# Appendix B

# Revision Problems

1. Find all triangles such that three given noncollinear points are the midpoints of the sides of the triangle.

2. Prove that :

   i. $\sigma_A\sigma_B = \sigma_B\sigma_C \iff B$ is the midpoint of $\overline{AC}$.

   ii. $\sigma_A\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}\sigma_B \iff \mathcal{L}$ is the perpendicular bisector of $\overline{AB}$.

3. Prove that any distance-preserving *mapping* $\alpha : \mathbb{E}^2 \to \mathbb{E}^2$ is a bijection (hence an isometry).

4. If $\square ABCD$ and $\square EFGH$ are congruent rectangles and $AB \neq BC$, then how many isometries are there that take one rectangle to the other ?

5. Show that the product of the reflections in the three angle bisectors of a triangle is a reflection in a line perpendicular to a side of the triangle.

6. If $\mathcal{L}, \mathcal{M}, \mathcal{N}$ are the perpendicular bisector of sides $\overline{AB}, \overline{BC}, \overline{CA}$, respectively, of $\triangle ABC$, then $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}$ is a reflection in which line ?

7. If $\mathcal{L}$ and $\mathcal{M}$ are distinct intersecting lines, find the locus of all points $P$ such that $\rho_{P,r}(\mathcal{L}) = \mathcal{M}$ for some $r$.

8. Let $A, B, C$ be three noncollinear points such that $m(\angle ABC) = 45$, $m(\angle BCA) = 105$ and $m(\angle CAB) = 30$. Find the fixed point of

$$\rho_{B,90}\rho_{A,60}.$$

9. Given a figure consisting of two points $P$ and $Q$, sketch a construction of the fixed point of $\tau_{P,Q}\rho_{P,45}$.

10. Given a figure consisting of three points $A, B, C$, sketch a construction of the fixed point of $\tau_{B,C}\rho_{A,120}$.

11. Indicate all pairs of *commuting* elements in the dihedral groups $\mathfrak{D}_3$ and $\mathfrak{D}_4$.

12. Show that any two parabolas are similar.

13. Find all stretch reflections taking point $A$ to point $B$. Also, find all stretch rotations taking point $A$ to point $B$.

14. Show that any given ellipse is the image of the unit circle under some affine transformation.

15. If $x' = ax + by + h$ and $y' = cx + dy + k$ are the equations of mapping $\alpha$ and $ad - bc = 0$, then show that $\alpha$ is not a collineation since all images are collinear.

16. Find equations for the strain of ratio $r$ about the line with equation $y = mx$.

17. Find

    (a) the equations for $\tau_{P,Q}^{-1}$ if $P = (a, b)$ and $Q = (h, k)$.

    (b) the equations for $\sigma_{\mathcal{L}}$ if $\mathcal{L}$ has equation $y = -x + 1$.

    (c) the image of the line with equation $x + y = 3$ under the transformation $\sigma_{\mathcal{L}}$, where $\mathcal{L}$ has equation $y = x - 1$.

    (d) the mage of the line with equation $x = 2$ under $\sigma_A\sigma_B$, where $A = (2, 1)$ and $B = (1, 2)$.

**Class test, March 1999**

18. Find

    (a) the equations for the rotation $\rho_{C,30}$ where $C = (2, -1)$.

    (b) $\mathcal{L}$, if $x' = \frac{3}{5}x + \frac{4}{5}y$ and $y' = \frac{4}{5}x - \frac{3}{5}y$ are equations for the isometry $\sigma_{\mathcal{L}}$.

    (c) the image of the point $P = (a, -b)$ and of the line $\mathcal{L}$ with equation $bx + ay = 0$ under the isometry

$$x' = ax - by - 1, \quad y' = bx + ay \quad (\text{with } a^2 + b^2 = 1).$$

**Class test, May 1999**

19. (a) Find equations of the dilation $\delta_{P,-3}$ about the point $P=(1,-1)$.
    (b) If $\sigma_P((x,y))=(-2x+3,-2y-4)$, find $P$.
    (c) Find equations of the strain of ratio $k$ about the line with equation $y=2x$.

**Exam, June 1999**

20. (a) If $\rho_{C,r}((x,y))=((\cos r)x-(\sin r)y+m,(\sin r)x+(\cos r)y+n)$, find $C$.
    (b) If $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}((x,y))=(x+6,y-3)$, find equations for lines $\mathcal{L}$ and $\mathcal{M}$.
    (c) Find all direct similarities $\alpha$ such that

$$\alpha((1,0))=(0,1) \qquad \text{and} \qquad \alpha((0,1))=(1,0).$$

**Exam, June 1999**

21. (a) Determine the preimage of the point $(-1,3)$ unde the mapping $(x,y)\mapsto\big(-x+\frac{y}{3},3x-y\big)$.
    (b) Write the equations for $\tau_{P,Q}^{-1}\sigma_P$ if $P=(1,1)$ and $Q=(3,3)$.
    (c) Find the image of the line with equation $x=2$ under $\sigma_A\sigma_B$, when $A=(2,1)$ and $B=(1,2)$.

**Class test, March 2000**

22. (a) What is the symmetry group of the capital letter **H** (written in most symmetric form) ? Describe this group (e.g. order of the group, generators, etc.).
    (b) Given the lines $(\mathcal{A})\ x+y=0$ and $(\mathcal{B})\ x+y=1$, find points $A$ and $B$ such that $\sigma_{\mathcal{B}}\sigma_{\mathcal{A}}=\sigma_B\sigma_A$.
    (c) Find equations for the rotation $\rho_{C,30}$ where $C=(2,-1)$.

**Class test, May 2000**

23. (a) What is the image of the line with equation $x-y+1=0$ under the reflection in the line with equation $2x+y=0$ ?
    (b) If $\rho_{C,r}((x,y))=(-y+h,x+k)$, find $C$ and $r$.
    (c) Find equations of the dilation $\delta_{P,-3}$ about the point $P=(1,-1)$.

(d) Find all stretch reflections taking point $A = (1,1)$ to point $B = (2,0)$.

(e) If $x' = (1-ab)x - b^2y, \quad y' = a^2x + (1+ab)y$ are equations for a shear about the line $\mathcal{L}$, find $\mathcal{L}$. What is the ratio of this shear ?

**Exam, June 2000**

24. Consider the transformation

$$\alpha : \mathbb{E}^2 \to \mathbb{E}^2, \qquad (x,y) \mapsto (x+y, y).$$

Show that $\alpha$ is a collineation. Is $\alpha$ an isometry ? Motivate your answer.

**Class test, March 2001**

25. (a) Write the equations for $\sigma_P \tau_{P,Q}^{-1}$ if $P = (a,b)$ and $Q = (c,d)$. What is this transformation ?

(b) Find the image of the line with equation $x = 2$ under $\sigma_A \sigma_B$, when $A = (2,1)$ and $B = (1,2)$.

**Class test, March 2001**

26. (a) What is the symmetry group of the capital letter **H** (written in most symmetric form) ? Describe this group (e.g. order of the group, generators, etc.).

(b) If $\sigma_{\mathcal{A}}\sigma_{\mathcal{B}}((x,y)) = (x-4, y+2)$, find equations for lines $\mathcal{A}$ and $\mathcal{B}$.

(c) Find equations for

- the relection $\sigma_{\mathcal{L}}$
- the rotation $\rho_{C,r}$

that map the $x$-axis onto the line $\mathcal{M}$ with equation $x - y + 2 = 0$.

**Class test, May 2001**

27. (a) What is the image of the line with equation $x - y + 1 = 0$ under the reflection in the line with equation $2x + y = 0$ ?

(b) If $\rho_{C,r}((x,y)) = (-y+h, x+k)$, find $C$ and $r$.

(c) Find all dilatations taking the circle with equation $x^2 + y^2 = 1$ to the circle with equation $(x-1)^2 + (y-2)^2 = 5$.

(d) Find equations for the strain of ratio $k$ about the line with equation $2x - y = 0$.

(e) If $x' = (1-ab)x - b^2y, \quad y' = a^2x + (1+ab)y$ are equations for a shear about the line $\mathcal{L}$, find $\mathcal{L}$. What is the ratio of this shear ?

**Exam, June 2001**

28. Consider the mapping

$$\alpha : \mathbb{E}^2 \to \mathbb{E}^2, \qquad (x,y) \mapsto (x, x^2 + y).$$

(a) Show that $\alpha$ is a transformation.

(b) Is $\alpha$ a collineation ? Motivate your answer.

(c) What is the *preimage* of the parabola with equation $y - x^2 = 0$ under the transformation $\alpha$ ?

(d) Find the points and lines *fixed* by the transformation $\alpha$.

**Class test, August 2002**

29. (a) Write the equations for the reflection $\sigma$ in the line with equation

$$(\sin r)x - (\cos r)y = 0.$$

Use these equations to verify that $\sigma$ is a transformation.

(b) Compute the (Cayley table for) the symmetry group of an equilateral triangle.

**Class test, August 2002**

30. (a) If $\sigma_{\mathcal{M}}\sigma_{\mathcal{L}}((x,y)) = (x+6, y-3)$, find equations for lines $\mathcal{L}$ and $\mathcal{M}$.

(b) If

$$x' = -\frac{\sqrt{3}}{2}x - \frac{1}{2}y + 1, \qquad y' = \frac{1}{2}x - \frac{\sqrt{3}}{2}y - \frac{1}{2}$$

are equations for the rotation $\rho_{C,r}$, then find the centre $C$ and the angle $r$.

**Class test, October 2002**

31. (a) Compute (the Cayley table for) the symmetry group of an equilateral triangle.

(b) Show that if

$$x' = (\cos r)x - (\sin r)y + h, \qquad y' = (\sin r)x + (\cos r)y$$

are equations for nonidentity rotation $\rho_{C,r}$, then

$$x_C = \frac{h}{2} \quad \text{and} \quad y_C = \frac{h}{2} f\left(\frac{r}{2}\right)$$

where $f$ is a function to be determined.

(c) Find all dilatations taking the circle with equation $x^2 + y^2 = 1$ to the circle with equation $(x-1)^2 + (y-2)^2 = 5$.

(d) Find equations for the strain of ratio $k$ about the line with equation $3x - y = 0$.

**Exam, November 2002**

32. Consider the mapping $\alpha : \mathbb{E}^2 \to \mathbb{E}^2$ given by

$$(x, y) \mapsto \left(\frac{1}{5}(-3x + 4y) + 4, \frac{1}{5}(4x + 3y) - 2\right).$$

(a) Verify that $\alpha$ is a transformation (on $\mathbb{E}^2$).

(b) Investigate whether $\alpha$ is a collineation. If so, find the image of the line $\mathcal{L}$ with equation $2x - y = 5$ under $\alpha$.

(c) Determine all points $P$ that are fixed by $\alpha$ (i.e. $\alpha(P) = P$).

(d) Find the image and the preimage of (the origin) $O = (0, 0)$ under $\alpha$.

(e) Write down the equations of the translation $\tau_{O,\alpha(O)}$ and the halfturn $\sigma_{\alpha(O)}$.

**Class test, August 2003**

33. Let $c, d \in \mathbb{R}$ and consider the transformation $a : \mathbb{E}^2 \to \mathbb{E}^2$ be given by the equations

$$x' = x + c \quad \text{and} \quad y' = -y + d.$$

(a) Look at the form of these equations and conclude that they represent an *odd isometry.* Explain.

(b) For what values of the parameters $c$ and $d$ is the given isometry $\alpha$

- a glide reflection ?
- a reflection ?

In each case, find the (equations of the) line fixed by $\alpha$ (i.e. the *axis* of the glide reflection and the *mirror* of the reflection, respectively).

**Class test, October 2003**

34.

(a) Given the point $C = (3, -2)$ and the line $\mathcal{L}$ with equation $x + y - 1 = 0$, find equations for $\delta_{C,2}$ (stretch), $\alpha_{\mathcal{L}}$ (reflection), and $\rho_{C,45}$ (rotation). Compute $\delta_{C,2}(\mathcal{L})$ and $\rho_{C,45}(\mathcal{L})$.

(b) Determine the collineation $\alpha$ such that

$$\alpha((0,0)) = (-1,4), \quad \alpha((-1,4)) = (-9,6), \quad \text{and} \quad \alpha((-9,16)) = (-13,22).$$

Is this collineation a similarity ? If yes, find its ratio. Identify $\alpha$.

(c) Find equations for $\alpha^2$ and then identify this transformation.

**Exam, November 2003**

35. Consider the mapping $\alpha : \mathbb{E}^2 \to \mathbb{E}^2$ given by

$$(x, y) \mapsto \left(\frac{1}{2}(x + \sqrt{3}y),\ \frac{1}{2}(\sqrt{3}x - y)\right).$$

(a) Verify that $\alpha$ is a transformation.

(b) Determine whether $\alpha$ is a collineation. If so, find the image of the line $\mathcal{L}$ with equation $\sqrt{3}x + y + 1 = 0$ under $\alpha$.

(c) Find *all* points $P$ that are fixed by $\alpha$ (i.e. $\alpha(P) = P$).

(d) Given the points $O = (0,0)$, $A = (1,0)$, $B = (1, \frac{1}{\sqrt{3}})$, and $C = (0, \frac{1}{\sqrt{3}})$

i. verify that the quadrilateral $\square OABC$ is a rectangle.

ii. determine *with justification* the image of $\square OABC$ under $\alpha$.

**Class test, September 2004**

36. Consider the points

$$O = (0,0), \quad A = (1,1) \quad \text{and} \quad B = (-1,1).$$

(a) Find the equations of all *four* isometries sending the segment $\overline{OA}$ onto the segment $\overline{OB}$.

(b) Identify these transformations. (Specify clearly, for a rotation : the centre and the directed angle, and for a reflection or glide reflection : the axis.)

(Hint : Use the general equations of an isometry.)

**Class test, October 2004**

37. Consider the points

$$A = (2,1), \quad B = (2,-2), \quad C = (-2,3), \quad D = (4,3), \quad P = \left(\frac{4}{5}, \frac{3}{5}\right), \quad Q = (0,-1)$$

and the line $\mathcal{L}$ with equation

$$x - y - 1 = 0.$$

(a) Find the equations of the following transformations :

i. the stretch $\delta_{P,2}$;

ii. the rotation $\rho_{P,90}$;

iii. The stretch rotation $\alpha_1 = \rho_{P,90}\delta_{P,2}$;

iv. the stretch $\delta_{Q,2}$;

v. the reflection $\sigma_{\mathcal{L}}$;

vi. the stretch reflection $\alpha_2 = \sigma_{\mathcal{L}}\delta_{Q,2}$.

(b) How many similarities are there sending the segment $\overline{AB}$ onto the segment $\overline{CD}$ ? Justify your claim.

(c) Find the equations of the unique *direct similarity* such that $A \mapsto C$ and $B \mapsto D$.

(d) Find the equations of the unique *opposite similarity* such that $A \mapsto D$ and $B \mapsto C$.

**Exam, November 2004**

38. Consider the points $O = (0,0)$ and $P = (0,2)$, the line $\mathcal{L}$ with equation $y - 1 = 0$, and the mappings

$$\begin{aligned} \alpha_1 : \quad (x,y) &\mapsto (x, x^2 + y) \\ \alpha_2 : \quad (x,y) &\mapsto (x, -y + 2) \\ \alpha_3 : \quad (x,y) &\mapsto \left(x - \frac{1}{2}y, -2x + y\right). \end{aligned}$$

(a) Find the point $\alpha_2(P)$ and the line $\alpha_2^{-1}(\mathcal{L})$.

(b) Find $\alpha_1(\mathcal{L})$ and $\alpha_3^{-1}(O)$.

(c) Which of the given mappings are *transformations* ? Which, if any, of these transformations is a *collineation* ? Justify your answers.

(d) Find the equations for the following (transformations) :

i. $\tau_{O,P}$.

ii. $\sigma_{\mathcal{L}}$.

iii. $\sigma_P$.

iv. $\tau_{O,P}\,\sigma_{\mathcal{L}}$.

**Class test, September 2005**

39. (a) Let $a, b, h, k \in \mathbb{R}$ such that $a^2 + b^2 = 1$. If

$$\begin{aligned} x' &= ax + by + h \\ y' &= bx - ay + k \end{aligned}$$

are the equations for isometry $\alpha$, show that $\alpha$ is a *reflection* if and only if

$$ah + bk + h = 0 \quad \text{and} \quad ak - bh = k.$$

(HINT : Find the equations for $\alpha^{-1}$ by using *Cramer's rule*.)

(b) Determine the fixed points (if any) of the following isometry

$$(x, y) \mapsto \left(\frac{1}{5}(-3x + 4y) + 4, \frac{1}{5}(-4x - 3y) + 2\right).$$

Hence identify this isometry.

(c) Find the *glide reflection* $\gamma$ such that

$$\gamma^2 = \tau_{A,B}, \quad \text{where } A = (1,1), B = (2,2).$$

**Class test, October 2005**

40. Consider the points

$$A = (4,2), \quad P = (h,k)$$

and the lines

$$\mathcal{L} : \quad y = 0, \qquad \mathcal{M} : \quad 2x = y = 3, \qquad \mathcal{N} : \quad 2x + y = 8.$$

(a) Write the equations for the following transformations :

i. the halfturn $\sigma_P$;

ii. the reflection $\sigma_{\mathcal{L}}$;

iii. the translation $\tau_{O,A}$;

iv. the rotation $\rho_{P,30}$;

v. the glide reflection $\sigma_{\mathcal{M}}\sigma_A$;

vi. the translation $\sigma_{\mathcal{N}}\sigma_{\mathcal{M}}$.

(b) Under what conditions the transformations $\sigma_P$ and $\sigma_{\mathcal{L}}$ do commute ?

(c) Find the point $C$ such that

$$\delta_{P,3}\delta_{A,\frac{1}{3}} = \tau_{A,C}.$$

(d) Let $B = \tau_{O,A}((0,-4))$. ($O$ is the origin.) Find the equations of all <u>four</u> isometries sending segment $\overline{OA}$ onto the segment $\overline{OB}$.

(Hint : Use the general equations for an isometry.)

**Exam, November 2005**