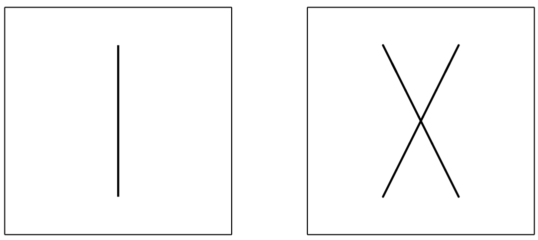
Chapter 3

Reflections and Rotations

Topics :

1. Equations for a Reflection
2. Properties of a Reflection
3. Rotations



3.1 Equations for a Reflection

A *reflection* will be defined as a transformation leaving invariant every point of a fixed line $\mathcal{L}$ and no other points. (An optical reflection along $\mathcal{L}$ in a mirror having both sides silvered, would yield the same result.) We make the following definition.

3.1.1 DEFINITION. **Reflection $\sigma_{\mathcal{L}}$ in line $\mathcal{L}$** is the mapping

$$\sigma_{\mathcal{L}} : \mathbb{E}^2 \to \mathbb{E}^2 , \quad P \mapsto \begin{cases} P\,, & \text{if point } P \text{ is on } \mathcal{L} \\ \\ Q\,, & \text{if point } P \text{ is off } \mathcal{L} \text{ and } \mathcal{L} \text{ is the} \\ & \text{perpendicular bisector of } \overline{PQ}. \end{cases}$$

The line $\mathcal{L}$ is usually referred to as the **mirror** of the reflection.

NOTE : We do not use the word *reflection* to denote the image of a point or of a set of points. A reflection is a transformation and never a set of points. Point $\sigma_{\mathcal{L}}(P)$ is the *image* of point P under the reflection $\sigma_{\mathcal{L}}$.

3.1.2 PROPOSITION. *Reflection $\sigma_{\mathcal{L}}$ is an involutory transformation that interchanges the halfplanes of $\mathcal{L}$. Reflection $\sigma_{\mathcal{L}}$ fixes point P if and only if P is on $\mathcal{L}$. Reflection $\sigma_{\mathcal{L}}$ fixes line $\mathcal{M}$ pointwise if and only if $\mathcal{M} = \mathcal{L}$. Reflection $\sigma_{\mathcal{L}}$ fixes line $\mathcal{M}$ if and only if $\mathcal{M} = \mathcal{L}$ or $\mathcal{M} \perp \mathcal{L}$.*

PROOF : It follows immediately from the definition that

$$\sigma_{\mathcal{L}} \neq \iota \quad \text{but} \quad \sigma_{\mathcal{L}}^2 = \iota$$

as the perpendicular bisector of $\overline{PQ}$ is the perpendicular bisector of $\overline{QP}$. Hence, $\sigma_{\mathcal{L}}$ is onto as $\sigma_{\mathcal{L}}(P)$ is the point mapped onto the given point P since $\sigma_{\mathcal{L}}(\sigma_{\mathcal{L}}(P)) = P$ for any point P. Also, $\sigma_{\mathcal{L}}$ is one-to-one as

$$\sigma_{\mathcal{L}}(A) = \sigma_{\mathcal{L}}(B) \quad \text{implies} \quad A = \sigma_{\mathcal{L}}(\sigma_{\mathcal{L}}(A)) = \sigma_{\mathcal{L}}(\sigma_{\mathcal{L}}(B)) = B\,.$$

Therefore, $\sigma_{\mathcal{L}}$ is an involutory transformation. Then, from the definition of $\sigma_{\mathcal{L}}$, it follows that $\sigma_{\mathcal{L}}$ interchanges the halfplanes of $\mathcal{L}$.

NOTE : In fact, any involutory mapping (on $\mathbb{E}^2$) is a transformation (and hence an involution).

Clearly, $\sigma_{\mathcal{L}}$ fixes point P if and only if P is on $\mathcal{L}$. Not only does $\sigma_{\mathcal{L}}$ fix line $\mathcal{L}$, but $\sigma_{\mathcal{L}}$ fixes every point on $\mathcal{L}$.

NOTE : In general, transformation α is said to **fix pointwise** set $\mathbf{S}$ of points if $\alpha(P) = P$ for every point P in $\mathbf{S}$; that is, if α leaves invariant (unchanged) every point in $\mathbf{S}$. Observe the difference between fixing a set and fixing a set pointwise. Every line perpendicular to $\mathcal{L}$ is fixed by $\sigma_{\mathcal{L}}$, but none of these lines is fixed pointwise as each contains only one fixed point.

Suppose line $\mathcal{M}$ is distinct from $\mathcal{L}$ and is fixed by $\sigma_{\mathcal{L}}$. Let $Q = \sigma_{\mathcal{L}}(P)$ for some point P that is on $\mathcal{M}$ but off $\mathcal{L}$. Then P and Q are both on $\mathcal{M}$ since $\mathcal{M}$ is fixed, and $\mathcal{L}$ is the perpendicular bisector of $\overline{PQ}$. Hence, $\mathcal{L}$ and $\mathcal{M}$ are perpendicular. $\square$

Exercise 41 Show that if the nonidentity mapping $\alpha : \mathbf{A} \to \mathbf{A}$ is *involutory* (i.e. α^2 is the identity mapping), then it is invertible.

NOTE : We have used the Greek letter *sigma* for both halfturns and reflections; the Greek letter *rho* is left free for use later with rotations. The Greek σ corresponds to the Roman s which begins the German word *Spiegelung*, meaning *reflection*. A halfturn is a sort of "reflection in a point". The similar notation for halfturns and reflections emphasizes the important property they do share, namely that of being involutions :

$$\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}^{-1}, \qquad \sigma_P = \sigma_P^{-1}.$$

What are the equations for a reflection ?

3.1.3 Proposition. *If line $\mathcal{L}$ has equation $ax+by+c=0$, then reflection $\sigma_{\mathcal{L}}$ has equations :*

$$\begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} \\[2ex] y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2} \, . \end{cases}$$

Proof : Let $P = (x,y)$ and $\sigma_{\mathcal{L}}(P) = (x',y') = Q$. For the moment, suppose that P is off $\mathcal{L}$. Now, the line through points P and Q is perpendicular to line $\mathcal{L}$. This geometric fact is expressed algebraically by the equation

$$b(x'-x) = a(y'-y).$$

Also,

$$\left(\frac{x+x'}{2}, \frac{y+y'}{2}\right) \quad \text{is the midpoint of } \overline{PQ} \text{ and is on } \mathcal{L}.$$

This geometric fact is expressed algebraically by the equation

$$a\left(\frac{x+x'}{2}\right) + b\left(\frac{y+y'}{2}\right) + c = 0\, .$$

Rewriting these two equations as

$$\begin{cases} bx' - ay' = bx - ay \\[2ex] ax' + by' = -2c - ax - by \end{cases}$$

we see we have two linear equations in two unknowns x' and y'. Solving these equations for x' and y' (by using Cramer's rule, for instance), we get

$$\begin{cases} x' = \dfrac{a^2x + b^2x - 2a^2x - 2aby - 2ac}{a^2+b^2} \\[2ex] y' = \dfrac{a^2y + b^2y - 2b^2y - 2abx - 2bc}{a^2+b^2} \, . \end{cases}$$

With these equations in the form

$$\begin{cases} x' = x - \dfrac{2a(ax+by+c)}{a^2+b^2} \\[2ex] y' = y - \dfrac{2b(ax+by+c)}{a^2+b^2} \end{cases}$$

it is easy to check that the equations also hold when P is on $\mathcal{L}$. This proves the result. $\square$

NOTE : Suppose we had defined a reflection as a transformation having equations given by PROPOSITION 3.1.3. Not only would this have seemed artificial, since these equations are not something you would think of examining in the first place, but just imagine trying to prove PROPOSITION 3.1.2 from these equations. Although this is conceptually easy, the actual computation involves a considerable amount of algebra.

3.2 Properties of a Reflection

We have already mentioned those properties of a reflection that follow immediately from the definition. Another important property is that *a reflection preserves distance*, which means the distance from $\sigma_{\mathcal{L}}(P)$ to $\sigma_{\mathcal{L}}(Q)$ is equal to the distance from P to Q, for all points P and Q. The following definition is fundamental.

3.2.1 DEFINITION. A transformation α is an **isometry** (or **congruent transformation**) if $P'Q' = PQ$ for all points P and Q, where $P' = \alpha(P)$ and $Q' = \alpha(Q)$.

In other words, an isometry is a distance-preserving transformation.

NOTE : (1) In fact, *any distance-preserving mapping is an isometry.* Such a mapping is one-to-one – because points at nonzero distance cannot have images at zero distance – but it is not clear that such a mapping is onto.

(2) The name *isometry* comes from the Greek *isos* (equal) and *metron* (measure). An isometry is also called a **rigid motion**.

The set of all isometries form a group. This group is denoted by $\mathfrak{Isom}$.

3.2.2 PROPOSITION. *Reflection $\sigma_{\mathcal{L}}$ is an isometry.*

PROOF : We shall consider several cases. Suppose P and Q are two points, $P' = \sigma_{\mathcal{L}}(P)$ and $Q' = \sigma_{\mathcal{L}}(Q)$. We must show $P'Q' = PQ$.

(a) If $\overleftrightarrow{PQ} = \mathcal{L}$ or if $\overleftrightarrow{PQ} \perp \mathcal{L}$, then the desired result follows immediately from the definition of $\sigma_{\mathcal{L}}$.

(b) Also, if $\overleftrightarrow{PQ}$ is parallel to $\mathcal{L}$ but distinct from $\mathcal{L}$, the result follows easily as $\Box PQQ'P'$ is a rectangle and so opposite sides $\overline{PQ}$ and $\overline{P'Q'}$ are congruent.

(c) Further, if one of P or Q, say P, is on $\mathcal{L}$ and Q is off $\mathcal{L}$, then $P'Q' = PQ$ follows from the fact that $P' = P$ and that $\mathcal{L}$ is the locus of all points equidistant from Q and Q'.

(d) Finally, suppose P and Q are both off $\mathcal{L}$ and that $\overleftrightarrow{PQ}$ intersects $\mathcal{L}$ at point R, but is not perpendicular to $\mathcal{L}$. So $RP = RP'$ and $RQ = RQ'$. *The desired result,* $P'Q' = PQ$, *then follows provided* R, P', Q' *are shown to be collinear.* $\square$

Exercise 42 Prove the preceding statement.

Exercise 43 Are translations and halfturns isometries ? Why ? (Hence the group of translations $\mathfrak{T}$ and the group $\mathfrak{H}$ are subgroups of $\mathfrak{Isom}$.)

Now that we know a reflection is an isometry, a long sequence of other properties dependent only on distance will follow.

3.2.3 Proposition. *An isometry is a collineation that preserves betweenness, midpoints, segments, rays, triangles, angles, angle measure, and perpendicularity.*

Proof : Since *these properties are shared by all isometries*, we shall consider a general isometry α.

(a) Suppose A, B, C are any three points and let $A' = \alpha(A)$, $B' = \alpha(B)$, $C' = \alpha(C)$. Since α preserves distance, if $AB + BC = AC$ then $A'B' + B'C' = A'C'$ as $A'B' = AB$, $B'C' = BC$, and $A'C' = AC$. Hence, $A - B - C$ implies $A' - B' - C'$; in other words, if B is between A and C, then B' is between A' and C'. We describe this by saying that α *preserves betweenness.*

(b) The special case $AB = BC$ in the argument above implies $A'B' = B'C'$. In other words, if B is the midpoint of A and C, then B' is the midpoint of A' and C'. Thus we say α *preserves midpoints.*

(c) More generally, since $\overline{AB}$ is the union of $A, B,$ and all points between A and B, then $\alpha(\overline{AB})$ is the union of A', B', and all points between A' and B'. So $\alpha(\overline{AB}) = \overline{A'B'}$ and we say α *preserves segments.*

(d) Likewise, since α is onto by definition and $AB^{\rightarrow}$ is the union of $\overline{AB}$ and all points C such that $A - B - C$, then $\alpha(AB^{\rightarrow})$ is the union of $\overline{A'B'}$ and all points C' such that $A' - B' - C'$. So $\alpha(AB^{\rightarrow}) = A'B'^{\rightarrow}$ and we say α *preserves rays.*

(e) Since $\overleftrightarrow{AB}$ is the union $AB^{\rightarrow}$ and $BA^{\rightarrow}$, then $\alpha(\overleftrightarrow{AB})$ is the union of $A'B'^{\rightarrow}$ and $B'A'^{\rightarrow}$, which is $\overleftrightarrow{A'B'}$. So α is a transformation that preserves lines ; in other words, α *is a collineation.*

(f) If $A, B, C,$ are not collinear, then $AB + BC > AC$ and so $A'B' + B'C' > A'C'$ and A', B', C' are not collinear. Then, since $\triangle ABC$ is a union of the three segments $\overline{AB}, \overline{BC}, \overline{CA}$, then we conclude that $\alpha(\triangle ABC)$ is just $\triangle A'B'C'$. So an isometry *preserves triangles.*

(g) It follows that α *preserves angles* as $\alpha(\angle ABC) = \angle A'B'C'$.

(h) Not only does α preserve angles, but α also *preserves angle measure.* That is, $m(\angle ABC) = m(\angle A'B'C')$ since $\triangle ABC \cong \triangle A'B'C'$ by SSS.

(i) Finally, if $\overrightarrow{BA} \perp \overrightarrow{BC}$ then $\overrightarrow{B'A'} \perp \overrightarrow{B'C'}$ since $m(\angle ABC) = 90$ implies $m(\angle A'B'C') = 90$. So α *preserves perpendicularity.* □

3.3 Rotations

We shall now formally define rotations in the most elementary manner.

3.3.1 DEFINITION. A **rotation about point C through directed angle** of r° is the transformation $\rho_{C,r}$ that fixes C and otherwise sends a point P to the point P', where $CP' = CP$ and r is the directed angle measure of the directed angle from $\overrightarrow{CP}$ to $\overrightarrow{CP'}$.

We agree that $\rho_{C,0}$ is the identity ι. Rotation $\rho_{C,r}$ is said to have **centre** C and **directed angle** r°.

3.3.2 PROPOSITION. *A rotation is an isometry.*

PROOF : Suppose $\rho_{C,r}$ sends points P and Q to P' and Q', respectively. If C, P, Q are collinear, then $PQ = P'Q'$ by the definition. If C, P, Q are not collinear, then $\triangle PCQ \cong \triangle P'C'Q'$ by SAS and $PQ = P'Q'$. So $\rho_{C,r}$ is a transformation that preserves distance. $\square$

3.3.3 PROPOSITION. *A nonidentity rotation fixes exactly one point, its centre. A rotation with centre C fixes every circle with centre C.*

PROOF : For distinct points C and P, circle C_P is defined to be the circle with centre C and radius CP. So $\overline{CP}$ is a radius of the circle C_P, and point P is on the circle. The result also follows immediately from the definition of a rotation. $\square$

Exercise 44 Show that (for point C and real numbers r and s)

$$\rho_{C,s}\rho_{C,r} = \rho_{C,r+s} \qquad \text{and} \qquad \rho_{C,r}^{-1} = \rho_{C,-r}.$$

3.3.4 COROLLARY. *The rotations with centre C form a commutative group.*

NOTE : (1) The involutory rotations are the halfturns, and (for any point C)

$$\rho_{C,180} = \sigma_C.$$

(2) Observe that, for example, $\rho_{C,30} = \rho_{C,390} = \rho_{C,-330}$. In general, for real numbers r and s, we have

$$r^\circ = s^\circ \iff r = s + 360\,k\,, \quad k \in \mathbb{Z}.$$

For distinct intersecting lines $\mathcal{L}$ and $\mathcal{M}$, there are *two* directed angles from $\mathcal{L}$ to $\mathcal{M}$. Clearly, these will have directed angle measures that differ by a multiple of 180. If r and s are the directed angle measures of the two directed angles from $\mathcal{L}$ to $\mathcal{M}$, then $(2r)^\circ = (2s)^\circ$, since numbers r and s differ by a multiple of 180. So, *if we are talking about the rotation through twice a directed angle from line $\mathcal{L}$ to line $\mathcal{M}$, then it makes no difference which of the two directed angles we choose.*

3.4 Exercises

Exercise 45 Given point P off line $\mathcal{L}$, *construct* $\rho_{P,60}(\mathcal{L})$.

Exercise 46 TRUE or FALSE ?

(a) If isometry α interchanges distinct points P and Q, then α fixes the midpoint of P and Q.

(b) $\sigma_{\mathcal{L}} = \sigma_P^{-1}$ if point P is on line $\mathcal{L}$.

(c) Reflection $\sigma_{\mathcal{L}}$ fixes the halfplanes of $\mathcal{L}$ but does not fix the halfplanes pointwise.

(d) Reflection $\sigma_{\mathcal{L}}$ fixes line $\mathcal{M}$ if and only if $\mathcal{L} \perp \mathcal{M}$.

(e) For line $\mathcal{L}$ and point P, $\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}^{-1} \neq \iota$ and $\sigma_P = \sigma_P^{-1} \neq \iota$.

(f) $\rho_{C,r}^{-1} = \rho_{C,-r} = \sigma_C$ for any point C.

Exercise 47 What are the images of $(0,0), (1,-3), (-2,1)$, and $(2,4)$ under the reflection in the line with equation $y = 2x - 5$?

Exercise 48 Describe the product of the reflection in $\overleftrightarrow{OO'}$ and the halfturn about O.

Exercise 49 PROVE or DISPROVE :

(a) $\sigma_{\mathcal{L}}\sigma_{\mathcal{M}} = \sigma_{\mathcal{M}}\sigma_{\mathcal{L}} \iff \mathcal{L} \perp \mathcal{M}$.

(b) $\sigma_P\sigma_{\mathcal{L}} = \sigma_{\mathcal{L}}\sigma_P \iff P \in \mathcal{L}$.

Exercise 50 PROVE or DISPROVE : If ρ is a rotation, then the cyclic group $\langle\rho\rangle$ is finite.

DISCUSSION : An *axiomatic system* provides an explicit foundation for a mathematical subject. Axiomatic systems include several parts : the logical language, rules of proof, undefined terms, axioms, definitions, theorems and proofs of theorems, and models.

Consider EUCLID's definition of a point as "that which has no part". This definition is more a philosophical statement about the nature of a point than a way to prove statements. EUCLID's definition of a straight line, "a line which lies evenly with the points on itself", is unclear as well as not useful. In essence, points and lines were so basic to EUCLID's work that there is no good way to define them. Mathematicians realized centuries ago the need for undefined terms to establish an unambiguous beginning. (Otherwise, each term would have to be defined with other terms, leading either to a cycle of terms or an infinite sequence of terms. Neither of these options is acceptable for carefully reasoned mathematics.) Of course, we then define all other terms from these initial, undefined terms. However, undefined terms are, by their nature, unrestricted. How can we be sure that two people *mean* the same thing when they use undefined terms ? In short, we can't. The axioms of a mathematical system become the "key" : they tell us how the undefined terms *behave*. (Axioms describe how to use terms and how they relate to one another, rather than telling us what the terms "really mean".) Indeed, mathematicians permit any *interpretation* of undefined terms, as long as all the axioms hold in that interpretation.

Unlike the Greek understanding of axioms as "self-evident truths", we do not claim the truth of axioms. However, this does not mean that we consider axioms to be false. Rather, we are free to chose axioms to formulate the fundamental relationships we want to investigate. From a logical point of view, the choice of axioms is arbitrary; in actuality, though, mathematicians carefully pick axioms to focus on particular features. *Axiomatic systems allow us to formulate and logically explore abstract relationships, freed from the specificity and imprecision of real situations.* There are two basic types of axiomatic systems. One completely chracterizes a particular mathematical system (for example, HILBERT's axioms characterize *Euclidean plane geometry* completely). The second focuses on the commmon features of a family of *structures* (e.g. groups, vector spaces, or metric spaces); such axiomatic systems unite a wide variety of examples within one powerful theoretical framework.

Mathematical definitions are built from undefined terms and previously defined terms.

In an axiomatic system, a *theorem* is a statement whose proof depends only on previously proven theorems, the axioms, the definitions, and the rules of logic. (This condition ensures that the entire edifice of theorems rests securely on the explicit axioms of the system.) Proofs of theorems in a axiomatic system cannot depend on diagrams, even though diagrams have been part of geometry since the ancient Greeks drew figures in the sand.

Axiomatic systems are a workable compromise between the austere formal languages of mathematical logic and Euclid's work with its many implicit assumptions. *Mathematicians need both the careful reasoning of proofs and the intuitive understanding of content.* Axiomatic systems provide more than a way to give careful proofs. They enable us to understand the relationship of particular concepts, to explore the consequences of assumptions, to contrast different systems, and to unify seemingly disparate situations under one framework. In short, *axiomatic systems are one important way in which mathematicians obtain insight.*

Mathematical models provide an explicit link between intuitions and undefined terms. The usual (Cartesian) model of Euclidean plane geometry is the set $\mathbb{R}^2$, where a *point* is interpreted as an ordered pair of (real) numbers and a *line* is interpreted as the locus of points that satisfy an appropriate (first degree) algebraic equation $ax + by + c = 0$. (In making a model, we are free to interpret the undefined terms in any way we want, provided that all the axioms hold under our interpretation. Note that the axioms are not by themselves true; a context is needed to give meaning to the axioms in order for them to be true or false.) Models do much more than provide concrete examples of axiomatic systems : they lead to important understandings about axiomatic systems. The most important property of an axiomatic system is *consistency*, which says that we cannot prove two statements that contradict each other. An axiomatic system is consistent if and only if it has a model.

I am coming more and more to the conviction that the necessity of our geometry cannot be demonstrated, at least neither by, nor for, the human intellect. [...] geometry should be ranked, not with arithmetic, which is purely aprioristic, but with mechanics.

Carl Friedrich Gauss