Appendix A

Answers and Hints to Selected Exercises

Geometric transformations

1. (M_1) and (M_2) are immediate, but (M_3) requires some work. (For a "clever" solution, you may think of the *dot product* of two points (vectors) in $\mathbb{R}^2$).

2. TRUE. Find the equation of the line.

3. The lines are *parallel* if and only if their direction vectors are collinear, and are *perpendicular* if and only if their direction vectors are orthogonal. Thus

$$\mathcal{L} \parallel \mathcal{M} \iff \begin{bmatrix} -b \\ a \end{bmatrix} = r \begin{bmatrix} -e \\ d \end{bmatrix} \quad \text{for some } r \in \mathbb{R} \setminus \{0\} \iff ae - bd = 0$$

and

$$\mathcal{L} \perp \mathcal{M} \iff \begin{bmatrix} -b \\ a \end{bmatrix} \bullet \begin{bmatrix} -e \\ d \end{bmatrix} = 0 \iff ad + be = 0.$$

4. TRUE. Find the equation of the line.

5. The line passing through P_2 and P_3 has equation

$$\begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix} = 0.$$

6. The set *cannot* be finite.

7. The mapping is invertible. (One can solve uniquely for x and y in terms of x' and y').

8. TRUE.

9. Recall (and use) the fact that a line is determined by two points.

10. Yes. (The mapping is invertible and coincides with its inverse).

11. Relation $PQ + QR = PR$ (equality in the triangle inequality) implies

$$Q - P = s\,(R - Q) \quad \text{for some } s > 0.$$

Conversely, we have $PQ = t\,PR$, $QR = (1-t)\,PR$, etc.

12. (a), (d), (f), (h), (i).

13. (d) $3ax + 2by + 6c = 0$; (f) $bx + 3ay + 3(c - 2a) = 0$; (h) $ax + by - c = 0$; (i) $ax + by + (c - 2a + 3b) = 0$.

14. (a) $y = -5x + 7$; (b) $y = -5x - 7$; (c) $y = 5x - 7$; (d) $x - 9y - 32 = 0$.

15. $TTTT$ $TTFT$ TF.

16. For instance, examples (8), (9), and (10) from **1.2.2**. (Find other examples.)

17. $x - 10y - 2 = 0$.

18. The necessary and sufficient condition for α to be a transformation is $ad - bc \neq 0$. Such a transformation is *always* a collineation.

19. Straightforward verification.

20. (a) $(x, y) \mapsto (x, y) + x(0, 1)$ is a shear (about the y-axis); the image of the unit square is a parallelogram. (b) $(x, y) \mapsto (y, x)$ is a reflection (in the angle bisector of the first quadrant); the image of the unit square is also the unit square. (c) $(x, y) \mapsto (x, y) + x^2(0, 1)$ is a generalized shear; the image of the unit square is a curvilinear quadrilateral (with two sides line segments). (d) $(x, y) \mapsto \left(x, \frac{y}{2}\right) \mapsto \left(x, -x + \frac{y}{2}\right) \mapsto \left(-x + \frac{y}{2}, x\right) \mapsto \left(-x + \frac{y}{2}, x + 2\right)$ is a product of transformations (strain + shear + reflection + translation); the image of the unit square is a parallelogram. (The decomposition is not unique. Find other decompositions, for instance : strain + shear + rotation + reflection.)

21. (a) $\beta\alpha = \gamma\alpha \Rightarrow \beta\alpha(\alpha^{-1}) = \gamma\alpha(\alpha^{-1}) \Rightarrow \beta(\alpha\alpha^{-1}) = \gamma(\alpha\alpha^{-1}) \Rightarrow \beta\iota = \gamma\iota \Rightarrow \beta = \gamma$. In particular, for $\gamma = \iota$, one has (c) $\beta\alpha = \alpha \Rightarrow \beta = \iota$. The parts (b), (d), and (e) can be proved analogously.

22. $TTTF$ FF.

23. True. (The group generated by the rotation of 1 rad is an infinite cyclic group.)

24. $TTFF$.

25. $a = b \in \mathbb{R} \setminus \{0\}$.

Translations and halfturns

26. Express condition (4) in coordinates : $(x_B - x_A)^2 + (y_B - y_A)^2 = (x_D - x_C)^2 + (y_D - y_C)^2$ and $\frac{y_B - y_A}{y_D - y_C} = \frac{x_B - x_A}{x_D - x_C} = t > 0$, etc.

27. If the points A, B, and C are collinear, then the parallelogram $\square CABD$ becomes a "degenerate" one. (What is a *degenerate* parallelogram ?)

28. A translation : τ^{-1}.

29. The LHS is a product of five halfturns that fixes Q.

30. $FFTT \quad TTTF \quad TT$.

31.
- The halfturn σ_A, where $A = (\frac{3}{2}, -4)$.
- $x' = x + a - g$ and $y' = y + c - h$.

32. TRUE.

33. $x' = x + a_5$ and $y' = y + b_5$.

34. $5x - y + 27 = 0$.

35.
- $\sigma_M \alpha \sigma_P(P) = P \Rightarrow \alpha \sigma_P = \sigma_M$.
- $\sigma_P \alpha = \sigma_N$ (P is the midpoint of M and N).

36. $\sigma_P(\mathcal{L})$ is the line with equation $y = 5x - 21$.

37. For $n \in \mathbb{Z} \setminus \{0\}, \quad \tau_{P,Q}^n \neq \iota$.

38. $\tau_{P,Q} \in \langle \tau_{R,S} \rangle \Rightarrow \exists m \in \mathbb{Z} : \tau_{P,Q} = \tau_{R,S}^m = \tau_{S,R}^{-m}$.

39. TRUE.

40. $(a) \quad X = (0, -1)$; $(b) \quad Y = (0, \frac{1}{2})$; $(c) \quad Z = (0, -2)$.

Reflections and rotations

41. If a product $\alpha_2 \alpha_1$ is invertible, then α_1 is one-to-one and α_2 is onto.

42. Show that certain angles are supplementary.

43. Yes.

45. A line is uniquely determined by two (distinct) points.

46. $TFFF \quad TF$.

47. The given reflection has the equations $x' = \frac{1}{5}(-3x + 4y) + 4, \quad y' = \frac{1}{5}(4x + 3y) - 2$. $(0,0) \mapsto (4,-2), \quad (1,-3) \mapsto (1,-3), \quad (-2,1) \mapsto (6,-3), \quad (2,4) \mapsto (6,2)$.

48. Reflection in the line through O and orthogonal to $\overleftrightarrow{OO'}$.

49. (a) FALSE. (Notice that the statement "$\sigma_{\mathcal{L}} \sigma_{\mathcal{M}} = \sigma_{\mathcal{M}} \sigma_{\mathcal{L}} \iff \mathcal{L} = \mathcal{M}$ or $\mathcal{L} \perp \mathcal{M}$" is TRUE.) (b) TRUE.

50. FALSE. (Find a counterexample.)

Isometries I

52. No. (Find a simple counterexample.)

53. $2x + y = 5$ and $4x - 3y = 10$.

54. $x' = -x + 4, \quad y' = -y + 6$ (halfturn) and $x' = x, \quad y' = y + 4$ (translation).

55. TRUE. $\sigma_{\mathcal{L}}\sigma_{\mathcal{L}} = \iota$

56. FALSE. (Consider an equilateral triangle.)

57. $TTTF \quad TFFF \quad F$.

58. If $Q = \sigma_{\mathcal{M}}(P) = \sigma_{\mathcal{N}}(P) \neq P$ then $\mathcal{M}$ and $\mathcal{N}$ are both perpendicular bisectors of $\overline{PQ}$, contradiction. Hence, $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}(P) = P \;\Rightarrow\; P \in \mathcal{M} \cap \mathcal{N}$.

59. TRUE.

61. $x' = x + 4, \quad y' = y + 2$.

62. $\sigma_{\mathcal{L}}\rho_{C,r}\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}\sigma_{\mathcal{M}} = \rho_{C,-r}$.

63. $TFTT \quad FTFT \quad TT$.

64. Consider the perpendicular bisector of $\overline{AC}$. There are two cases; in each case construct the desired rotation.

65. TRUE. $\tau = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}} = (\sigma_{\mathcal{N}}\sigma_{\mathcal{L}})(\sigma_{\mathcal{L}}\sigma_{\mathcal{M}})$.

66. TRUE.

67. Let $C \in \mathcal{L}$. If $\rho_{C,r}(\mathcal{L}) = \mathcal{L}$, then let $\mathcal{M} \perp \mathcal{L}$ and $\rho_{C,r} = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$. We have

$$\mathcal{L} = \rho_{C,r}(\mathcal{L}) = \sigma_{\mathcal{N}}(\mathcal{L}) \Rightarrow (1)\;\; \mathcal{L} = \mathcal{N} \quad \text{or} \quad (2)\;\; \mathcal{L} \perp \mathcal{N}.$$

In the first case, $\rho_{C,r}$ is a halfturn (and so any line through C is fixed), whereas in the second case, $\rho_{C,r}$ is the identity transformation (and so any line is fixed).

68. $(\mathcal{M}) \quad 2x - y + c = 0$ and $(\mathcal{N}) \quad 4x - 2y + 2c - 15 = 0$ (parallel lines).

69. If the lines are neither concurent nor parallel, then $\sigma_{\mathcal{C}}\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}$ is the product of a (nonidentity) rotation and a reflection (in whatever order). Such a product *cannot* be a reflection : assume the contrary and derive a contradiction.

70. $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}} = (\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}})^{-1} = \sigma_{\mathcal{L}}\sigma_{\mathcal{M}}\sigma_{\mathcal{N}}$.

Isometries II

71. There are several cases.

72. $\left(\alpha\beta\alpha^{-1}\right)^2 = \iota \iff \alpha\beta^2\alpha^{-1} = \iota \iff \beta^2 = \iota$.

73. Consider the identity $\alpha\rho_{C,r}\alpha^{-1} = \rho_{\alpha(C),\pm r}$ (where α is *any* isometry).

74. Any translation is a product of two (special) rotations.

75. A translation fixing line $\mathcal{C}$ commutes with (the reflection) $\sigma_{\mathcal{C}}$.

76. $TTTT \quad FT$.

77. TRUE.

78. Let $\rho_1 = \rho_{C_1,r} = \sigma_{\mathcal{C}}\sigma_{\mathcal{A}}$ with $\{C_1\} = \mathcal{A}\cap\mathcal{C}$, and $\rho_2 = \rho_{C_2,s} = \sigma_{\mathcal{B}}\sigma_{\mathcal{C}}$ with $\{C_2\} = \mathcal{B}\cap\mathcal{C}$. Then

- $\rho_2\rho_1 = \sigma_{\mathcal{B}}\sigma_{\mathcal{A}} = \rho_{C,r+s}$ and
- $\rho_2^{-1}\rho_1 = \sigma_{\mathcal{B}'}\sigma_{\mathcal{A}} = \rho_{C',r-s} \quad (r \neq s)$.

The points C_1, C, and C' are collinear : they all lie on the line $\mathcal{A}$.

79. Let $\tau = \tau_{A,B}$ (that is, $\tau(A) = B$). Then

$$\tau\sigma_{\mathcal{C}} = \sigma_{\mathcal{C}}\tau \iff \sigma_{\mathcal{C}}\tau\sigma_{\mathcal{C}}^{-1} = \tau \iff \tau_{\sigma_{\mathcal{C}(A)},\sigma_{\mathcal{C}(B)}} = \tau_{A,B} \iff \overleftrightarrow{AB} \parallel \mathcal{C} \iff \tau(\mathcal{C}) = \mathcal{C}.$$

80. $FFFT \quad FF$.

81. TRUE. Let $\gamma = \sigma_{\mathcal{C}}\tau_{A,B}$. We can choose $A, B \in \mathcal{C}$ such that the given point M is the midpoint of the segment $\overline{AB}$.

82. TRUE. $\gamma = \sigma_P\sigma_{\mathcal{L}} = \sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$.

83. Translations and halfturns are dilatations. Reflections are not. Neither are glide reflections : if $\sigma_{\mathcal{L}}\sigma_P$ where a dilatation δ, then $\sigma_{\mathcal{L}}$ would be the dilatation $\delta\sigma_P$.

84. Let $\tau = \tau_{A,B}$ (that is, $\tau(A) = B$). Then, for the glide reflection $\gamma = \sigma_{\mathcal{L}}\sigma_A$,

$$\gamma^2 = \sigma_{\mathcal{L}}\sigma_A\sigma_{\mathcal{L}}\sigma_A = \left(\sigma_{\mathcal{L}}\sigma_A\sigma_{\mathcal{L}}^{-1}\right)\sigma_A = \sigma_{\sigma_{\mathcal{L}}(A)}\sigma_A = \sigma_M\sigma_A = \tau_{A,B} = \tau.$$

85.
- $\rho_{O,90}$: $x' = -y$ and $y' = x$.
- $\rho_{O,180}$: $x' = -x$ and $y' = -y$.
- $\rho_{O,270}$: $x' = y$ and $y' = -x$.

86. Since α is an odd isometry, we have that α is a reflection $\iff$ $\alpha = \alpha^{-1}$. The equations for α^{-1} are $x' = ax + by - (ah + bk)$ and $y' = bx - ay + ak - bh$.

87. $TFTT \quad TTT$.

88. $r = 150$. Let $P = (u, v)$. Then $1 = (1-\cos r)u + (\sin r)v$ and $-\frac{1}{2} = (1-\cos r)v - (\sin r)u$ imply $u = \frac{4-\sqrt{3}}{4}$ and $v = \frac{3-2\sqrt{3}}{4}$.

89. C is the *only* point fixed by $\rho_{C,r}$ with $r \neq 0$. The coordinates of C are

$$x_C = \frac{(1-\cos r)h - (\sin r)k}{2(1-\cos r)} \quad \text{and} \quad y_C = \frac{(1-\cos r)k + (\sin r)h}{2(1-\cos r)}\,.$$

90. $\mathcal{L}$ is the *only* line fixed by $\sigma_{\mathcal{L}}$. An equation for $\mathcal{L}$ is $(a-1)x+by+h=0$ or, equivalently, $bx-(a+1)y+k=0$.

Symmetry

92. (a) Yes. (b) No.

93. A *bounded* figure cannot have two points of symmetry, because if it had, it would be invariant under a nonidentity translation.

94. (a) $\mathfrak{C}_1$. (b) $\mathfrak{D}_1$.

95. (a) $\mathfrak{C}_2$ or $\mathfrak{D}_2$.

97. $\mathfrak{D}_2$.

98. $\mathfrak{D}_2$.

99. $FTTF$.

100. $\mathfrak{D}_3$.

103. There are *ten* equivalence classes :

- **A**, **M**, **T**, **U**, **V**, **W**.
- **B**, **C**, **D**, **E**, **K**.
- **F**, **G**, **J**, **P**, **R**.
- **H**, **I**.
- **L**.
- **N**, **S**, **Z**.
- **O**.
- **Q**.
- **X**.
- **Y**.

104. (c) Use the relation $\sigma\rho^k=\rho^{-k}\sigma$ to show that $\sigma\rho^{n-1}\neq\rho^{n-1}\sigma$.

105. The polygons cannot be regular. (Find more than one example in each case.)

Similarities

107. The inverse of a similarity is also a similarity.

109. The function is invertible.

112. In each case we have

- $C=rA+(1-r)B=B+r(A-B)\in\overleftrightarrow{AB}, \quad C\neq B.$

- $C = \frac{s(1-r)}{1-rs}A + \frac{1-s}{1-rs}B \in \overleftrightarrow{AB}$.
- $C = \frac{1}{1-r}B + \frac{r}{r-1}A \in \overleftrightarrow{AB}$.

113. Let $P = (h,k)$. Then $x' = -2(x-h)+h,\ \ y' = -2(y-k)+k \ \ \Rightarrow \ \ h = 1$ and $k = -\frac{4}{3}$.

114. A stretch reflection fixes *exactly one* point and *exactly two* lines. A stretch rotation fixes *no* line.

115. $TTTF \quad FTT$.

116. FALSE.

117. TRUE. α is 1-1 and onto ($\alpha\beta$ is onto $\Rightarrow$ α is onto). See NOTE 2 : Distance-preserving mappings.

118. $(a)\quad P = \left(-\frac{7}{2}, \frac{5}{2}\right)$; $(b)\quad t = 5$; $(c)\quad x = -15$; $(d)\quad P + \delta_{C,r}(B) = Q + \delta_{C,r}(A)$; in particular, $P = \delta_{C,r}(A)$ and $Q = \delta_{C,r}(B)$; (e) $\delta_{B,s}(A)$; (f) $x = r$; (g) $C = \tau^2_{B,A}(B)$.

119. FALSE.

120. $r = \frac{5\sqrt{2}}{4}$.

121. α is a direct similarity with equations $x' = x - 2y + 1, \quad y' = 2x + y$. $\alpha((-1,6)) = (-12,4)$. (α is a stretch rotation; find its center and the angle of rotation.)

122. An involutory similarity is an isometry ($\alpha = \beta\delta \Rightarrow \delta = \iota$).

123. TRUE. (Consider first the case when the circles are equal (i.e $A = C$ and $AB = CD$)).

124. $\sigma_{\mathcal{L}} = \delta_{P,r}\sigma_{\mathcal{L}}\delta^{-1}_{P,r} = \sigma_{\delta_{P,r}(\mathcal{L})} \iff \mathcal{L} = \delta_{P,r}(\mathcal{L}) \iff P \in \mathcal{L}$.

Affine transformations

126. You may use (in a clever way) the *cross product* of two vectors in $\mathbb{R}^3$.

127. (b) $\begin{vmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

128. $[\alpha\beta] = [\alpha][\beta]$ (closure property); $[\alpha^{-1}] = [\alpha]^{-1}$ (inverse property).

129.

$$[\delta_{P,r}] = \begin{bmatrix} 1 & 0 \\ (1-r)\mathbf{v} & rI \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} h \\ k \end{bmatrix}.$$

130. (a) For α_k : if $k = 1$ ($\alpha_k = \iota$), then every point is fixed; if $k \neq 1$, then only the points on the y-axis are fixed. (b) $PQR = 15$. (c) $\alpha_k(P)\alpha_k(Q)\alpha_k(R) = 15 \cdot |k|$; $\beta_k(P)\beta_k(Q)\beta_k(R) = 15$.

131. $TFFT \quad TTF$.

132. $\underline{P_0P'} = k \cdot \underline{P_0P}, \quad k \neq 0$.

133. $x = \frac{1}{\Delta}(dx' - by') - \frac{1}{\Delta}(dh - bk)$ and $y = \frac{1}{\Delta}(-cx' + ay') - \frac{1}{\Delta}(-ch + ak)$ $(\Delta = ad - bc \neq 0)$. We have

$$[\alpha] = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & A \end{bmatrix} \quad \text{and} \quad [\alpha^{-1}] = [\alpha]^{-1} = \begin{bmatrix} 1 & 0 \\ -A^{-1}\mathbf{v} & A^{-1} \end{bmatrix}.$$

134. TRUE.

135. A similarity is a product of a stretch and an isometry, whereas a stretch is a product of two strains (about perpendicular lines).

136. The equations of the given transformation (shear) are *not* of the form

$$x' = (ar)x - (bs)y + h \quad \text{and} \quad y' = \pm\left((br)x + (as)y\right) + k.$$

137. $x' = x - y, \quad y' = y$ and $x' = x, \quad y' = x + y$. Only one point is fixed by the product of these two transformations (shears).

138. TRUE. (A similarity that preserves area must be an isometry.)

139. TRUE. (An involutory affine transformation is an isometry.)

140. $x' = 2x$ and $y' = \frac{1}{2}y$. (Find other examples.)