Chapter 4

Applications

Topics :

1. The Brachistochrone Problem
2. The Elastic Problem
3. Dubins' Problem
4. Other Problems

4.1 The Brachistochrone Problem

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Two very interesting classic problems will be considered next. If the curvature function $\kappa(\cdot)$ (of a curve in the Euclidean plane $\mathbb{E}^2$) is regarded as a control function, many classic variational problems in geometry become optimal control problems (OCPs).

4.2 The Elastic Problem

Consider the following problem : Given points $x_0, x_1 \in \mathbb{E}^2$ and unit tangent vectors $v_0, v_1 \in T_0\mathbb{E}^2 = \mathbb{R}^2$, find a (differentiable) curve $\gamma : [0, T] \to \mathbb{E}^2$ such that :

- γ is parametrized by arc length.
- γ has *curvature* $\kappa(\cdot)$ (almost everywhere).
- γ satisfies the *boundary conditions* :

$$\gamma(0) = x_0, \quad \dot{\gamma} = v_0, \quad \gamma(T) = x_1, \quad \dot{\gamma}(T) = v_1.$$

- γ *minimizes* the (cost) functional

$$\mathcal{J} = \frac{1}{2}\int_0^T \kappa^2(t)\, dt.$$

This (variational) problem, known as the **elastic problem**, goes back to Leonhard Euler (1707-1783), and the solution curves are called the *elastica.*

Note : The *elastic problem* has a rich classical heritage inspired by the following physical situation : a thin elastic rod, when subjected to bending only, assumes the shape of an elastica in its equilibrium position. In this context, Euler made the initial study of the (planar) elastica in 1744. Much of the development in the theory of the elastic rods is based on a discovery of Gustav R. Kirchhoff (1824-1887) (known as *the kinetic analogue of the elastic problem*) that the equations for the equilibrium configurations of an elastic rod are the same as the equations for the (Lagrange's) *spinning top*.

The geometric significance of minimizing $\frac{1}{2}\int \kappa^2\, dt$ (or, more generally, *any* functional of κ) was recognized by Wilhelm Blaschke (1885-1962) under the name of *Radon's problem* (after the name of the mathematician Johann Radon (1887-1956)).

Investigations of motion of the rigid body (the kinetic analogue of the elastic problem) in *non-Euclidean spaces* were done by William K. Clifford (1845-1879) as early as 1874.

Euler's elastic problem admits a natural formulation (as a OCP) on the matrix Lie group $\mathsf{SE}(2)$ (of proper rigid motions on $\mathbb{E}^2$). Recall that $\mathsf{SE}(2)$ is the semidirect product of $\mathbb{R}^2$ with $\mathsf{SO}(2)$ which can also be regarded as the subgroup of $\mathsf{GL}(3,\mathbb{R})$ consisting of 3×3 matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 \\ x_1 & \alpha & -\beta \\ x_2 & \beta & \alpha \end{bmatrix}$$

with $(x_1, x_2) \in \mathbb{R}^2$ and $\alpha^2 + \beta^2 = 1$. This group can also be viewed as the set of all pairs $(x, \underline{b})$, with x a point in $\mathbb{E}^2$ and $\underline{b}$ a positively-oriented (orthonormal) frame at x. Also recall that the Lie algebra $\mathfrak{se}(2)$ of $\mathsf{SE}(2)$ consists of 3×3 matrices of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ a_1 & 0 & -a_3 \\ a_2 & a_3 & 0 \end{bmatrix}$$

with $(a_1, a_2, a_3) \in \mathbb{R}^3$. Let

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

denote the standard basis (so that any element in the Lie algebra is writen $a_1A_1 + a_2A_2 + a_3A_3$). Corresponding to each element A in the Lie algebra, $\vec{A}$ denotes the left-invariant vector field $\vec{A} : g \mapsto gA$. We shall consider the following (left-invariant) control system on $\mathsf{SE}(2)$:

$$\dot{g} = \vec{A}_1(g) + u(t)\vec{A}_3(g)$$

with

$$g(t) = \begin{bmatrix} 1 & 0 \\ x(t) & R(t) \end{bmatrix}, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \text{and} \quad R(t) \in \mathsf{SO}(2).$$

NOTE : This control system is the classic *Serret-Frenet control system*, associated with a curve $x(\cdot)$ parametrized by its arc length. The state equation of the system can also be written as

$$\dot{x} = R(t)e_1, \qquad \dot{R} = R(t)\begin{bmatrix} 0 & -u(t) \\ u(t) & 0 \end{bmatrix}.$$

Observe that the rotation matrix $R(t)$, when parametrized by the angle θ, yields the following differential system in $\mathbb{E}^3$:

$$\dot{x}_1 = \cos\theta, \quad \dot{x}_2 = \sin\theta, \quad \dot{\theta} = u.$$

It follows that $\ddot{x}(t) = \dot{R}(t)e_1 = u(t)R(t)e_2$ and therefore

$$\|\ddot{x}(t)\| = |u(t)|$$

and the control $u(\cdot)$ is equal to the *geodesic curvature*.

Our OCP is the following :

$$\dot{g} = g\,(A_1 + uA_3)\,, \qquad g \in \mathsf{SE}\,(2),\; u \in \mathbb{R}$$

$$g(0) = (x_0, R(0))\,, \quad g(t_1) = (x_1, R(t_1)) \qquad (x_0, x_1, R(0), R(t_1) \;\text{ fixed})$$

$$\frac{1}{2}\int_0^{t_1} u^2(t)\, dt \;\to\; \text{min.}$$

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We shall determine the extremals for Euler's elastic problem. Denote by H_1, H_2, H_3 the Hamiltonians of the left-invariant vector fields $\vec{A}_1, \vec{A}_2, \vec{A}_3$. respectively. (Because $[\vec{A}_1, \vec{A}_2] = 0$, $[\vec{A}_1, \vec{A}_3] = \vec{A}_2$, and $[\vec{A}_2, \vec{A}_3] = -\vec{A}_1$, it follows that the Poisson brackets of H_1, H_2, and H_3 satisfy the same relations.) It follows that the *regular extremals* are the integral curves of the Hamiltonian vector field $\vec{H}$, defined by

$$H := \frac{1}{2}H_3^2 + H_1$$

and that along each extremal curve $\xi(\cdot)$ the corresponding control $u(\cdot)$ is equal to $H_3(\xi(\cdot))$.

For *abnormal extremals*,

$$H_3(\xi(t)) = 0 \quad \text{and} \quad \{H_3, H_1\}\,(\xi(t)) = 0.$$

Because $\{H_3, H_1\} = -H_2$, it follows that $H_2(\xi(t)) = 0$.

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4.3 Dubins' Problem

In 1957 L.E. Dubins considered (and solved) the problem of finding the (parametrized) curves of *minimal length* that would connect two given *config-*

urations (x_0, v_0) and (x_1, v_1) (in the *tangent bundle* of $\mathbb{E}^2$) and would satisfy the additional constraint that $|\kappa(t)| \leq k_0$ (almost everywhere).

NOTE : DUBINS proved that *optimal arcs are concatenations of circular arcs (with constant curvature* k_0 *) and straight line segments.* Moreover, he proved that optimal arcs consists of at most three pieces and that the line segment – if there is any – has to be in the middle. This reduces finding the optimal arcs to a finite problem. There are at most six candidates for optimal arcs. So all one has to do is to determine these arcs and compare their lengths.

One of the well-known interpretations of this problem is to think of a car moving with constant speed in the plane subject to the constraint that it cannot make arbitrarily sharp turns (see also the *unicycle*).

Indeed, consider a car moving in the plane. The car can move forward with a fixed *linear velocity* and simultaneously rotate with a bounded *angular velocity*. Given unitial and terminal positions, and orientation of the car in the plane, *the problem is to drive the car from the initial configuration to the terminal one in minimal time.*

Admissible paths of the car are (geometric) curves with bounded curvature. Suppose that curves are parametrized by arc length; then our problem can be stated geometrically : Given two points in the plane and two unit velocity vectors attached respectively at these points, one has to find a (parametrized) curve in the plane that starts at the first point with the first velocity vector and comes to the second point with second velocity vector, has curvature bounded by a given constant, and has the minimal length among all such curves.

NOTE : If curvature is unbounded, then the problem, in general, has no solution. Indeed, the infimum of lengths of all curves that satisfy the boundary conditions without bound on curvature is the distance between the initial and terminal points :

the segment of the straight line through these points can be approximated by smooth curves with the required boundary conditions. But this infimum is not attained when the boundary velocity vectors do not lie on the line through the boundary points and are not collinear one to another.

After rescaling, we obtain a time-optimal control problem (T-OCP) :

$$\boxed{\begin{array}{c} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ u \end{bmatrix}, \qquad (x,\theta) = (x_1, x_2, \theta) \in \mathbb{E}^2 \times \mathbb{S}^1,\ |u| \leq 1 \\ x(0) = x_0, \theta(0) = \theta_0, x(t_1) = x_1, \theta(t_1) = \theta_1 \qquad (x_0, \theta_0, x_1, \theta_1 \ \text{fixed}) \\ t_1 = \int_0^{t_1} 1\, dt \ \to \ \min. \end{array}}$$

Note : The problem of Dubins also admits a natural formulation (as a T-OCP) on the special Euclidean group $\mathsf{SE}(2)$, associated with the Serret-Frenet control system.

Existence of solutions is guaranted by the Filippov's Theorem.

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We apply *Pontryagin's Maximum Priciple* (PMP). We have $(x_1, x_2, \theta) \in M = \mathbb{E}^2 \times \mathbb{S}^1$ and let (ξ_1, ξ_2, μ) be the corresponding coordinates of the adjoint vector. Then

$$\lambda = (x, \theta, \xi, \mu) \in T^*M$$

and the control-dependent Hamiltonian is

$$\mathcal{H}_u(\lambda) = \xi_1 \cos\theta + \xi_2 \sin\theta + \mu u.$$

The Hamiltonian system of PMP yields

$$\begin{aligned} \dot{\xi} &= 0 \\ \dot{\mu} &= \xi_1 \sin\theta - \xi_2 \cos\theta \end{aligned}$$

and the maximality condition reads

$$\mu(t)u(t) = \max_{|u|\leq 1} \mu(t)u.$$

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NOTE : T-OCPs constitute one of the basic concerns of optimal control theory. Minimal-time problems go back to the beginnings of the calculus of variations. JOHAN BERNOULLI's solution of the *brachistochrone problem*in 1697 was based on Fermat's principle of least time, which postulates that "light traverses any medium in the least possible time". Since then such problems have remained important sources of inspiration.

4.4 Other Problems

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An OCP on $\mathsf{SO}(3)$

Let $\mathsf{SO}(3)$ be the rotation group. (Recall that $\mathsf{SO}(3)$ is a compact and connected matrix Lie group, of dimension 3, whose associated Lie algebra $\mathfrak{so}(3)$ consists of all 3×3 skew-symmetric matrices). A *driftless, left-invariant control system* on $\mathsf{SO}(3)$ can be written in the following form :

$$\dot{g} = g\left(u_1 A_1 + u_2 A_2 + u_3 A_3\right), \qquad g \in \mathsf{SO}(3)$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

form the *standard basis* of $\mathfrak{so}(3)$ (see **Exercise 162**). The Lie algebra structure of $\mathfrak{so}(3)$ is given by the following table for the Lie bracket (commutator) :

$[\cdot,\cdot]$	A_1	A_2	A_3
A_1	0	A_3	$-A_2$
A_2	$-A_3$	0	A_1
A_3	A_2	$-A_1$	0

NOTE : The *minus Lie-Poison structure* on $\mathfrak{so}(3)^*$ is given by

$$\Pi = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}.$$

Exercise 25 Show that there are *only* four different driftless, left-invariant control systems on $\mathsf{SO}(3)$, and these are :

(1) $\dot{g} = g\,(u_1A_1 + u_2A_2)$.

(2) $\dot{g} = g\,(u_1A_1 + u_3A_3)$.

(3) $\dot{g} = g\,(u_2A_2 + u_3A_3)$

(4) $\dot{g} = g\,(u_1A_1 + u_2A_2 + u_3A_3)$.

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