## Chapter 1

## Geometric Transformations

## Topics :

1. The Euclidean Plane $\mathbb{E}^{2}$
2. Transformations
3. Properties of Transformations


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### 1.1 The Euclidean Plane $\mathbb{E}^{2}$

Consider the Euclidean plane (or two-dimensional space) $\mathbb{E}^{2}$ as studied in high school geometry.

Note : It is customary to assign different meanings to the terms set and space. Intuitively, a space is expected to possess a kind of arrangement or order that is not required of a set. The necessity of a structure in order for a set to qualify as a space may be rooted in the feeling that a notion of "proximity" (in some sense not necessarily quantitative) is inherent in our concept of a space. Thus a space differs from the mere set of its elements by possessing a structure which in some way (however vague) gives expression to that notion. A direct quantitative measure of proximity is introduced on an abstract set $\mathbf{S}$ by associating with each ordered pair $(x, y)$ of its elements (called "points") a non-negative real number, denoted by $d(x, y)$, and called the "distance" from $x$ to $y$.

On this "geometric space" one introduces Cartesian coordinates which are used to define a one-to-one correspondence

$$
P \mapsto\left(x_{P}, y_{P}\right)
$$

between $\mathbb{E}^{2}$ and the set $\mathbb{R}^{2}$ of all ordered pairs of real numbers. This mapping preserves distances between points of $\mathbb{E}^{2}$ and their images in $\mathbb{R}^{2}$. It is the existence of such a coordinate system which makes the identification of $\mathbb{E}^{2}$ and $\mathbb{R}^{2}$ possible. Thus we can say that

$$
\mathbb{R}^{2} \text { may be identified with } \mathbb{E}^{2} \text { plus a coordinate system. }
$$

Note : The geometers before the 17th century did not think of the Euclidean plane $\mathbb{E}^{2}$ as a "space" of ordered pairs of real numbers. In fact it was defined axiomatically beginning with undefined objects such as points and lines together with a list of their properties - the axioms - from which the theorems of geometry where then deduced.

The identification of $\mathbb{E}^{2}$ and $\mathbb{R}^{2}$ (or, more generally, of $\mathbb{E}^{n}$ and $\mathbb{R}^{n}$ ) came about after the invention of analytic geometry by P. Fermat (1601-1665) and R.

Descartes (1596-1650) and was eagerly seized upon since it is very tricky and difficult to give a suitable definition of Euclidean space (of any dimension) in the spirit of Euclid. This difficulty was certainly recognized for a long time, and has interested many great mathematicians. It lead in part to the discovery of non-Euclidean geometries (like spherical and hyperbolic geometries) and thus to manifolds.

We make the following definition.
1.1.1 Definition. The Euclidean plane $\mathbb{E}^{2}$ is the set $\mathbb{R}^{2}$ together with the Euclidean distance between points $P=\left(x_{P}, y_{P}\right)$ and $Q=\left(x_{Q}, y_{Q}\right)$ given by

$$
d(P, Q)=P Q:=\sqrt{\left(x_{Q}-x_{P}\right)^{2}+\left(y_{Q}-y_{P}\right)^{2}} .
$$

Since the Euclidean distance $d: \mathbb{E}^{2} \times \mathbb{E}^{2} \rightarrow \mathbb{R}$ is the only distance function to be considered in this course, we shall called it, simply, the distance.

> Numbers will be denoted by lowercase Roman letters.

Note : The set $\mathbb{R}^{2}$ has the structure of a vector space (over $\mathbb{R}$ ). This means that the set $\mathbb{R}^{2}$ is endowed with a rule for addition

$$
\left(x_{1}, y_{1}\right)+\left(x_{2}, y_{2}\right):=\left(x_{1}+x_{2}, y_{1}+y_{2}\right)
$$

and a rule for scalar multiplication

$$
r(x, y):=(r x, r y)
$$

such that these operations satisfy the eight axioms below (for all $\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right),\left(x_{3}, y_{3}\right) \in$ $\mathbb{R}^{2}$ and all $\left.r, s \in \mathbb{R}\right)$ :
$(\mathrm{VS} 1) \quad\left(\left(x_{1}, y_{1}\right)+\left(x_{2}, y_{2}\right)\right)+\left(x_{3}, y_{3}\right)=\left(x_{1}, y_{1}\right)+\left(\left(x_{2}, y_{2}\right)+\left(x_{3}, y_{3}\right)\right) ;$
$(\mathrm{VS} 2) \quad\left(x_{1}, y_{1}\right)+\left(x_{2}, y_{2}\right)=\left(x_{2}, y_{2}\right)+\left(x_{1}, y_{1}\right)$;
(VS3) $\quad\left(x_{1}, y_{1}\right)+(0,0)=\left(x_{1}, y_{1}\right)$;
(VS4) $\quad\left(x_{1}, y_{1}\right)+\left(-x_{1},-y_{1}\right)=(0,0)$;
(VS5) $\quad r\left(\left(x_{1}, y_{1}\right)+\left(x_{2}, y_{2}\right)\right)=r\left(x_{1}, y_{1}\right)+r\left(x_{2}, y_{2}\right)$;
(VS6) $\quad(r+s)\left(x_{1}, y_{1}\right)=r\left(x_{1}, y_{1}\right)+s\left(x_{1}, y_{1}\right)$;
(VS7) $\quad r\left(s\left(x_{1}, y_{1}\right)\right)=r s\left(x_{1}, y_{1}\right)$;
$(\mathrm{VS} 8) \quad 1\left(x_{1}, y_{1}\right)=\left(x_{1}, y_{1}\right)$.
Hence, the Euclidean plane $\mathbb{E}^{2}$ is a (real, 2-dimensional) vector space.
A point $P$ of $\mathbb{E}^{2}$ is an ordered pair $(x, y)$ of real numbers.

## Points will be denoted by uppercase Roman letters.

Exercise 1 Verify that (for $P, Q, R \in \mathbb{E}^{2}$ ):

$$
\begin{equation*}
P Q \geq 0, \quad \text { and } \quad P Q=0 \Longleftrightarrow P=Q \tag{M1}
\end{equation*}
$$

(M2) $\quad P Q=Q P$;
(M3) $\quad P Q+Q R \geq P R$.
Relation (M3) is known as the triangle inequality.

Note : A set $\mathbf{S}$ equipped with a function $d: \mathbf{S} \times \mathbf{S} \rightarrow \mathbb{R}, \quad(P, Q) \mapsto P Q$ satisfying conditions (M1)-(M3) is called a metric space (with metric d). Hence the Euclidean plane $\mathbb{E}^{2}$ is not only a vector space. It is also a metric space.

It is important to realize that in order to do geometry we need a structure which provides lines. In Euclidean geometry, a (straight) line may be defined as either a curve with zero acceleration (i.e. such that the tangent vector to the curve is constant along the curve) or a curve which represents the shortest path between points.

In our (Cartesian) model of Euclidean plane it is convenient to define a line by specifying its (Cartesian) equation.

A line $\mathcal{L}$ in $\mathbb{E}^{2}$ is a set of points satisfying an equation $a x+b y+c=0$, where $a, b, c$ are real numbers with not both $a=0$ and $b=0$ (i.e. $a^{2}+b^{2} \neq$ $0)$.

Note : The triplets $(a, b, c)$ and $(r a, r b, r c), r \neq 0$ determine the same line. A point $P=\left(x_{P}, y_{P}\right)$ lies on the line with equation $a x+b y+c=0$ if (and only if) the coordinates of the point satisfy the equation of the line : $a x_{p}+b y_{p}+c=0$. If this is the case, we also say that the given line passes through the point $P$.

## Lines will be denoted by uppercase calligraphic letters.

Exercise 2 PROVE or DISPROVE : Through any two different points $P_{1}=\left(x_{1}, y_{1}\right)$ and $P_{2}=\left(x_{2}, y_{2}\right)$, there passes a unique line.

## Notation and terminology (review)

We introduce some basic geometric notation and terminology. This should be read now to emphasize the basic notation and used later as a reference.

- By the triangle inequality, $A B+B C \geq A C$. $A-B-C$ is read "point $B$ is between points $A$ and $C$ ", and means $A, B, C$ are three distinct points such that $A B+B C=A C$.
- $\overleftrightarrow{A B}$ is the unique line determined by two distinct points $A$ and $B$.
- $\overline{A B}$ is a line segment and consists of $A, B$ and all points between $A$ and $B$.
- $A B^{\rightarrow}$ is a ray from $A$ through $B$ and consists of all points in $\overline{A B}$ together with all points $P$ such that $A-B-P$.
- $\angle A B C$ is an angle and is the union of noncollinear rays $B A^{\rightarrow}$ and $B C \rightarrow$. $m(\angle A B C)$ is the degree measure of $\angle A B C$ and is a number between 0 and 180.
- $\triangle A B C$ is a triangle and is the union of noncollinear segments $\overline{A B}, \overline{B C}$, and $\overline{C A}$.
- "œ" is read "is congruent to" and has various meanings depending on context.
$-\overline{A B} \cong \overline{C D} \Longleftrightarrow A B=C D ;$
$-\angle A B C \cong \angle D E F \Longleftrightarrow m(\angle A B C)=m(\angle D E F) ;$
$-\triangle A B C \cong \triangle D E F \Longleftrightarrow \overline{A B} \cong \overline{D E}, \overline{B C} \cong \overline{E F}, \overline{A C} \cong \overline{D F}, \angle A \cong$ $\angle D, \angle B \cong \angle E, \angle C \cong \angle F$. Not all six corresponding parts must be checked to show triangles congruent. The familiar congruence theorems for triangles $\triangle A B C$ and $\triangle D E F$ are:
* (SAS) : If $\overline{A B} \cong \overline{D E}, \angle A \cong \angle D$, and $\overline{A C} \cong \overline{D F}$, then $\triangle A B C \cong$ $\triangle D E F$;
* (ASA) : If $\angle A \cong \angle D, \overline{A B} \cong \overline{D E}$, and $\angle B \cong \angle E$, then $\triangle A B C \cong$ $\triangle D E F$;
* (SAA) :If $\overline{A B} \cong \overline{D E}, \angle B \cong \angle E$, and $\angle C \cong \angle F$, then $\triangle A B C \cong$ $\triangle D E F$;
* (SSS) :If $\overline{A B} \cong \overline{D E}, \overline{B C} \cong \overline{E F}$, and $\overline{C A} \cong \overline{F D}$, then $\triangle A B C \cong$ $\triangle D E F$.
- The Exterior Angle Theorem states that given $\triangle A B C$ and $B-C-D$, then $m(\angle A C D)$
$=m(\angle A)+m(\angle B)$. So for $\triangle A B C$ we have $m(\angle A)+m(\angle B)+m(\angle C)=180$.
- Given $\triangle A B C$ and $\triangle D E F$ such that $\angle A \cong \angle D, \angle B \cong \angle E$, and $\angle C \cong$ $\angle F$, then $\triangle A B C \sim \triangle D E F$, where " $\sim$ " is read "is similar to". If two of these angle congruences hold, then the third congruence necessarily holds and the triangles are similar ; this result is known as the Angle-Angle Similarity Theorem. Two triangles are also similar if and only if their corresponding sides are proportional.
- At times, we shall need to talk about directed angles and directed angle measure, say from $A B^{\rightarrow}$ to $A C^{\rightarrow}$, with counterclockwise orientation chosen as positive, and clockwise orientation chosen as negative. In general, for real numbers $r$ and $s$, we agree that $r^{\circ}=s^{\circ} \Longleftrightarrow r=s+360 k$ for some integer $k$.
- Given line $\mathcal{L}$, the points of the plane are partitioned into three sets, namely the line itself and the two halfplanes of the line.
- Lines $\mathcal{L}_{1}$ and $\mathcal{L}_{2}$ are parallel if either $\mathcal{L}_{1}=\mathcal{L}_{2}$ or else $\mathcal{L}_{1}$ and $\mathcal{L}_{2}$ have no points in common.
- The locus of all points equidistant from two points $A$ and $B$ is the perpendicular bisector of $A$ and $B$, which is the line through the midpoint of $\overline{A B}$ and perpendicular to $\overline{A B}$.

Exercise 3 Show that the lines

$$
(\mathcal{L}) \quad a x+b y+c=0 \quad \text { and } \quad(\mathcal{M}) \quad d x+e y+f=0
$$

are parallel if and only if $a e-b d=0$, and are perpendicular if and only if $a d+b e=0$.

Exercise 4 PROVE or DISPROVE: Through any point $P$ off a line $\mathcal{L}$, there passes a unique line parallel to the given line $\mathcal{L}$.

Exercise 5 Show that three points $P_{1}=\left(x_{1}, y_{1}\right), P_{2}=\left(x_{2}, y_{2}\right)$ and $P_{3}=\left(x_{3}, y_{3}\right)$ are collinear if and only if

$$
\left|\begin{array}{ccc}
1 & 1 & 1 \\
x_{1} & x_{2} & x_{3} \\
y_{1} & y_{2} & y_{3}
\end{array}\right|=0 .
$$

### 1.2 Transformations

One of the most important concepts in geometry is that of a transformation.
Note : Transformations are a special class of functions. Consider two sets $\mathbf{S}$ and T. A function (or mapping) $\alpha$ from $\mathbf{S}$ to $\mathbf{T}$ is a rule that associates with each element $s$ of $\mathbf{S}$ a unique element $t=\alpha(s)$ of $\mathbf{T}$; the element $\alpha(s)$ is called the image of $s$ under $\alpha$, and $s$ is a preimage of $\alpha(s)$. The set $\mathbf{S}$ is called the domain (or source) of $\alpha$, and the set $\mathbf{T}$ is the codomain (or target) of $\alpha$. The set of all $\alpha(s)$ with $s \in S$ is called the image (or range) of $\alpha$ and is denoted by $\alpha(\mathbf{S})$. If any two different elements of the domain have different images under $\alpha$ (that is, if $\alpha\left(s_{1}\right)=\alpha\left(s_{2}\right)$ implies that $s_{1}=s_{2}$ ), then $\alpha$ is one-to-one (or injective). If all elements of the codomain are images under $\alpha$ (that is, if $\alpha(\mathbf{S})=\mathbf{T}$ ), then $\alpha$ is onto (or surjective). If a function is injective and surjective, it is said to be bijective.

Exercise 6 If there exists a one-to-one mapping $f: \boldsymbol{A} \rightarrow \boldsymbol{A}$ which is not onto, what can be said about the set $\boldsymbol{A}$ ?

When both the domain and codomain of a mapping are "geometrical" the mapping may be referred to as a transformation. We shall find it convenient to use the word transformation ONLY IN THE SPECIAL SENSE of a bijective mapping of a set (space) onto itself. We make the following definition.
1.2.1 Definition. A transformation on the plane is a bijective mapping of $\mathbb{E}^{2}$ onto itself.

## Transformations will be denoted by lowercase Greek letters.

For a given transformation $\alpha$, this means that for every point $P$ there is a unique point $Q$ such that $\alpha(P)=Q$ and, conversely, for every point $S$ there is a unique point $R$ such that $\alpha(R)=S$.

Note : Not every mapping on $\mathbb{E}^{2}$ is a transformation. Suppose a mapping $\alpha$ is given by $(x, y) \mapsto\left(\alpha_{1}(x, y), \alpha_{2}(x, y)\right)$. Then $\alpha$ is a bijection (i.e. a transformation) if and only if, given the equations (of $\alpha$ )

$$
\begin{aligned}
x^{\prime} & =\alpha_{1}(x, y) \\
y^{\prime} & =\alpha_{2}(x, y),
\end{aligned}
$$

one can solve uniquely for (the "old" coordinates) $x$ and $y$ in terms of (the "new" coordinates) $x^{\prime}$ and $y^{\prime}: x=\beta_{1}\left(x^{\prime}, y^{\prime}\right)$ and $y=\beta_{2}\left(x^{\prime}, y^{\prime}\right)$.
1.2.2 Examples. The following mappings on $\mathbb{E}^{2}$ are transformations:

1. $(x, y) \mapsto(x, y) \quad$ (identity);
2. $\quad(x, y) \mapsto(-x, y) \quad$ (reflection);
3. $(x, y) \mapsto(x-1, y+2) \quad$ (translation);
4. $(x, y) \mapsto(-y, x) \quad$ (rotation);
5. $(x, y) \mapsto(2 x, 2 y) \quad$ (dilation);
6. $(x, y) \mapsto(x+y, y) \quad$ (shear);
7. $(x, y) \mapsto\left(-x+\frac{y}{2}, x+2\right) \quad$ (affinity);
8. $(x, y) \mapsto\left(x, x^{2}+y\right) \quad$ (generalized shear);
9. $(x, y) \mapsto\left(x, y^{3}\right)$;
10. $\quad(x, y) \mapsto(x+|y|, y)$.
1.2.3 EXAMPLES. The following mappings on $\mathbb{E}^{2}$ are not transformations:
11. $(x, y) \mapsto(x, 0)$;
12. $(x, y) \mapsto(x y, x y)$;
13. $(x, y) \mapsto\left(x^{2}, y\right)$;
14. $(x, y) \mapsto\left(-x+\frac{y}{2}, 2 x-y\right)$;
15. $(x, y) \mapsto\left(e^{x} \cos y, e^{x} \sin y\right)$.
1.2.4 Example. Consider the mapping

$$
\beta: \mathbb{E}^{2} \rightarrow \mathbb{E}^{2}, \quad(x, y) \mapsto\left(x^{\prime}, y^{\prime}\right)=\left(x^{2}-y^{2}, 2 x y\right) .
$$

Let us first use polar coordinates $r, t$ so that

$$
x=r \cos t, \quad y=r \sin t, \quad 0 \leq t \leq 2 \pi .
$$

By using some trigonometric identities, we can express $\beta((x, y))$ as

$$
\beta((r \cos t, r \sin t))=\left(r^{2} \cos 2 t, r^{2} \sin 2 t\right), \quad 0 \leq t \leq 2 \pi .
$$

From this it follows that under $\beta$ the image curve of the circle of radius $r$ and center at the origin counterclockwise once is the circle of radius $r^{2}$ and center at the origin counterclockwise twice. Thus the effect of $\beta$ is to wrap the plane $\mathbb{E}^{2}$ smoothly around itself, leaving the origin fixed, since $\beta((0,0))=(0,0)$, and therefore $\beta$ is surjective but not injective.

Exercise 7 Verify that the mapping

$$
(x, y) \mapsto\left(x-\frac{2 a}{a^{2}+b^{2}}(a x+b y+c), y-\frac{2 b}{a^{2}+b^{2}}(a x+b y+c)\right)
$$

is a transformation.

## Collineations

1.2.5 Definition. A transformation $\alpha$ with the property that if $\mathcal{L}$ is a line, then $\alpha(\mathcal{L})$ is also a line is called a collineation.

Note : We take the view that a line is a set of points and so $\alpha(\mathcal{L})$ is the set of all points $\alpha(P)$ with point $P$ on line $\mathcal{L}$; that is,

$$
\alpha(\mathcal{L})=\{\alpha(P) \mid P \in \mathcal{L}\} \subset \mathbb{E}^{2}
$$

Clearly, $\alpha(P) \in \alpha(\mathcal{L}) \Longleftrightarrow P \in \mathcal{L}$.
1.2.6 Example. The mapping

$$
\alpha: \mathbb{E}^{2} \rightarrow \mathbb{E}^{2}, \quad(x, y) \mapsto\left(x, y^{3}\right)
$$

is a transformation as $(u, \sqrt[3]{v})$ is the unique point sent to $(u, v)$ for given numbers $u$ and $v$ (given the equations $u=x$ and $v=y^{3}$, one can solve uniquely for $x$ and $y$ in terms of $u$ and $v$ ). However, $\alpha$ is not a collineation, since the line with equation $y=x$ is not sent to a line, but rather to the cubic curve with equation $y=x^{3}$.
1.2.7 Example. The mapping

$$
\beta: \mathbb{E}^{2} \rightarrow \mathbb{E}^{2}, \quad(x, y) \mapsto\left(-x+\frac{y}{2}, x+2\right)
$$

is a collineation. Indeed, from (the equations of $\beta$ )

$$
\begin{aligned}
x^{\prime} & =-x+\frac{y}{2} \\
y^{\prime} & =x+2
\end{aligned}
$$

we get (uniquely)

$$
\begin{aligned}
& x=y^{\prime}-2 \\
& y=2 x^{\prime}+2 y^{\prime}-4
\end{aligned}
$$

Hence $\beta$ is a transformation.
Now consider the line $\mathcal{L}$ with equation $a x+b y+c=0$, and let $P^{\prime}=\left(x^{\prime}, y^{\prime}\right)$ denote the image of the (arbitrary) point $P=(x, y)$ under (the transformation) $\beta$. Recall that

$$
P^{\prime}=\left(x^{\prime}, y^{\prime}\right) \in \beta(\mathcal{L}) \Longleftrightarrow P=(x, y) \in \mathcal{L}
$$

Then

$$
a\left(y^{\prime}-2\right)+b\left(2 x^{\prime}+2 y^{\prime}-4\right)+c=0
$$

or, equivalently,

$$
(2 b) x^{\prime}+(a+2 b) y^{\prime}+c-4 b-2 a=0 .
$$

(Observe that $(2 b)^{2}+(a+2 b)^{2} \neq 0$ since $a^{2}+b^{2} \neq 0$.) So the line $\mathcal{L}$ with equation $a x+b y+c=0$ goes to the line with equation $(2 b) x+(a+2 b) y+$ $c-4 b-2 a=0$. Hence $\beta$ is a collineation.

Exercise 8 PROVE or DISPROVE : Collineations preserve parallelness among lines (i.e. the images of two parallel lines under a given collineation are also parallel lines).

### 1.3 Properties of Transformations

Various sets of transformations correspond to important geometric properties. We will look at properties of sets of transformations that make them algebraically interesting. Let $\mathfrak{G}$ be a set of transformations.

Sets of transformations will be denoted by uppercase Gothic letters.
1.3.1 Definition. The transformation defined by

$$
\iota: \mathbb{E}^{2} \rightarrow \mathbb{E}^{2}, \quad P \mapsto P
$$

is called the identity transformation.
Note : No other transformation is allowed to use the Greek letter iota. The identity transformation may seem of little importance by itself, but its presence simplifies investigations about transformations, just as the number 0 simplifies addition of numbers.

If $\iota$ is in the set $\mathfrak{G}$, then $\mathfrak{G}$ is said to have the identity property.
Recall that $\alpha$ is a transformation if (and only if) for every point $P$ there is a unique point $Q$ such that $\alpha(P)=Q$ and, conversely, for every point $S$
there is a point $R$ such that $\alpha(R)=S$. From this definition we see that the mapping $\alpha^{-1}: \mathbb{E}^{2} \rightarrow \mathbb{E}^{2}$, defined by

$$
\alpha^{-1}(A)=B \Longleftrightarrow \alpha(B)=A
$$

is a transformation, called the inverse of $\alpha$.
Note : We read " $\alpha^{-1}$ " as "alpha inverse". If (the transformation) $\alpha$ is given by

$$
(x, y) \mapsto\left(x^{\prime}, y^{\prime}\right)=\left(\alpha_{1}(x, y), \alpha_{2}(x, y)\right)
$$

with $x=\beta_{1}\left(x^{\prime}, y^{\prime}\right)$ and $y=\beta_{2}\left(x^{\prime}, y^{\prime}\right)$, then (the transformation)

$$
\beta:(x, y) \mapsto\left(\beta_{1}(x, y), \beta_{2}(x, y)\right)
$$

is the inverse of $a$; that is, $\beta=\alpha^{-1}$.
If $\alpha^{-1}$ is also in $\mathfrak{G}$ for every transformation $\alpha$ in our set $\mathfrak{G}$ of transformations, then $\mathfrak{G}$ is said to have the inverse property.

Whenever two transformations are brought together they might form new transformations. In fact, one transformation might form new transformations by itself, as we can see by considering $\alpha=\beta$ below.
1.3.2 Definition. Given two transformations $\alpha$ and $\beta$, the mapping

$$
\beta \alpha: \mathbb{E}^{2} \rightarrow \mathbb{E}^{2}, \quad P \mapsto \beta(\alpha(P))
$$

is called the product of the transformation $\beta$ by the transformation $\alpha$.
Note : Transformation $\alpha$ is applied first and then transformation $\beta$ is applied. We read " $\beta \alpha$ " as "the product beta-alpha".
1.3.3 Proposition. The product of two transformations is itself a transformation.

Proof: Let $\alpha$ and $\beta$ be two transformations. Since for every point $C$ there is a point $B$ such that $\alpha(B)=C$ and for every point $B$ there is a point $A$ such that $\alpha(A)=B$, then for every point $C$ there is a point $A$
such that $\beta \alpha(A)=\beta(\alpha(A))=\beta(B)=C$. So $\beta \alpha$ is an onto mapping. Also, $\beta \alpha$ is one-to-one, as the following argument shows. Suppose $\beta \alpha(P)=\beta \alpha(Q)$. Then $\beta(\alpha(P))=\beta(\alpha(Q))$ by the definition of $\beta \alpha$. So $\alpha(P)=\alpha(Q)$ since $\beta$ is one-to-one. Then $P=Q$ as $\alpha$ is one-to-one. Therefore, $\beta \alpha$ is both one-to-one and onto.

If our set $\mathfrak{G}$ has the property that the product $\beta \alpha$ is in $\mathfrak{G}$ whenever $\alpha$ and $\beta$ are in $\mathfrak{G}$, then $\mathfrak{G}$ is said to have the closure property. Since both $\alpha^{-1} \alpha(P)=P$ and $\alpha \alpha^{-1}(P)=P$ for every point $P$, we see that

$$
\alpha^{-1} \alpha=\alpha \alpha^{-1}=\iota .
$$

Hence if $\mathfrak{G}$ is a nonempty set of transformations having both the inverse property and the closure property, then $\mathfrak{G}$ must necessarily have the identity property.

Our set $\mathfrak{G}$ of transformations is said to have the associativity property, as any elements $\alpha, \beta, \gamma$ in $\mathfrak{G}$ satisfy the associativity law :

$$
\gamma(\beta \alpha)=(\gamma \beta) \alpha .
$$

Indeed, for every point $P$,

$$
(\gamma(\beta \alpha))(P)=\gamma(\beta \alpha(P))=\gamma(\beta(\alpha(P)))=(\gamma \beta)(\alpha(P))=((\gamma \beta) \alpha)(P) .
$$

## Groups of transformations

The important sets of transformations are those that simultaneously satisfy the closure property, the associativity property, the identity property, and the inverse property. Such a set is called a group (of transformations).

Note: We mention all four properties because it is these four properties that are used for the definition of an abstract group in algebra. However, when we want to check that a nonempty set $\mathfrak{G}$ of transformations forms a group, we need check only the closure property and the inverse property.
1.3.4 Proposition. The set of all transformations forms a group.

Proof : The closure property and the inverse property hold for the set of all transformations.

Exercise 9 Let $\alpha$ be a collineation. Show that, given a line $\mathcal{L}$, there exists a line $\mathcal{M}$ such that $\alpha(\mathcal{M})=\mathcal{L}$.
1.3.5 Proposition. The set of all collineations forms a group.

Proof: We suppose $\alpha$ and $\beta$ are collineations. Suppose $\mathcal{L}$ is a line. Then $\alpha(\mathcal{L})$ is a line since $\alpha$ is a collineation, and $\beta(\alpha(\mathcal{L}))$ is then a line since $\beta$ is a collineation. Hence, $\beta \alpha(\mathcal{L})$ is a line, and $\beta \alpha$ is a collineation. So the set of collineations satisfies the closure property. There is a line $\mathcal{M}$ such that $\alpha(\mathcal{M})=\mathcal{L}$. So

$$
\alpha^{-1}(\mathcal{L})=\alpha^{-1}(\alpha(\mathcal{M}))=\alpha^{-1} \alpha(\mathcal{M})=\iota(\mathcal{M})=\mathcal{M} .
$$

Hence, $\alpha^{-1}$ is a collineation, and the set of all collineations satisfies the inverse property. The set is not empty as the identity is a collineation. Therefore, the set of all collineations forms a group.

If every element of transformation group $\mathfrak{G}^{\prime}$ is an element of transformation group $\mathfrak{G}$, then $\mathfrak{G}^{\prime}$ is a subgroup of $\mathfrak{G}$. All of our groups will be subgroups of the group of all collineations. These transformation groups will be a very important part of our study of geometry.

Note : The word group now has a technical meaning and should never be used as a general collective noun in place of the word set.

Transformations $\alpha$ and $\beta$ may or may not satisfy the commutativity law: $\alpha \beta=\beta \alpha$. If the commutativity law is always satisfied by the elements from a group, then that group is said to be commutative (or Abelian). The term Abelian is after the Norwegian mathematician N.H. Abel (1801-1829).

## Orders and generators

Given a transformation $\alpha$, the product $\alpha \alpha \ldots \alpha$ ( $n$ times) is denoted by $\alpha^{n}$. As expected, we define $\alpha^{0}$ to be $\iota$. Also, we write

$$
\left(\alpha^{-1}\right)^{n}=\alpha^{-n}, \quad n \in \mathbb{Z} .
$$

If group $\mathfrak{G}$ has exactly $n$ elements, then $\mathfrak{G}$ is said to be finite and have order $n$; otherwise, $\mathfrak{G}$ is said to be infinite. Analogously, if there is a smallest positive integer $n$ such that $\alpha^{n}=\iota$, then transformation $\alpha$ is said to have order $n$; otherwise $\alpha$ is said to have infinite order.
1.3.6 Example. Let $\rho$ be a rotation of $\frac{360}{n}$ degrees about the origin with $n$ a positive integer and let

$$
\tau: \mathbb{E}^{2} \rightarrow \mathbb{E}^{2}, \quad(x, y) \mapsto(x+1, y)
$$

Then

- $\rho$ has order $n$,
- the set $\left\{\rho, \rho^{2}, \ldots, \rho^{n}\right\}$ forms a group,
- $\tau$ has infinite order,
- the set $\left\{\tau^{k}: k \in \mathbb{Z}\right\}$ forms an infinite group.

If every element of a group containing $\alpha$ is a power of $\alpha$, then we say that the group is cyclic with generator $\alpha$ and denote the group as $\langle\alpha\rangle$.
1.3.7 Example. If $\rho$ is a rotation of $36^{\circ}$, then $\langle\rho\rangle$ is a cyclic group of order 10. Note that this same group is generated by $\beta$ where $\beta=\rho^{3}$. In fact, we have

$$
\langle\rho\rangle=\left\langle\rho^{3}\right\rangle=\left\langle\rho^{7}\right\rangle=\left\langle\rho^{9}\right\rangle .
$$

So a cyclic group may have more than one generator.

Note : Since the powers of a transformation always commute (i.e. $\alpha^{m} \alpha^{n}=$ $\alpha^{m+n}=\alpha^{n} \alpha^{m}$ for integers $m$ and $n$ ), we see that a cyclic group is always Abelian.

If $\mathfrak{G}=\langle\alpha, \beta, \gamma, \ldots$,$\rangle , then every element of group \mathfrak{G}$ can be written as a product of powers of $\alpha, \beta, \gamma, \ldots$ and $\mathfrak{G}$ is said to be generated by $\{\alpha, \beta, \gamma, \ldots\}$.

## Involutions and multiplication tables

Among the particular transformations that will command our attention are the involutions.
1.3.8 Definition. A transformation $\alpha$ is an involution if $\alpha^{2}=\iota$ but $\alpha \neq \iota$.

Note : The identity transformation is not an involution by definition.
1.3.9 ExAmple. The following transformations are involutions :

1. $(x, y) \mapsto(y, x)$;
2. $(x, y) \mapsto(-x+2 a,-y+2 b)$;
3. $\quad(x, y) \mapsto\left(\frac{1}{2}(x+\sqrt{3} y), \frac{1}{2}(\sqrt{3} x-y)\right)$.
1.3.10 Proposition. A nonidentity transformation $\alpha$ is an involution if and only if $\alpha=\alpha^{-1}$.

Proof : $(\Rightarrow)$ Assume the nonidentity transformation $\alpha$ is an involution. Then $\alpha^{2}=\iota$. By multiplying both sides by $\alpha^{-1}$, we get

$$
\alpha^{-1}(\alpha \alpha)=\alpha^{-1} \iota \Longleftrightarrow\left(\alpha^{-1} \alpha\right) \alpha=\alpha^{-1} \Longleftrightarrow \iota \alpha=\alpha^{-1} \Longleftrightarrow \alpha=\alpha^{-1}
$$

$(\Leftarrow)$ Conversely, assume the nonidentity transformation $\alpha$ is such that $\alpha=$ $\alpha^{-1}$. Then by multiplying both sides by $\alpha$, we get

$$
\alpha^{2}=\alpha \alpha=\alpha \alpha^{-1}=\iota .
$$

Exercise 10 Determine whether the transformation

$$
(x, y) \mapsto\left(x-\frac{2 a}{a^{2}+b^{2}}(a x+b y+c), y-\frac{2 b}{a^{2}+b^{2}}(a x+b y+c)\right)
$$

is an involution.
A multiplication table for a finite group is often called a Cayley table for the group. This is in honour of the English mathematician A. CAYLEY (1821-1895). In a Cayley table, the product $\beta \alpha$ is found in the row headed " $\beta$ " and the column headed " $\alpha$ ".
1.3.11 EXAMPLE. Consider the group $\mathfrak{C}_{4}$ that is generated by a rotation $\rho$ of $90^{\circ}$ about the origin. The Cayley table for $\mathfrak{C}_{4}$ is given below :

| $\mathfrak{C}_{4}$ | $\iota$ | $\rho$ | $\rho^{2}$ | $\rho^{3}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\iota$ | $\iota$ | $\rho$ | $\rho^{2}$ | $\rho^{3}$ |
| $\rho$ | $\rho$ | $\rho^{2}$ | $\rho^{3}$ | $\iota$ |
| $\rho^{2}$ | $\rho^{2}$ | $\rho^{3}$ | $\iota$ | $\rho$ |
| $\rho^{3}$ | $\rho^{3}$ | $\iota$ | $\rho$ | $\rho^{2}$ |

Clearly, $\mathfrak{C}_{4}$ is a group of order 4 (it is easy to check the closure property and the inverse property). Group $\mathfrak{C}_{4}$ is cyclic and is generated by $\rho$. Since

$$
\left(\rho^{3}\right)^{2}=\rho^{6}=\rho^{2}, \quad\left(\rho^{3}\right)^{3}=\rho^{9}=\rho, \quad \text { and } \quad\left(\rho^{3}\right)^{4}=\rho^{12}=\iota,
$$

then $\mathfrak{C}_{4}$ is also generated by $\rho^{3}$. So

$$
\mathfrak{C}_{4}=\langle\rho\rangle=\left\langle\rho^{3}\right\rangle
$$

Note, also, that group $\mathfrak{C}_{4}$ contains the one involution $\rho^{2}$.
1.3.12 Example. Consider the group $\mathfrak{V}_{4}=\left\{\iota, \sigma_{O}, \sigma_{h}, \sigma_{v}\right\}$, where

$$
\begin{gathered}
\iota((x, y))=(x, y), \quad \sigma_{O}((x, y))=(-x,-y) \\
\sigma_{h}((x, y))=(x,-y), \quad \sigma_{v}((x, y))=(-x, y)
\end{gathered}
$$

The Cayley table for $\mathfrak{V}_{4}$ can be computed algebraically without any geometric interpretation.

| $\mathfrak{\mathfrak { V }}_{4}$ | $\iota$ | $\sigma_{h}$ | $\sigma_{v}$ | $\sigma_{O}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\iota$ | $\iota$ | $\sigma_{h}$ | $\sigma_{v}$ | $\sigma_{O}$ |
| $\sigma_{h}$ | $\sigma_{h}$ | $\iota$ | $\sigma_{O}$ | $\sigma_{v}$ |
| $\sigma_{v}$ | $\sigma_{v}$ | $\sigma_{O}$ | $\iota$ | $\sigma_{h}$ |
| $\sigma_{O}$ | $\sigma_{O}$ | $\sigma_{v}$ | $\sigma_{h}$ | $\iota$ |

Group $\mathfrak{V}_{4}$ is Abelian but not cyclic. Every element of $\mathfrak{V}_{4}$ except the identity is an involution.

### 1.4 Exercises

Exercise 11 Let $P, Q$, and $R$ be three distinct points. Prove that

$$
P Q+Q R=P R \Longleftrightarrow Q=(1-t) P+t R \quad \text { for some } 0<t<1 .
$$

(The line segment $\overline{P R}$ consists of $P, R$ and all points between $P$ and $R$. Hence

$$
\overline{P R}=\{(1-t) P+t R \mid 0 \leq t \leq 1\} .)
$$

Exercise 12 Which of the following mappings defined on the Euclidean plane $\mathbb{E}^{2}$ are transformations?
(a) $(x, y) \mapsto\left(x^{3}, y^{3}\right)$.
(b) $(x, y) \mapsto(\cos x, \sin y)$.
(c) $(x, y) \mapsto\left(x^{3}-x, y\right)$.
(d) $(x, y) \mapsto(2 x, 3 y)$.
(e) $(x, y) \mapsto(-x, x+3)$.
(f) $(x, y) \mapsto(3 y, x+2)$.
(g) $(x, y) \mapsto\left(\sqrt[3]{x}, e^{y}\right)$.
(h) $(x, y) \mapsto(-x,-y)$.
(i) $(x, y) \mapsto(x+2, y-3)$.

Exercise 13 Which of the transformations in the exercise above are collineations ? For each collineation, find the image of the line with equation $a x+b y+c=0$.

Exercise 14 Find the image of the line with equation $y=5 x+7$ under collineation $\alpha$ if $\alpha((x, y))$ is :
(a) $(-x, y)$.
(b) $(x,-y)$.
(c) $(-x,-y)$.
(d) $(2 y-x, x-2)$.

Exercise 15 TRUE or FALSE ? Suppose $\alpha$ is a transformation on the plane.
(a) If $\alpha(P)=\alpha(Q)$, then $P=Q$.
(b) For any point $P$ there is a unique point $Q$ such that $\alpha(P)=Q$.
(c) For any point $P$ there is a point $Q$ such that $\alpha(P)=Q$.
(d) For any point $P$ there is a unique point $Q$ such that $\alpha(Q)=P$.
(e) For any point $P$ there is a point $Q$ such that $\alpha(Q)=P$.
(f) A collineation is necessarily a transformation.
(g) A transformation is necessarily a collineation.
(h) A collineation is a mapping that is one-to-one.
(i) A collineation is a mapping that is onto.
(j) A transformation is onto but not necessarily one-to-one.

Exercise 16 Give three examples of transformations on the plane that are not collineations.

Exercise 17 Find the preimage of the line with equation $y=3 x+2$ under the collineation

$$
\alpha: \mathbb{E}^{2} \rightarrow \mathbb{E}^{2}, \quad(x, y) \mapsto(3 y, x-y)
$$

## Exercise 18 If

$$
\left\{\begin{array}{l}
x^{\prime}=a x+b y+h \\
y^{\prime}=c x+d y+k
\end{array}\right.
$$

are the equations for mapping $\alpha: \mathbb{E}^{2} \rightarrow \mathbb{E}^{2}$, then what are the necessary and sufficient conditions on the coefficients for $\alpha$ to be a transformation? Is such a transformation always a collineation?

Exercise 19 Let $\mathbf{P}=\left\{P_{1}, \ldots, P_{n}\right\}$ be a finite set of points (in the plane), and let $C$ be its centre of gravity, namely

$$
C:=\frac{1}{n}\left(P_{1}+\cdots+P_{n}\right) .
$$

Consider a transformation $\alpha: \mathbb{E}^{2} \rightarrow \mathbb{E}^{2}$ of the form

$$
(x, y) \mapsto(a x+b y+h, c x+d y+k) \quad \text { with } a d-b c \neq 0
$$

and let $P_{i}^{\prime}=\alpha\left(P_{i}\right), i=1,2, \ldots n$ and $C^{\prime}=\alpha(C)$. Show that

$$
C^{\prime}=\frac{1}{n}\left(P_{1}^{\prime}+\cdots+P_{n}^{\prime}\right)
$$

Exercise 20 Sketch the image of the unit square under the following transformations :
(a) $(x, y) \mapsto(x, x+y)$.
(b) $(x, y) \mapsto(y, x)$.
(c) $(x, y) \mapsto\left(x, x^{2}+y\right)$.
(d) $(x, y) \mapsto\left(-x+\frac{y}{2}, x+2\right)$.

Exercise 21 Prove that if $\alpha, \beta$, and $\gamma$ are elements in a group, then
(a) $\beta \alpha=\gamma \alpha$ implies $\beta=\gamma$;
(b) $\beta \alpha=\beta \gamma$ implies $\alpha=\gamma$;
(c) $\beta \alpha=\alpha$ implies $\beta=\iota$;
(d) $\beta \alpha=\beta$ implies $\alpha=\iota$;
(e) $\beta \alpha=\iota$ implies $\beta=\alpha^{-1}$ and $\alpha=\beta^{-1}$.

## Exercise 22 TRUE or FALSE?

(a) If $\alpha$ and $\beta$ are transformations, then $\alpha=\beta$ if and only if $\alpha(P)=\beta(P)$ for every point $P$.
(b) Transformation $\iota$ is in every group of transformations.
(c) If $\alpha \beta=\iota$, then $\alpha=\beta^{-1}$ and $\beta=\alpha^{-1}$ for transformations $\alpha$ and $\beta$.
(d) " $\alpha \beta$ " is read "the product beta-alpha".
(e) If $\alpha$ and $\beta$ are both in group $\mathfrak{G}$, then $\alpha \beta=\beta \alpha$.
(f) $(\alpha \beta)^{-1}=\alpha^{-1} \beta^{-1}$ for transformations $\alpha$ and $\beta$.

Exercise 23 PROVE or DISPROVE : There is an infinite cyclic group of rotations.

## Exercise 24 TRUE or FALSE?

(a) $\langle\iota\rangle$ is a cyclic group of order 1 .
(b) $\langle\gamma\rangle=\left\langle\gamma^{-1}\right\rangle$ for any transformation $\gamma$.
(c) An Abelian group is always cyclic, but a cyclic group is not always Abelian.
(d) If $\langle\alpha\rangle=\langle\beta\rangle$, then $\alpha=\beta$ or $\alpha=\beta^{-1}$.

Exercise 25 Find all $a$ and $b$ such that the transformation

$$
(x, y) \mapsto\left(a y, \frac{x}{b}\right)
$$

is an involution.

## DISCUSSION : The Euclidean plane can be approached in many ways.

 One can take the view that plane geometry is about points, lines, circles, and proceed from "self-evident" properties of these figures (axioms) to deduce the less obvious properties as theorems. This was the classical approach to geometry, also known as synthetic. It was based on the conviction that geometry describes actual space and, in particular, that the theory of lines and circles describes what one can do with ruler and compass. To develop this theory, Euclid (c. 300 b.c.) stated certain plausible properties of lines and circles as axioms and derived thorems from them by pure logic. Actually he occasionally made use of unstated axioms; nevertheless his approach is feasible and it was eventually made rigorous by David Hilbert (1862-1943).EucLid's approach has some undeniable advantages. Above all, it presents geometry in a pure and self-contained manner, without use of "non-geometric" concepts. One feels that the "real reason" for geometric theorems are revealed in such a system. Visual intuition not only supplies the axioms, it also prompt the steps in a proof, so that some extremely short and elegant proofs result.

Nevertheless, with the enormous growth of mathematics over the last two centuries, Euclid's approach has become isolated and inefficient. It is isolated because Euclidean geometry is no longer the geometry of space and the basis for most of
mathematics. Nowadays, numbers and sets are regarded as more fundamental than points and lines. They form a much broader basis, not only for geometry, but for mathematics as a whole. Moreover, geometry can be built more efficiently on this basis because the powerful techniques of algebra and analysis can be brought into play.

The construction of geometry from numbers and sets is implicit in the coordinate geometry of René Descartes (1596-1650), though Descartes, in fact, took the classical view that points, lines, and curves had a prior existence, and he regarded coordinates and equations as merely a convenient way to study them. Perhaps the first to grasp the deeper value of the coordinate approach was Bernhard Riemann (1826-1866), who wrote the following : "It is well known that geometry assumes as given not only the concept of space, but also the basic principles of construction in space. It gives only nominal definitions of these things; their determination being in the form of axioms. As a result, the relationships between these assumptions are left in the dark; one does not see whether, or to what extent, connections between them are necessary, or even whether they are a priori possible."

Riemann went on to outline a very general approach to geometry in which "points" in an " $n$-dimesional space" are $n$-tuples of numbers, and all geometric relations are determined by a metric on this space, a differentiable function giving the "distance" between two "points". This analytic approach allows a vast range of spaces to be considerd simultaneously, and Riemann found that their geometric properties were largely controlled by a property of the metric he called its curvature.

The concept of curvature illuminates the axioms of Euclidean geometry by showing them to hold only in the presence of zero curvature. In particular, the Euclidean plane is a two-dimensional space of zero curvature (though not the only one). It also becomes obvious what the natural alternatives to Euclidean geometry are - those of constant positive and negative curvature - and one can pinpoint precisely where change of curvature causes a change in axioms.

RiEmann set up analytic machinery to study spaces whose curvature varies from point to point. However, simpler machinery suffices for spaces of constant curvature. The reason is that the geometry of these spaces is reflected in isometries (distancepreserving transformations) and isometries turn out to be easily understood. This approach is due to Felix Klein (1849-1925). The concept of isometry actually fills a gap in EucliD's approach to geometry, where the idea of "moving" one figure
until it coincides with another is used without being formally recognized. Thus, when geometry is based on coordinates and isometries, it is possible to enjoy the benefits of both the analytic and synthetic approaches.

A point is that which has no parts.
Euclid

A "point" is much more subtle object than naive intuition suggests.
John Stillwell

