problem statement

Consider a control system of the form

$$
\dot{x}=F(x, u), \quad x \in M, u \in U \subseteq \mathbb{R}^{m}
$$

where the state space $M$ is a smooth manifold and the control set $U$ is an arbitrary subset of $\mathbb{R}^{m}$. We shall assume that

- for each $u \in U$, the mapping $F_{u}=F(\cdot, u): M \rightarrow T M$ is a smooth vector field on $M$
- the mapping $F: M \times U \rightarrow T M$ is continuous (or, most often, smooth). The class of admissible controls $\mathcal{U}$ is the set of all (essentially) bounded measurable $U$-valued mappings (defined on some compact interval $\left[t_{0}, t_{1}\right]$ ). (For simplicity, one can consider piecewise continuous controls.)

Note : Let $J$ be an interval in $\mathbb{R}$ and $U$ an arbitrary subset of $\mathbb{R}^{m}$.
(a) A piecewise constant mapping $\omega: J \rightarrow U$ is one that is constant in each element $J_{i}$ of a finite partition of $J$ into subintervals.
(b) A mapping $u: J \rightarrow U$ is measurable if there exists some sequence $\left(\omega_{r}\right)_{r \geq 1}$ of piecewise constant mappings so that $\omega_{r} \rightarrow u$ almost everywhere (i.e. the set $\left\{t \in J \mid \omega_{r}(t) \nrightarrow u(t)\right\}$ has measure zero). Clearly, piecewise continuous mappings are measurable (and, in general, $\varphi \circ u$ is measurable if $u$ is measurable and $\varphi$ is continuous).
(c) A mapping $u: J \rightarrow U$ is (essentially) bounded if it is measurable and there exists a compact subset $K \subseteq U$ such that $u(t) \in K$ for almost all $t \in J$. Piecewise continuous mappings (with $J$ compact) are (essentially) bounded.

If $u(\cdot)$ is an admissible control, there is always a sequence $\left(\omega_{r}\right)_{r \geq 1}$ of piecewise constant mappings, converging almost everywhere to $u(\cdot)$. (Often one can obtain approximations by more regular controls. For instance, if $U$ is convex, then each piecewise constant control can be approximated almost everywhere by continuous controls, and hence every (essentially) bounded measurable control can be approximated by a sequence of continuous controls. If, in addition, $U$ is open, then one can
approximate (as long as the interval $J$ is finite) by analytic, and even polynomial, controls.)

We shall use $\mathcal{F}$ to denote the family of (smooth) vector fields $\mathcal{F}=$ $\left\{F_{u} \mid u \in U\right\}$ generated by $F$. A continuous curve $x(\cdot):\left[t_{0}, t_{1}\right] \rightarrow M$ is called a trajectory of $\mathcal{F}$ if there exists a partition $t_{0}=\tau_{0}<\tau_{1}<\cdots<\tau_{m}=t_{1}$ and vector fields $X_{1}, \ldots, X_{m}$ in $\mathcal{F}$ such that the restriction of $x(\cdot)$ to each open interval $\left(\tau_{i-1}, \tau_{i}\right)$ is smooth and (for $\left.t \in\left(\tau_{i-1}, \tau_{i}\right)\right)$

$$
\dot{x}(t)=X_{i}(x(t)), \quad i=1,2, \cdots m
$$

Note : Because the elements of $\mathcal{F}$ are parametrized by controls, it follows that each $X_{i}$ is equal to $F_{u_{i}}$ for some $u_{i} \in U$. Hence $x(\cdot)=x_{u}(\cdot)$ is an integral curve of the time-varying vector field (on $M)(t, x) \mapsto F(x, u(t)$ ), with $u(\cdot)$ equal to the piecewise constant control which takes constant value $u_{i}$ in each subinterval $\left[\tau_{i-1}, \tau_{i}\right]$ (see Definition 4.3.2).

We shall refer to a trajectory-control pair $(x(\cdot), u(\cdot))$ as a controlled trajectory. (In some cases, a trajectory $x(\cdot)$ cannot arise from more than one control $u(\cdot)$, so it is not necessary to distinguish between "trajectories" and "controlled trajectories".)

In order to compare admissible controls one with another (on an interval $\left.\left[t_{0}, t_{1}\right]\right)$, introduce a cost functional

$$
\mathcal{J}(u):=\int_{t_{0}}^{t_{1}} L(x(t), u(t)) d t
$$

(The integrand $L: M \times U \rightarrow \mathbb{R}$, called the Lagrangian, satisfies the same regularity assumptions as $F$.) Let $x_{0}, x_{1} \in M$. We formulate the following problem :
"MINIMIZE THE COST FUNCTIONAL $\mathcal{J}$ IN THE CLASS OF ALL CONTROLLED TRAJECTORIES $(x(\cdot), u(\cdot))$ SUCH THAT

$$
x\left(t_{0}\right)=x_{0} \quad \mathrm{AND} \quad x\left(t_{1}\right)=x_{1} . "
$$

A controlled trajectory $(x(\cdot), u(\cdot)):\left[t_{0}, t_{1}\right] \rightarrow M \times U$ such that $x\left(t_{0}\right)=x_{0}$ and $x\left(t_{1}\right)=x_{1}$ is said to transfer (or steer) the initial point (state) $x_{0}$ to the final point (state) $x_{1}$ over the time interval $\left[t_{0}, t_{1}\right]$.

We shall refer to this problem as the optimal control problem (OCP) :

$$
\begin{gathered}
\dot{x}=F(x, u), \quad x \in M, u \in U \\
x\left(t_{0}\right)=x_{0}, \quad x\left(t_{1}\right)=x_{1} \quad\left(x_{0}, x_{1} \text { fixed }\right) \\
\mathcal{J}=\int_{t_{0}}^{t_{1}} L(x(t), u(t)) d t \rightarrow \min
\end{gathered}
$$

Note : The length of time required to transfer $x_{0}$ to $x_{1}$ is not fixed in advance. On the other hand, if the controlled trajectory $(x(\cdot), u(\cdot))$ transfers $x_{0}$ to $x_{1}$ over the interval $\left[t_{0}, t_{1}\right]$, then the "time-shifted" controlled trajectory $(\bar{x}(\cdot), \bar{u}(\cdot))$ with $\bar{x}(t)=x\left(t+t_{0}\right)$ and $\bar{u}(t)=u\left(t+t_{0}\right)$, transfers $x_{0}$ to $x_{1}$ over the interval $\left[0, t_{1}-t_{0}\right]$, and the cost of the transfer along $(\bar{x}(\cdot), \bar{u}(\cdot))$ is the same as the cost of transfer along $(x(\cdot), u(\cdot))$. Hence, the initial time $t_{0}$ can always be taken to be 0 .

One study two types of problems : with fixed (final time) $t_{1}$, and with free $t_{1}$. A solution $(\bar{x}(\cdot), \bar{u}(\cdot))$ of the OCP is called optimal; the admissible control $\bar{u}(\cdot)$ is called and optimal control, and the corresponding trajectory (curve) $\bar{x}(\cdot)$ is the optimal trajectory.

Note : The optimal control problem (OCP) is the minimization problem for $\mathcal{J}(u)$ with constraints on $u$ given by the (state equation of the) control system and the
fixed endpoints conditions. These constraints cannot usually be resolved w.r.t. u, hence solving optimal control problems requires special techniques.

## Existence of optimal solutions

Optimal control problems on the state space $M$ can be essentially reduced to the study of attainable sets of some auxiliary control systems. Indeed, the integrant $L$ of the cost functional $\mathcal{J}$ (to be minimized) and the dynamics $F$ of the control system can be viewed as (defining) an extended control system on $M_{\text {ext }}:=\mathbb{R} \times M$ :

$$
\begin{aligned}
\dot{\xi} & =L(x, u) \\
\dot{x} & =F(x, u), \quad(\xi, x) \in M_{\mathrm{ext}}, u \in U .
\end{aligned}
$$

Then trajectories $x_{\text {ext }}(\cdot)$ of the extended control system (with initial conditions $\left.x_{\text {ext }}(0)=\left(0, x_{0}\right)\right)$ are expressed through trajectories of the initial control system as

$$
x_{\mathrm{ext}}(t)=\left(\int_{0}^{t} L(x(\tau), u(\tau)) d \tau, x(t)\right)
$$

(Trajectories of the extended control systems are curves $t \mapsto(\xi(t), x(t))$ in $\mathbb{R} \times M$ parametrized by the control functions $u(\cdot)$.

It turns out that optimal trajectories of the OCP on $M$ (more precisely, their lift to the extended state space $M_{\text {ext }}$ ) must come to the boundary of the attainable set $\boldsymbol{A}_{\text {ext }}\left(\left(0, x_{0}\right), t_{1}\right)$. Hence, in order to find optimal trajectories, we find first those coming to the boundary of $\boldsymbol{A}_{\text {ext }}\left(\left(0, x_{0}\right), t_{1}\right)$, and then select the optimal ones among them.

Note : The first step is much more important than the second one, so solving OCPs essentially reduces to the study of (dynamics of boundary of) attainable sets.

Due to the reduction of OCPs to the study of attainable sets, existence of optimal solutions to OCPs is reduced to compactness of attainable sets. For control systems

$$
\dot{x}=F(x, u), \quad x \in M, u \in U
$$

sufficient conditions for compactness of attainable sets are given in the following proposition (given without proof), due to the Russian mathematician Alexei F. Filippov (1923-).
3.1.1 Proposition. Let the control set $U \subset \mathbb{R}^{m}$ be compact. Assume that
(i) There exists a compact set $K \subset M$ such that $F(x, u)=0$ for $x \notin K$ and $u \in U$.
(ii) The velocity sets

$$
F_{U}(x):=\{F(x, u) \mid u \in U\} \subseteq T_{x} M, \quad x \in M
$$

are convex.

Then the attainable sets $\boldsymbol{\mathcal { A }}\left(x_{0}, t\right)$ and $\boldsymbol{\mathcal { A }}\left(x_{0}, \leq T\right)$ are compact for all $x_{0} \in$ $M$, and $T>0$.

Note : In Filippov's theorem, the hypothesis of common support of the vector fields (in the right-hand side) is essential to ensure the completeness of vector fields (and also the uniform boundedness of velocities). On a manifold, sufficient conditions for completeness of vector fields cannot be given in terms of boundedness of the vector field and its derivatives: a constant (parallel) vector field is not complete on a bounded domain in $\mathbb{R}^{n}$. Nevertheless, one can prove compactness of attainable sets for many control systems without the assumption of common compact support. (If for such a system we have a priori bounds on the solution, then we can multiply its
right-hand side by a "cut-off" function, and obtain a control system with vector fields having compact support. We can apply Filippov's theorem to this new system; since trajectories of the initial and new control systems coincide in a domain of interest for us, we obtain a conclusion on compactness of attainable sets for the initial system.)

For control systems on $M=\mathbb{E}^{n}$, there exist well-known sufficient conditions for completeness of vector fields : if the right-hand side grows at infinity not faster than a linear one (i.e., for some constant $C$,

$$
\left.\|F(x, u)\| \leq C(1+\|x\|), \quad x \in \mathbb{E}^{n}, u \in U\right)
$$

then the (time-varying) vector fields $F_{u}$ are complete. These conditions provide an a priori bound for solutions : any solution $x(\cdot)$ of the control system

$$
\dot{x}=F(x, u), \quad x \in \mathbb{E}^{n}, u \in U
$$

with the right-hand side satisfying the above condition, admits the bound

$$
\|x(t)\| \leq e^{2 C t}(\|x(0)\|+1), \quad t \geq 0
$$

Filippov's theorem (plus the preceding remark) implies the following sufficient condition for compactness of attainable sets for control systems on $\mathbb{E}^{n}$.
3.1.2 Corollary. Let the control set $U \subset \mathbb{R}^{m}$ be compact. Assume that
(i) There exists a constant $C$ such that

$$
\|F(x, u)\| \leq C(1+\|x\|), \quad x \in \mathbb{E}^{n}, u \in U
$$

(ii) The velocity sets

$$
F_{U}(x):=\{F(x, u) \mid u \in U\} \subseteq T_{x} \mathbb{E}^{n}, \quad x \in \mathbb{E}^{n}
$$

are convex.

Then the attainable sets $\boldsymbol{A}\left(x_{0}, t\right)$ and $\boldsymbol{A}\left(x_{0}, \leq T\right)$ are compact for all $x_{0} \in$ $M$, and $T>0$.

### 3.2 Pontryagin's Maximum Principle

The optimal control problem (OCP) is to find an optimal solution (i.e. an optimal controlled trajectory), assuming that the latter exists. An indirect approach to this problem consists in first determining some properties of optimal trajectories that will be sufficiently distinctive to narrow the class of candidates for optimal solutions to a small class of curves. Pontryagin's Maximum Principle (PMP) provides a list of necessary conditions that an optimal trajectory must fulfill. We begin with an initial formulation of the maximum principle for optimal control problems in $\mathbb{E}^{n}$.

Note: Rather than seeking the most general conditions under which the principle is valid, we shall follow the original presentation of Lev S. Pontryagin and his coworkers, namely V.G. Boltyanskii, R.V. Gamkrelidze, and E.F. Mishchenko. This level of generality is sufficient for many important applications and is, at the same time, relatively free of the technicalities that could obscure its geometric content.

## The maximum principle in $\mathbb{E}^{n}$

Let

$$
F=\left(F_{1}, F_{2}, \ldots, F_{n}\right): M \times U \rightarrow \mathbb{E}^{n}
$$

be a given mapping (of $n+m$ variables $x_{1}, \ldots, x_{n}, u_{1}, \ldots, u_{m}$ ), where $M$ is an open subset of (the Euclidean $n$-space) $\mathbb{E}^{n}$ and $U$ is an arbitrary subset of $\mathbb{R}^{m}$. Consider the control system (on $M$ ) described by (the state equation)

$$
\dot{x}=F(x, u), \quad x \in M, u \in U .
$$

Note : The Euclidean space $\mathbb{E}^{n}$ is viewed as a smooth $n$-manifold, whereas the Cartesian $n$-space $\mathbb{R}^{m}$ is viewed here only as a topological space. The state space $M$ is a smooth submanifold of $\mathbb{E}^{n}$ (of dimension $n$ ).

We shall assume that

- for each $u \in U$, the mapping $F_{u}=F(\cdot, u): M \rightarrow \mathbb{E}^{n}$ is smooth
- the mappings $F, \frac{\partial F}{\partial x_{i}}: M \times U \rightarrow \mathbb{E}^{n}$ are continuous (with respect to the canonical topology on $\mathbb{E}^{n}$ and the induced topology on $M \times$ $\left.U \subset \mathbb{E}^{n} \times \mathbb{R}^{m}\right)$.

By an admissible control $u(\cdot)$ we mean a $U$-valued mapping defined on some compact interval $\left[t_{0}, t_{1}\right]$ that is (essentially) bounded and measurable on $\left[t_{0}, t_{1}\right]$. When an admissible control $u(\cdot):\left[t_{0}, t_{1}\right] \rightarrow U$ is substituted in the (right-hand side of the) state equation, a system of (time-varying) ODEs

$$
\dot{x}_{i}=F_{i}(x(t), u(t)), \quad i=1,2, \ldots, n
$$

results. CARATHÉODORY's Existence and Uniqueness Theorem guarantees that, for any admissible control $u(\cdot)$ and any $x_{0} \in M$ : (i) there exists, on some interval $J \subseteq\left[t_{0}, t_{1}\right]$ such that $t_{0} \in J$, a solution curve through $x_{0}$ (i.e. an absolutely continuous mapping $x(\cdot): J \rightarrow M$ such that $\dot{x}_{i}(t)=$ $F_{i}(x(t), u(t))$ for almost all $t \in J$ and $x\left(t_{0}\right)=x_{0}$; (ii) if $x_{1}(\cdot): J_{1} \rightarrow M$ and $x_{2}(\cdot): J_{2} \rightarrow M$ are two such solution curves, then they coincide on $J_{1} \cap J_{2}$.

Note : A mapping $\xi: J=[a, b] \rightarrow \mathbb{R}^{n}$ is said to be absolutely continuous if it satisfies the following property : for each $\varepsilon>0$ there is a $\delta>0$ such that, for every finite sequence of points

$$
a \leq a_{1}<b_{1}<a_{2}<\cdots<a_{k}<b_{k} \leq b
$$

so that $\sum_{i=1}^{k}\left(b_{i}-a_{i}\right)<\delta$, it holds that $\sum_{i=1}^{k}\left\|\xi\left(b_{i}\right)-\xi\left(a_{i}\right)\right\|<\varepsilon$. The mapping $\xi$ is absolutely continuous if and only if it is an indefinite integral (i.e. there is some
integrable mapping $h$ such that

$$
\xi(t)=\xi(a)+\int_{a}^{t} h(\tau) d \tau
$$

for all $t \in J)$. An absolutely continuous mapping is differentiable almost everywhere, and $\dot{\xi}(t)=h(t)$ holds for almost all $t$.

Solution curves can always be continued to the maximal interval of existence. Assuming that $J$ is the maximal interval, then the solution curve (through $x_{0}$ ) is unique (up to a set of measure zero). We shall refer to it as an integral curve (or sometimes a trajectory) of the original control system, that corresponds to $u(\cdot)$.

For any integral curve $x(\cdot): J \rightarrow M$, let $A(t)$ denote the matrix

$$
A(t):=\left[\frac{\partial F_{i}}{\partial x_{j}}(x(t), u(t))\right] \in \mathbb{R}^{n \times n}
$$

Each entry $A_{i j}: J \rightarrow \mathbb{R}$ is an (essentially) bounded measurable function. The following linear system of ODEs is called the variational system along the trajectory $x(\cdot)$ (or, more precisely, along the pair $(x(\cdot), u(\cdot)))$ :

$$
\dot{v}_{i}=\sum_{j=1}^{n} A_{i j}(t) v_{j}, \quad i=1,2, \ldots, n
$$

It follows from the theory of linear differential equations that for each $v_{0} \in \mathbb{R}^{n}$, there exists an absolutely continuous curve $v(\cdot): J \rightarrow M$ such that $v\left(t_{0}\right)=v_{0}$ and which satisfies the above conditions (linear ODEs) for almost all $t \in J$.

The adjoint variational system (along $x(\cdot)$ ) is given by

$$
\dot{p}_{i}=-\sum_{j=1}^{n} p_{j} A_{j i}(t), \quad i=1,2, \ldots, n
$$

The solution curves for the adjoint system are also defined on the entire interval $J$ for each initial value $p_{0} \in \mathbb{R}^{n}$.

Exercise 21 Verify that the solution curves $v(\cdot)$ and $p(\cdot)$ of the variational system and the adjoint variational system, respectively, satisfy (for almost $t \in J$ )

$$
p_{1}(t) v_{1}(t)+\cdots+p_{n}(t) v_{n}(t)=\text { constant. }
$$

Note : The pair of differential systems

$$
\begin{aligned}
\dot{x}_{i} & =F_{i}(x(t), u(t)) \\
\dot{p}_{i} & =-\sum_{j=1}^{n} p_{j} \frac{\partial F_{j}}{\partial x_{i}}(x(t), u(t)), \quad i=1,2, \ldots, n
\end{aligned}
$$

can be expressed in terms of a single function $H$, given by

$$
H(x, p, u):=p_{1} F_{1}(x, u)+\cdots+p_{n} F_{n}(x, u)
$$

by the formulas

$$
\dot{x}_{i}=\frac{\partial H}{\partial p_{i}}(x, p, u) \quad \text { and } \quad \dot{p}_{i}=-\frac{\partial H}{\partial x_{i}}(x, p, u) \quad(i=1,2, \ldots, n)
$$

(valid for any admissible control $u(\cdot)$ ).

Consider the optimal control problem

$$
\begin{gathered}
\dot{x}=F(x, u), \quad x \in M \subseteq \mathbb{E}^{n}, u \in U \\
x(0)=x_{0}, \quad x\left(t_{1}\right)=x_{1} \quad\left(x_{0}, x_{1} \text { fixed }\right) \\
\int_{0}^{t_{1}} L(x(t), u(t)) d t \rightarrow \text { min. }
\end{gathered}
$$

The extended control system (on $M_{\text {ext }}=\mathbb{R} \times M$ ) defines its family of Hamiltonians $\mathcal{H}_{u}: \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ (parametrized by the control functions), given by

$$
\mathcal{H}_{u}(x, p):=p_{0} L(x, u)+p_{1} F_{1}(x, u)+\cdots+p_{n} F_{n}(x, u) .
$$

Note : In a slight abuse of notation, we write (in the left-hand side) $x$ instead of $x_{\mathrm{ext}}=(\xi, x) \in \mathbb{R} \times M$ and also $p=\left[\begin{array}{llll}p_{0} & p_{1} & \ldots & p_{n}\end{array}\right] \in \mathbb{R}^{1 \times(n+1)}\left(=\left(\mathbb{R}^{n+1}\right)^{*}\right)$.

For each admissible control $u(\cdot)$, let $\overrightarrow{\mathcal{H}}_{u}$ denote the Hamiltonian vector field that corresponds to (the Hamiltonian function) $\mathcal{H}_{u} . \overrightarrow{\mathcal{H}}_{u}$ is the vector field (on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ ) defined by

$$
\overrightarrow{\mathcal{H}}_{u}(x, p):=\left[\begin{array}{c}
\frac{\partial \mathcal{H}_{u}}{\partial p}(x, p) \\
\\
-\frac{\partial \mathcal{H}_{u}}{\partial x}(x, p)
\end{array}\right] \in \mathbb{R}^{2(n+1) \times 1} .
$$

We have

$$
\overrightarrow{\mathcal{H}}_{u}=\mathbb{J} \cdot \nabla \mathcal{H}_{u}
$$

where $\mathbb{J}=\left[\begin{array}{cc}0 & I_{n+1} \\ -I_{n+1} & 0\end{array}\right] \in \mathbb{R}^{2(n+1) \times 2(n+1)}$ and $\nabla \mathcal{H}_{u}$ is the naive gradient of $\mathcal{H}_{u}$; that is, the row matrix (the matrix of the derivative) $d \mathcal{H}_{u}$ written as a column matrix : $\nabla \mathcal{H}_{u}=\left[\begin{array}{ll}\frac{\partial \mathcal{H}_{u}}{\partial x} & \frac{\partial \mathcal{H}_{u}}{\partial p}\end{array}\right]^{T}$.

Note : Consider a vector space $E=V \times V^{*}$, where $V$ is a (real) vector space and $V^{*}$ is its dual. Define the canonical symplectic form $\Omega$ on $E$ by

$$
\Omega\left(\left(v_{1}, \alpha_{1}\right),\left(v_{2}, \alpha_{2}\right)\right):=\alpha_{2}\left(v_{1}\right)-\alpha_{1}\left(v_{2}\right)
$$

where $v_{1}, v_{2} \in V$ and $\alpha_{1}, \alpha_{2} \in V^{*}$. Then the induced linear mapping $\Omega^{b}: E \rightarrow E^{*}$, defined by

$$
\Omega^{b}\left(v_{1}, \alpha_{1}\right)\left(v_{2}, \alpha_{2}\right):=\Omega\left(\left(v_{1}, \alpha_{1}\right),\left(v_{2}, \alpha_{2}\right)\right),
$$

is one-to-one. (If $V$ is finite dimensional, then so is $E$ and $\Omega^{b}$ is an isomorphism. In this case the matrix of $\Omega^{b}$ is $\mathbb{J}^{T}$.) A vector field $X: E \rightarrow E$ is called Hamiltonian if

$$
\Omega^{b}(X(v, \alpha))=d H(v, \alpha)
$$

for all $(v, \alpha) \in E$, for some $C^{1}$ function $H: E \rightarrow \mathbb{R}$. (Here $d H=D H$ is alternative notation for the derivative of $H$.) If such an $H$ exists, we write $X=X_{H}$ (or $X=\vec{H}$ )
and call $H$ a Hamiltonian function for $X$. In a number of important examples, $H$ need not be defined on all of $E$. If (the vector space) $E$ is finite dimensional, then the existence of $X_{H}$ is guaranteed for any given $\left(C^{1}\right)$ function $H$. Moreover, $X_{H}$ is unique (since the mapping $\Omega^{b}$ is one-to-one).

All these considerations carry over to any symplectic vector space $(E, \Omega)$. Here $E$ is a real Banach space, and $\Omega: E \times E \rightarrow \mathbb{R}$ is a non-degenerate skew-symmetric (continuous) bilinear form; non-degeneracy of $\Omega$ is equivalent to the injectivity of $\Omega^{b}$. If $E$ is finite dimensional, then the induced mapping $\Omega$ is an isomorphism, and the dimension is even, since the determinant of a skew-symmetric matrix with an odd number of rows (and columns) is zero.

The integral curves of $\overrightarrow{\mathcal{H}}_{u}$ satisfy

$$
\left[\begin{array}{c}
\dot{x} \\
\dot{p}
\end{array}\right]=\overrightarrow{\mathcal{H}}_{u}(x, p), \quad(x, p) \in \mathbb{R}^{n+1} \times\left(\mathbb{R}^{n+1}\right)^{*}
$$

that is, they satisfy the following system of ODEs (for almost all $t$ ):

$$
\begin{aligned}
\dot{\xi} & =\frac{\partial \mathcal{H}_{u}}{\partial p_{0}}=L(x, u) \\
\dot{x}_{i} & =\frac{\partial \mathcal{H}_{u}}{\partial p_{i}}=F_{i}(x, u), \quad 1=1,2, \ldots, n \\
\dot{p}_{0} & =-\frac{\partial \mathcal{H}_{u}}{\partial \xi}=p_{0} \frac{\partial L}{\partial \xi}(x, u)+\sum_{j=1}^{n} p_{j} \frac{\partial F_{j}}{\partial \xi}(x, u) \\
\dot{p}_{i} & =-\frac{\partial \mathcal{H}_{u}}{\partial x_{i}}=p_{0} \frac{\partial L}{\partial x_{i}}(x, u)+\sum_{j=1}^{n} p_{j} \frac{\partial F_{j}}{\partial x_{i}}(x, u), \quad i=1,2, \ldots, n .
\end{aligned}
$$

The maximal Hamiltonian associated with each integral curve $(x(\cdot), p(\cdot))$ is defined by

$$
\mathcal{H}(x, p):=\sup _{u \in U} \mathcal{H}_{u}(x, p)
$$

The maximum principle consists of necessary conditions for optimality. We shall consider first OCPs with free final time.
3.2.1 Theorem. If $(\bar{x}(\cdot), \bar{u}(\cdot)$ ) is an optimal solution of our OCP (with free final time $t_{1}>0$ ), then there exists a nonzero, absolutely continuous curve $p=\left(p_{0}, p_{1}, \ldots, p_{n}\right):\left[0, t_{1}\right] \rightarrow\left(\mathbb{R}^{n+1}\right)^{*}$ such that :
(MP1) $(\bar{x}(\cdot), p(\cdot))$ is a solution curve of the differential system

$$
\left[\begin{array}{c}
\dot{x} \\
\dot{p}
\end{array}\right]=\overrightarrow{\mathcal{H}}_{\bar{u}}(x, p), \quad(x, p) \in \mathbb{R}^{n+1} \times\left(\mathbb{R}^{n+1}\right)^{*} .
$$

(MP2) $\quad \mathcal{H}_{\bar{u}}(\bar{x}, p)=\mathcal{H}(\bar{x}, p)$ for almost all $t \in\left[0, t_{1}\right]$.
(MP3) $\quad p_{0}\left(t_{1}\right) \leq 0$ and $\mathcal{H}\left(\bar{x}\left(t_{1}\right), p\left(t_{1}\right)\right)=0$.
(Furthermore, it can be shown that

$$
\mathcal{H}_{\bar{u}}(\bar{x}, p)=\text { const }, \quad t \in\left[0, t_{1}\right]
$$

and the coordinate $p_{0}(\cdot)$ associated with the adjoint curve $t \mapsto p(t)$ is constant.)

Note: (1) $p_{0}$ can always be normalized and so we can assume that $p_{0}=-1$ or 0 . It is then convenient to reduce the Hamiltonians to $M \times \mathbb{R}^{n}$ and regard $p_{0}$ as parameter.
(2) If we have a maximization problem instead of a minimization problem (OCP), then the inequality $p_{0}\left(t_{1}\right) \leq 0$ should be reversed.

A curve $t \mapsto(x(t), p(t), u(t))$ in $M \times\left(\mathbb{R}^{n}\right)^{*} \times U$ is called an extremal triple if there exists a constant $p_{0} \leq 0$ such that $x(\cdot), p(\cdot), u(\cdot)$, and $p_{0}$ satisfy conditions (MP1)-(MP3) of the maximum principle and, in addition, satisfy $p(t) \neq 0$ whenever $p_{0}=0$. We shall also say that $(x(\cdot), p(\cdot))$ is the extremal curve generated by $u(\cdot)$. (Sometimes we may also refer to $u(\cdot)$ as the extremal control.)

Note : It is known that the maximal Hamiltonian $\mathcal{H}=\sup _{u \in U} \mathcal{H}_{u}$ is constant along each extremal curve $(x(\cdot), p(\cdot))$. Consequently, condition (MP3) of the maximum principle can be replaced by

$$
\mathcal{H}(x, p) \equiv 0
$$

The passage to OCPs with fixed final time is as follows : Suppose that $x_{0}, x_{1} \in M$ and (final time) $t_{1}>0$ are fixed in advance, and that $(x(\cdot), u(\cdot))$ is an optimal solution of our OCP; that is, $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a solution curve that minimizes the cost (functional)

$$
\int_{0}^{t_{1}} L(x(t), u(t)) d t
$$

among all other solutions that transfer $x_{0}$ to $x_{1}$ in $t_{1}$ units of time.
Let $x_{n+1}(t)=t$ be another coordinate attached to the solution curve $t \mapsto x(t)=\left(x_{1}(t), \ldots x_{n}(t)\right)$. Denote by $\tilde{x}_{0}=\left(x_{0}, 0\right)$ and $\tilde{x}_{1}=\left(x_{1}, t_{1}\right)$ the points in $M \times \mathbb{R} \subseteq \mathbb{E}^{n+1}$ defined by the boundary conditions $x_{0}$ and $x_{1}$. Then $\tilde{x}(\cdot)=\left(x_{1}(\cdot), \ldots, x_{n}(\cdot), x_{n+1}(\cdot)\right)$, and $u(\cdot)$ is a solution curve for the OCP for the extended system (on $(\mathbb{R} \times M) \times \mathbb{R})$ :

$$
\begin{aligned}
\dot{\xi} & =L(x, u) \\
\dot{x}_{i} & =F_{i}(x, u), \quad i=1,2, \ldots, n \\
\dot{x}_{n+1} & =1
\end{aligned}
$$

An extended (controlled) trajectory $(\tilde{x}(\cdot), u(\cdot))$ of the foregoing system can transfer $\tilde{x}_{0}$ to $\tilde{x}_{1}$ only in $t_{1}$ units of time. Therefore, the adjoint curve defined by the maximum principle is defined on $\left[0, t_{1}\right]$. Let $p_{n+1}(\cdot)$ denote the component of the adjoint curve that corresponds to the optimal solution $(\tilde{x}(\cdot), u(\cdot))$ (of the extended system). Then $\dot{p}_{n+1}=0$ (since the original
system is autonomous) and therefore $p_{n+1}(t)=$ constant. Hence condition (MP3) becomes

$$
p_{0} L(x, u)+\sum_{i=1}^{n} p_{i}(t) F_{i}(x(t), u(t))+p_{n+1}=0
$$

for almost all $t \in\left[0, t_{1}\right]$. Thus (for almost all $t$ )

$$
\sum_{i=0}^{n} p_{i}(t) F_{i}(x(t), u(t))=\mathcal{H}_{u(t)}(x(t), p(t))=\text { constant } .
$$

The foregoing argument shows that the necessary optimality conditions given by PMP, for fixed final time, differ from the conditions corresponding to a free (variable) final time only by the constant defined by $\mathcal{H}(x, p)$.

Note : Nonautonomous (i.e. time-varying) problems can be reduced to autonomous ones by a similar trick, leading to necessary optimality conditions, except that the maximal Hamiltonian $\mathcal{H}(x(t), u(t))$ may no longer be constant along extremal curves.

Now consider the time-optimal control problem (T-OCP):

$$
\begin{gathered}
\dot{x}=F(x, u), \quad x \in M \subseteq \mathbb{E}^{n}, u \in U \\
x(0)=x_{0}, \quad x\left(t_{1}\right)=x_{1} \quad\left(x_{0}, x_{1} \text { fixed }\right) \\
t_{1}=\int_{0}^{t_{1}} 1 d t \rightarrow \mathrm{~min} .
\end{gathered}
$$

For this problem, Pontryagin's Maximum Principle (PMP) takes the following form:
3.2.2 Corollary. If $(\bar{x}(\cdot), \bar{u}(\cdot))$ is an optimal solution of our T-OCP, then there exists a nonzero, absolutely continuous curve $p=\left(p_{1}, \ldots, p_{n}\right)$ : $\left[0, t_{1}\right] \rightarrow\left(\mathbb{R}^{n}\right)^{*}$ such that :
(1) $(\bar{x}(\cdot), p(\cdot))$ is a solution curve of the differential system

$$
\left[\begin{array}{c}
\dot{x} \\
\dot{p}
\end{array}\right]=\overrightarrow{\mathcal{H}}_{\bar{u}}(x, p), \quad(x, p) \in \mathbb{R}^{n} \times\left(\mathbb{R}^{n}\right)^{*}
$$

(2) $\mathcal{H}_{\bar{u}}(\bar{x}, p)=\mathcal{H}(\bar{x}, p)$ for almost all $t \in\left[0, t_{1}\right]$.
(3) $\mathcal{H}(\bar{x}, p) \geq 0$ for almost all $t \in\left[0, t_{1}\right]$.

Proof : Apply Theorem 6.2.1 (and the remark after it) by taking $L \equiv 1$. Then the Hamiltonian system (1) and the maximality condition (2) follow. Inequality (3) is equivalent to conditions

$$
\mathcal{H}_{\bar{u}}(\bar{x}, p)+p_{0}=0 \quad \text { and } \quad p_{0} \leq 0 .
$$

The condition $p \neq 0$ is obtained as follows : if $p=0$, then $\mathcal{H}_{\bar{u}}(\bar{x}, p)=0$ and hence $p_{0}=0$. But the pair $\left(p_{0}, p\right) \in(\mathbb{R})^{*} \times\left(\mathbb{R}^{n}\right)^{*}$ must be nontrivial. Consequently, $p \neq 0$.

### 3.3 Simple Examples

We consider several optimal control problems (which can be solved by appying Pontryagin's Maximum Principle). We start with some simple concrete problems.

## The fastest stop of a train at a station

Consider a train moving on a railway. The problem is to drive the train to a station and stop it there in a minimal time.

Describe position of the train by a coordinate $z$ on the real line; the origin $0 \in \mathbb{R}$ corresponds to the station. Assume that the train moves without friction, and we can control acceleration of the train by applying a force bounded by absolute value. Using rescaling if necessary, we can assume that the absolute value of acceleration is bounded by 1 . We write the equation of motion as

$$
\ddot{z}=u, \quad z \in \mathbb{R},|u| \leq 1
$$

or, in the standard form (for $x_{1}=z$ and $x_{2}=\dot{z}$ ),

$$
\begin{aligned}
& \dot{x}_{1}=x_{2} \\
& \dot{x}_{2}=u, \quad x \in \mathbb{E}^{2},|u| \leq 1 .
\end{aligned}
$$

Our time-optimal control problem (T-OCP) is

$$
\begin{gathered}
\dot{x}=\left[\begin{array}{c}
x_{2} \\
u
\end{array}\right], \quad x \in \mathbb{E}^{2},|u| \leq 1 \\
x(0)=x_{0}, \quad x\left(t_{1}\right)=0 \quad\left(x_{0} \text { fixed }\right) \\
t_{1}=\int_{0}^{t_{1}} 1 d t \rightarrow \min
\end{gathered}
$$

First we verify existence of optimal controls by Filippov's theorem.

Now we apply Pontryagin's Maximum Principle (PMP). Introduce canonical coordinates on the cotangent bundle :

$$
\begin{aligned}
T^{*} \mathbb{E}^{2} & =\mathbb{R}^{2} \times\left(\mathbb{R}^{2}\right)^{*} \\
& =\left\{(x, p) \left\lvert\, x=\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]\right., p=\left[\begin{array}{ll}
p_{1} & p_{2}
\end{array}\right]\right\} .
\end{aligned}
$$

The control-dependent Hamiltonian function of PMP is

$$
\mathcal{H}_{u}(x, p)=\left[\begin{array}{ll}
p_{1} & p_{2}
\end{array}\right]\left[\begin{array}{l}
x_{2} \\
u
\end{array}\right]=p_{1} x_{2}+p_{2} u
$$

and the corresponding Hamiltonian system has the form

$$
\begin{aligned}
\dot{x} & =\frac{\partial \mathcal{H}_{u}}{\partial p} \\
\dot{p} & =-\frac{\partial \mathcal{H}_{u}}{\partial x} .
\end{aligned}
$$

So for any point of the Euclidean plane there exists exactly one extremal trajectory steering this point to the origin. Since optimal trajectories exist, then the solutions found are optimal.

## Control of a linear oscillator

Consider a linear oscillator whose motion can be controlled by a force bounded in absolute value. The equation of motion (after appropriate rescaling) is

$$
\ddot{z}+z=u, \quad z \in \mathbb{R},|u| \leq 1
$$

or, in the canonical form :

$$
\begin{aligned}
& \dot{x}_{1}=x_{2} \\
& \dot{x}_{2}=-x_{1}+u, \quad x \in \mathbb{E}^{2},|u| \leq 1 .
\end{aligned}
$$

We consider the following time-optimal control problem (T-OCP) :

$$
\begin{gathered}
\dot{x}=\left[\begin{array}{c}
x_{2} \\
-x_{1}+u
\end{array}\right], \quad x \in \mathbb{E}^{2},|u| \leq 1 \\
x(0)=x_{0}, \quad x\left(t_{1}\right)=0 \quad\left(x_{0} \text { fixed }\right) \\
t_{1}=\int_{0}^{t_{1}} 1 d t \rightarrow \text { min. }
\end{gathered}
$$

By Filippov's theorem, optimal control exists.

We apply Pontryagin's Maximum Principle (PMP) : the control-dependent Hamiltonian function is

$$
\mathcal{H}_{u}(x, p)=p_{1} x_{2}-p_{2} x_{1}+p_{2} u, \quad(x, p) \in T^{*} \mathbb{E}^{2}=\mathbb{R}^{2} \times\left(\mathbb{R}^{2}\right)^{*}
$$

and the Hamiltonian system reads

$$
\begin{aligned}
\dot{x}_{1} & =x_{2} \\
\dot{x}_{2} & =-x_{1}+u \\
\dot{p}_{1} & =p_{2} \\
\dot{p}_{2} & =-p_{1} .
\end{aligned}
$$

The time-optimal control problem is solved : in the part of the Euclidean plane over the switching curve the optimal control is $\bar{u}=-1$ and below this curve $\bar{u}=+1$. Through any point of the plane passes one optimal trajectory which corresponds to this optimal control rule. After finite number of switchings, any optimal trajectory comes to the origin.

## The cheapest stop of a train

Again we control the motion of a train. Now the goal is to stop the train at the fixed instant of time with a minimum expenditure of energy (which is assumed to be proportional to the integral of squared acceleration).

So the T-OCP is as follows :

$$
\begin{gathered}
\dot{x}=\left[\begin{array}{c}
x_{2} \\
u
\end{array}\right], \quad x \in \mathbb{E}^{2},|u| \leq 1 \\
x(0)=x_{0}, \quad x\left(t_{1}\right)=0 \quad\left(x_{0}, t_{1}>0 \text { fixed }\right) \\
\frac{1}{2} \int_{0}^{t_{1}} u^{2}(t) d t \rightarrow \text { min. }
\end{gathered}
$$

Filippov's theorem cannot be applied directly, since the rhs of the control system is not compact.

In order to find (the) optimal control, we apply PMP. The Hamiltonian function is

$$
\mathcal{H}_{u}(x, p)=\frac{p_{0}}{2} u^{2}+p_{1} x_{2}+p_{2} u, \quad(x, p) \in \mathbb{R}^{2} \times\left(\mathbb{R}^{2}\right)^{*} .
$$

Along optimal trajectories

$$
p_{0} \leq 0 \quad \text { and } \quad p_{0}=\text { constant } .
$$

So through the initial point (state) $x_{0}$ passes a unique extremal trajectory (ariving at the origin). It is a curve $t \mapsto\left(x_{1}(t), x_{2}(t)\right), t \in\left[0, t_{1}\right]$, where $x_{1}(\cdot)$ is a cubic polynomial that satisfies certain boundary conditions (see above) and $x_{2}(t)=\dot{x}_{1}(t)$. In view of existence, this is an optimal trajectory.

We control a linear oscillator (e.g. a pendulum with a small amplitude) by an unbounded force $u(\cdot)$, but take into account expenditure of energy measured by the inegral $\frac{1}{2} \int_{0}^{t_{1}} u^{2}(t) d t$. Our optimal control problem (OCP) is :

$$
\begin{gathered}
\dot{x}=\left[\begin{array}{c}
x_{2} \\
-x_{1}+u
\end{array}\right], \quad x \in \mathbb{E}^{2}, u \in \mathbb{R} \\
x(0)=x_{0}, \quad x\left(t_{1}\right)=0 \quad\left(x_{0}, t_{1}>0 \text { fixed }\right) \\
\frac{1}{2} \int_{0}^{t_{1}} u^{2}(t) d t \rightarrow \min .
\end{gathered}
$$

Existence of optimal control(s) can be proved by the same argument as in the previous example.

The Hamiltonian function of PMP is

$$
\mathcal{H}_{u}(x, p)=\frac{p_{0}}{2} u^{2}+p_{1} x_{2}-p_{2} x_{1}+p_{2} u .
$$

The corresponding Hamiltonian system yields

$$
\begin{aligned}
\dot{p}_{1} & =p_{2} \\
\dot{p}_{2} & =-p_{1} .
\end{aligned}
$$

In the same way as the previous example, we show that there are no abnormal extremals. Hence we can assume that $p_{0}=-1$.

### 3.4 The Linear Time-Optimal problem

Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ be matrices, and let $U$ be a compact convex polytope in $\mathbb{R}^{m}$. (The polytope $U$ is the convex hull of a finite set of points $a_{1}, \ldots, a_{k}$ in $\mathbb{R}^{m}: U=\operatorname{conv}\left\{a_{1}, \ldots, a_{k}\right\}$. We assume that the points $a_{i}$ do not belong to the convex hull of all the other points $a_{j}, j \neq i$ so that each $a_{i}$ is a vertex of the polytope $U$.)

We consider the following time-optimal control problem (T-OCP) :

$$
\begin{gathered}
\dot{x}=A x+B u, \quad x \in \mathbb{R}^{n}, u \in U \\
x(0)=x_{0}, \quad x\left(t_{1}\right)=x_{1} \quad\left(x_{0}, x_{1} \text { fixed }\right) \\
t_{1}=\int_{0}^{t_{1}} 1 d t \rightarrow \min .
\end{gathered}
$$

Such a problem is called a linear time-optimal control problem.
Note: T-OCPs constitute one of the basic concerns of optimal control theory. Minimal-time problems go back to the beginnings of the calculus of variations. Johan Bernoulli's solution of the brachistochrone problem in 1697 was based on Fermat's principle of least time, which postulates that "light traverses any medium in the least possible time". Since then such problems have remained important sources of inspiration.

We shall assume that the following (general position) condition holds : For any edge $\left[a_{i}, a_{j}\right]$ of (the polytope) $U$, the vector $e_{i j}:=a_{j}-a_{i}$ is such that

$$
\operatorname{span}\left\{B e_{i j}, A B e_{i j}, \ldots, A^{n-1} B e_{i j}\right\}=\mathbb{R}^{n} .
$$

(This condition is equivalent to the controllability of the linear control system $\dot{x}=A x+B u$ with the set of control parameters $u \in \mathbb{R} e_{i j}$. The condition can
be achieved by a small perturbation of the matrices $A, B$.)
Existence of optimal control for any points $x_{0}, x_{1}$ such that $x_{1} \in \boldsymbol{A}\left(x_{0}\right)$ is guaranteed by Filippov's Theorem.

NOTE : For the analogous problem with an unbounded set of control parameters, optimal control may not exist.

Optimal control in the linear time-optimal control problem is "bang-bang" : it is piecewiwe constant and takes values in the vertices of the polytope $U$.

The next three examples are special cases of linear time-optimal control problems.

## Time-optimal control of linear mechanical systems

Consider the problem of controlling a linear mechanical system

$$
\ddot{z}+k \dot{z}+\beta z=u(t)
$$

by an external force $u(\cdot)$ restricted in magnitude : $|u| \leq \varepsilon$. Here $k$ and $\beta$ are some non-negative constants (see Example 5.1.1). The equivalent first-order system (induced by the state variables $x_{1}:=z$ and $x_{2}:=\dot{z}$ ) is given by

$$
\dot{x}_{1}=x_{2}, \quad \dot{x}_{2}=-\beta x_{1}-k x_{2}+u
$$

or, in vector notation,

$$
\dot{x}=A x+b u, \quad x=\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right] \in \mathbb{R}^{2}
$$

where

$$
A=\left[\begin{array}{cc}
0 & 1 \\
-\beta & -k
\end{array}\right] \quad \text { and } \quad b=e_{2}
$$

Exercise 22 Show that the foregoing linear control system is controllable if and only if $k=0$.

We shall consider only the cases when $k=0$.

Case $I: \beta=0$. In this case, $\ldots$

Case II : $\beta \neq 0$. This case corresponds to the control of a linear harmonic oscillator through an external force $u(\cdot)$.

The next example represents a class of OCPs very "popular" in applications.

### 3.5 The Linear-Quadratic Problem

Minimizing the integral of a quadratic form over the trajectories of a linear control system, known as the linear quadratic problem, was one of the earliest OCPs (Kalman, 1960).

We consider linear control systems with quadratic cost :

$$
\begin{gathered}
\dot{x}=A x+B u, \quad x \in \mathbb{R}^{n}, u \in \mathbb{R}^{m} \\
x(0)=x_{0}, \quad x\left(t_{1}\right)=x_{1} \quad\left(x_{0}, x_{1}, t_{1}>0 \text { fixed }\right) \\
\frac{1}{2} \int_{0}^{t_{1}}\langle P u, u\rangle+\langle Q x, u\rangle+\langle R x, x\rangle d t \rightarrow \text { min. }
\end{gathered}
$$

Here $A, B, P, Q, R$ are matrices of appropriate dimensions, $P$ and $R$ are symmetric (i.e. $P^{T}=P$ and $R^{T}=R$ ), and the angle brackets $\langle\cdot, \cdot\rangle$ denote the standard inner product in $\mathbb{R}^{n}$ and $\mathbb{R}^{m}$.

Note : One can write the "Lagrangian" $L$ (i.e. the integrand of the cost functional) in the (matrix) form

$$
L(x(t), u(t))=\frac{1}{2}\left[u^{T}(t) P u(t)+u^{T}(t) Q x(t)+x^{T}(t) R x(t)\right] .
$$

One can show that the condition $P \geq 0$ (i.e. the matrix $P$ is positive semi-definite) is necessary for the existence of optimal control. We do not discuss here the case of degenerate $P$ and assume that $P>0$ (i.e. the matrix $P$ is positive definite).

Exercise 23 Verify that the substitution (of variables) $u \mapsto v=P^{1 / 2} u$ transforms the cost functional $\mathcal{J}(u)=\int_{0}^{t_{1}} L(x(t), u(t)) d t$ into a similar one with the identity matrix $I$ instead of $P$.

We assume that $P=I$. Another change of variables "kills" the matrix $Q$.

Exercise 24 Find a change of variables that would transform the cost functional into a similar one with the zero matrix $O$ instead of $Q$.

Hence we can write the cost functional as follows :

$$
\mathcal{J}(u)=\frac{1}{2} \int_{0}^{t_{1}}\|u(t)\|^{2}+\langle Q x(t), u(t)\rangle d t .
$$

We further assume that the linear control system $\dot{x}=A x+B u$ is controllable

$$
\operatorname{rank}\left[\begin{array}{llll}
B & A B & \cdots & A^{n-1} B
\end{array}\right]=n
$$

### 3.6 Optimal Control on Matrix Lie Groups

In approaching variational problems on Lie groups, optimal control theory finds itself on the same ground as Hamiltonian mechanics : facing an already developed theory of Hamiltonian systems that it needs to understand and absorb in order to arrive at the proper solutions of its own problems. This theory of Hamiltonian systems on Lie groups is based on a particular realization of the cotangent bundle of a Lie group $G$ as the product of $G$ and the dual of its Lie algebra $\mathfrak{g}$. (For vector spaces $V$, which are also commutative Lie groups, that realization of the cotangent bundle coincides with the usual representation : $T^{*} V=V \times V^{*}$.)

Let $G$ be a (real) Lie group with Lie algebra $\mathfrak{g}$, and let $e$ denote the group identity of $G$. Recall that $\mathfrak{g}$ is (isomorphic to) the tangent space to $G$ at $e: \mathfrak{g}=T_{e} G$. Let $T^{*} G$ denote the cotangent bundle of $G$.

## The symplectic structure of $T^{*} G$

For each element $g \in G$, let $L_{g}$ denote the left-translation by $g$ (i.e. $\left.x \mapsto L_{g}(x):=g x\right)$. The tangent mapping $d L_{g}=\left(L_{g}\right)_{*}$ maps (the tangent space) $T_{x} G$ onto $T_{g x} G$ for each $x$ in $G$. Let $d L_{g}^{*}$ denote the dual mapping of $d L_{g}$. Then (for each $x \in G$ )

$$
d L_{g}^{*}: T_{g x}^{*} G \rightarrow T_{x}^{*} G
$$

For each (tangent covector) $p \in T_{g x}^{*} G$ we have

$$
d L_{g}^{*}(p)=p \circ d L_{g}
$$

In particular, at $x=g, d L_{g^{-1}}^{*} \operatorname{maps} T_{e}^{*} G$ onto $T_{g}^{*} G$.

The correspondence

$$
(g, p) \longleftrightarrow d L_{g^{-1}}^{*}(p)
$$

realizes $T^{*} G$ as $G \times \mathfrak{g}^{*}$.
Note : In this representation of $T^{*} G$, the Hamiltonians of left-invariant vector fields are linear functionals on $\mathfrak{g}^{*}$, and the Hamiltonians of right-invariant vector fields are functions that depend on both factors $G$ and $\mathfrak{g}^{*}$. The explicit expressions are as follows :

The Hamiltonian $H_{X}$ of a left-invariant vector field $X$ is given by

$$
H_{X}(g, p):=p(X(e))
$$

and the Hamiltonian of a right-invariant vector field $X$ is given by

$$
H_{X}^{\prime}(g, p):=p\left(d L_{g^{-1}} X(g)\right)=p\left(d L_{g^{-1}} d R_{g}(X(e))\right) .
$$

$T^{*} G$ could also have been realized as $G \times \mathfrak{g}^{*}$ in terms of the right multiplications $x \mapsto R_{g}(x)=x g$. Then the correspondence would be given by

$$
(g, f) \longleftrightarrow d R_{g^{-1}}^{*} f
$$

and therefore the Hamiltonians of right-invariant vector fields would become linear functionals on $\mathfrak{g}^{*}$. These representations are equally suitable for applications. (The left-invariant realization is better for the applications that follow.)

The tangent bundle of $G \times \mathfrak{g}^{*}$ is naturally identified with $T G \times T \mathfrak{g}^{*}$. We shall further identify $T G$ with $G \times \mathfrak{g}$ via the correspondence

$$
(g, X) \longleftrightarrow d L_{g}(X)
$$

for each $g \in G$ and $X \in \mathfrak{g}$. Since $T \mathfrak{g}^{*}=\mathfrak{g}^{*} \times \mathfrak{g}^{*}$, we get

$$
\begin{aligned}
T\left(T^{*} G\right) & =T\left(G \times \mathfrak{g}^{*}\right)=T G \times T \mathfrak{g}^{*} \\
& =(G \times \mathfrak{g}) \times\left(\mathfrak{g}^{*} \times \mathfrak{g}^{*}\right)
\end{aligned}
$$

In this realization, each element $\left((g, X),\left(p, Y^{*}\right)\right)$ is a tangent vector $\left(X, Y^{*}\right)$ based at $(g, p)$ in $T^{*} G$. With these conventions, vector fields on $T^{*} G$ will be represented by pairs $\left(X, Y^{*}\right)$, with $X$ taking values in $\mathfrak{g}$, and $Y^{*}$ taking values in $\mathfrak{g}^{*}$.

Note : Having identified $T^{*} G$ with $G \times \mathfrak{g}^{*}$, functions on $T^{*} G$ become functions on $G \times \mathfrak{g}^{*}$.

Left-invariant control systems and co-adjoint orbits

## Casimir functions and the conservation laws

## Lie-Poisson reduction and the Maximum Principle

