## Chapter 5

## Isometries II

Topics :

1. Even and Odd Isometries
2. Classification of Isometries
3. Equations for Isometries


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### 5.1 Even and Odd Isometries

A product of two reflections is a translation or a rotation. By considering the fixed points of each, we see that neither a translation nor a rotation can be equal to a reflection. Thus, for lines $\mathcal{L}, \mathcal{M}, \mathcal{N}$

$$
\sigma_{\mathcal{N}} \sigma_{\mathcal{M}} \neq \sigma_{\mathcal{L}}
$$

When a given isometry is expressed as a product of reflections, the number of reflections is not invariant. Although the product of two reflections cannot be a reflection, we know that in some cases a product of three reflections is a reflection. (We shall see this is possible only because both 3 and 1 are odd integers.) We make the following definitions.
5.1.1 Definition. An isometry that is a product of an even number of reflections is said to be even.
5.1.2 Definition. An isometry that is a product of an odd number of reflections is said to be odd.

Note : It is intuitively clear that the product of an even number of reflections preserves the sense of a clockwise oriented circle in the plane, whereas the product of an odd number of reflections reverses it. We say that even isometries are orientationpreserving and that odd isometries are orientation-reversing isometries.

We shall refer to the property of an isometry of being even or odd as the parity. But is this concept "well-defined" ? Observe that, since an isometry is a product of reflections, an isometry is even or odd. Of course, no integer can be both even and odd, but is it not conceivable some product of ten reflections could equal to some product of seven reflections? We shall show this is impossible.

Exercise 71 Show that if $P$ is a point and $\mathcal{A}$ and $\mathcal{B}$ are lines, then there are lines $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{C}$ passing through $P$ such that $\sigma_{\mathcal{B}} \sigma_{\mathcal{A}}=\sigma_{\mathcal{D}} \sigma_{\mathcal{C}}$.

Based on this simple fact, we can now prove the following
5.1.3 Proposition. A product of four reflections is a product of two reflections.

Proof : Suppose product $\sigma_{\mathcal{S}} \sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is given. We want to show this product is equal to a product of two reflections. Let $P$ a point on line $\mathcal{P}$. There are lines $\mathcal{Q}^{\prime}$ and $\mathcal{R}^{\prime}$ such that $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}}=\sigma_{\mathcal{R}^{\prime}} \sigma_{\mathcal{Q}^{\prime}}$ with $P$ on $\mathcal{Q}^{\prime}$. Also, there are lines $\mathcal{R}^{\prime \prime}$ and $\mathcal{M}$ such that $\sigma_{\mathcal{S}} \sigma_{\mathcal{R}^{\prime}}=\sigma_{\mathcal{M}} \sigma_{\mathcal{R}^{\prime \prime}}$ with $P$ on $\mathcal{R}^{\prime \prime}$. Since $\mathcal{P}, \mathcal{Q}^{\prime}, \mathcal{R}^{\prime \prime}$ are concurent at $P$, then there is a line $\mathcal{L}$ such that $\sigma_{\mathcal{R}^{\prime \prime}} \sigma_{\mathcal{Q}^{\prime}} \sigma_{\mathcal{P}}=\sigma_{\mathcal{L}}$. Therefore,

$$
\sigma_{\mathcal{S}} \sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}=\sigma_{\mathcal{S}} \sigma_{\mathcal{R}^{\prime}} \sigma_{\mathcal{Q}^{\prime}} \sigma_{\mathcal{P}}=\sigma_{\mathcal{M}} \sigma_{\mathcal{R}^{\prime \prime}} \sigma_{\mathcal{Q}^{\prime}} \sigma_{\mathcal{P}}=\sigma_{\mathcal{M}} \sigma_{\mathcal{L}}
$$

Note : Not only are there lines such that the given product of four reflections is equal to $\sigma_{\mathcal{M}} \sigma_{\mathcal{L}}$, but our proof even tells us how to find such lines.
5.1.4 Proposition. An even isometry is a product of two reflections. An odd isometry is a reflection or a product of three reflections. No isometry is both even and odd.

Proof : Given a long product of reflections, we can use Proposition 5.1.3 repeatedly to replace the first four reflections by two reflections until we have obtained a product with less than four reflections. By repeated application of the result to an even isometry, we can reduce the even isometry to a product of two reflections. Also, by repeated application of the result to an odd isometry, we can reduce the odd isometry to a product of three reflections or to a reflection. Therefore, to show an isometry cannot be both even and odd, we need to show only that a product of two reflections cannot equal a reflection or a product of three reflections. Assume there are lines $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}$ such that $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}=\sigma_{\mathcal{S}} \sigma_{\mathcal{T}}$. Then, we have shown above that there are lines $\mathcal{L}$ and $\mathcal{M}$ such that

$$
\sigma_{\mathcal{M}} \sigma_{\mathcal{L}}=\sigma_{\mathcal{S}} \sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}=\sigma_{\mathcal{S}} \sigma_{\mathcal{S}} \sigma_{\mathcal{T}}=\sigma_{\mathcal{T}}
$$

We have a contradiction since $\sigma_{\mathcal{M}} \sigma_{\mathcal{L}}$ is a translation or a rotation and cannot be equal to reflection $\sigma_{\mathcal{T}}$. A product of two reflections is never equal to a reflection or a product of three reflections.
5.1.5 Proposition. An even involutory isometry is a halfturn; an odd involutory isometry is a reflection.

Proof: The even isometries are the translations and the rotations. Since the involutory isometries are the halfturns and the reflections, the result follows.
5.1.6 Proposition. The even isometries form a group $\mathfrak{I s o m}^{+}$.

Proof: An isometry and its inverse have the same parity, since the inverse of a product of reflections is the product of the reflections in reverse order. So the set $\mathfrak{I s o m}^{+}$of all even isometries has the inverse property. Further, the set $\mathfrak{I s o m}^{+}$has the closure property since the sum of two even integers is even. So the even isometries form a group.

Note: $\mathfrak{I s o m}^{+}$will always denote the group of even isometries. So $\mathfrak{I s o m}^{+}$ consists of the translations and the rotations.

## Some "technical" results

Exercise 72 Suppose that $\alpha$ and $\beta$ are two isometries. Prove that
(a) $\alpha \beta \alpha^{-1}$ is an involution $\Longleftrightarrow \beta$ is an involution.
(b) $\alpha \beta \alpha^{-1}$ and $\beta$ must have the same parity.

Note: In general, $\alpha \beta \alpha^{-1}$ is called the conjugate of $\beta$ by $\alpha$.
5.1.7 Proposition. If $P$ is a point, $\mathcal{L}$ is a line, and $\alpha$ is an isometry, then

$$
\alpha \sigma_{\mathcal{L}} \alpha^{-1}=\sigma_{\alpha(\mathcal{L})} \quad \text { and } \quad \alpha \sigma_{P} \alpha^{-1}=\sigma_{\alpha(P)} .
$$

Proof: Since $\sigma_{P}$ is an even involutory isometry, so is $\alpha \sigma_{P} \alpha^{-1}$ (Exercise 72). By Proposition 5.1.5, $\alpha \sigma_{P} \alpha^{-1}$ must be a halfturn. Since halfturn
$\alpha \sigma_{P} \alpha^{-1}$ fixes point $\alpha(P)$, then $\alpha \sigma_{P} \alpha^{-1}$ must be the halfturn about $\alpha(P)$; that is,

$$
\alpha \sigma_{P} \alpha^{-1}=\sigma_{\alpha(P)} .
$$

In similar fashion, since $\alpha \sigma_{\mathcal{L}} \alpha^{-1}$ is an odd involutory isometry, then $\alpha \sigma_{\mathcal{L}} \alpha^{-1}$ is a reflection. Hence, since $\alpha \sigma_{\mathcal{L}} \alpha^{-1}$ clearly fixes every point $\alpha(P)$ on line $\alpha(\mathcal{L})$, then $\alpha \sigma_{\mathcal{L}} \alpha^{-1}$ must be the reflection in the line $\alpha(\mathcal{L})$. That is,

$$
\alpha \sigma_{\mathcal{L}} \alpha^{-1}=\sigma_{\alpha(\mathcal{L})}
$$

5.1.8 Proposition. If $\alpha$ is an isometry, then

$$
\alpha \tau_{A, B} \alpha^{-1}=\tau_{\alpha(A), \alpha(B)} \quad \text { and } \quad \alpha \rho_{C, r} \alpha^{-1}=\rho_{\alpha(C), \pm r}
$$

where the positive sign applies when $\alpha$ is even and the negative sign applies when $\alpha$ is odd.

Proof: If $M$ is the midpoint of $\overline{A B}$, then point $\alpha(M)$ is the midpoint of $\overline{\alpha(A) \alpha(B)}$. Also,

$$
\tau_{A, B}=\sigma_{M} \sigma_{A} \quad \text { and } \quad \tau_{\alpha(A), \alpha(B)}=\sigma_{\alpha(M)} \sigma_{\alpha(A)} .
$$

Now

$$
\begin{aligned}
\alpha \tau_{A, B} \alpha^{-1} & =\alpha \sigma_{M} \sigma_{A} \alpha^{-1}=\left(\alpha \sigma_{M} \alpha^{-1}\right)\left(\alpha \sigma_{A} \alpha^{-1}\right)=\sigma_{\alpha(M)} \sigma_{\alpha(A)} \\
& =\tau_{\alpha(A), \alpha(B)} .
\end{aligned}
$$

That is,

$$
\alpha \tau_{A, B} \alpha^{-1}=\tau_{\alpha(A), \alpha(B)} .
$$

Finding the conjugate of a rotation is slightly more complicated.
We first examine the conjugate of $\rho_{C, r}$ by $\sigma_{\mathcal{L}}$. Let $\mathcal{M}$ be the line through $C$ that is perpendicular to $\mathcal{L}$. Then there exists a line $\mathcal{N}$ through $C$ such that

$$
\rho_{C, r}=\sigma_{\mathcal{N}} \sigma_{\mathcal{M}}
$$

Now $\sigma_{\mathcal{L}}(\mathcal{M})=\mathcal{M}$ and $\sigma_{\mathcal{L}}(\mathcal{N})$ intersect at $\sigma_{\mathcal{L}}(C)$, and a directed angle from $\sigma_{\mathcal{L}}(\mathcal{M})$ to $\sigma_{\mathcal{L}}(\mathcal{N})$ is the negative of a directed angle from $\mathcal{M}$ to $\mathcal{N}$. (This explains the negative sign on the far right in the following calculation.) We have

$$
\begin{aligned}
\sigma_{\mathcal{L}} \rho_{C, r} \sigma_{\mathcal{L}}^{-1} & =\sigma_{\mathcal{L}} \sigma_{\mathcal{N}} \sigma_{\mathcal{M}} \sigma_{\mathcal{L}}^{-1}=\left(\sigma_{\mathcal{L}} \sigma_{\mathcal{N}} \sigma_{\mathcal{L}}^{-1}\right)\left(\sigma_{\mathcal{L}} \sigma_{\mathcal{M}} \sigma_{\mathcal{L}}^{-1}\right)=\sigma_{\sigma_{\mathcal{L}(\mathcal{N})}} \sigma_{\sigma_{\mathcal{L}}(\mathcal{M})} \\
& =\rho_{\sigma_{\mathcal{L}}(C),-r}
\end{aligned}
$$

If $\alpha=\sigma_{\mathcal{T}} \sigma_{\mathcal{S}}$, then

$$
\alpha \rho_{C, r} \alpha^{-1}=\sigma_{\mathcal{T}}\left(\sigma_{\mathcal{S}} \rho_{C, r} \sigma_{\mathcal{S}}^{-1}\right) \sigma_{\mathcal{T}}^{-1}=\rho_{\alpha(C), r}
$$

If $\alpha=\sigma_{\mathcal{T}} \sigma_{\mathcal{S}} \sigma_{\mathcal{R}}$, then the sign in front of $r$ is back to a negative sign again.

## Commuting isometries

By taking $\alpha=\rho_{D, s}$ in Proposition 5.1.8, we can show that nonidentity rotation $\rho_{D, s}$ does not commute with nonidentity rotation $\rho_{C, r}$ unless $D=C$. We leave this as an exercise.

Exercise 73 Prove that nonidentity rotations with different centres do not commute.

We are also in a position to answer the question "When do reflections commute ?"
5.1.9 Proposition. $\quad \sigma_{\mathcal{M}} \sigma_{\mathcal{N}}=\sigma_{\mathcal{N}} \sigma_{\mathcal{M}}$ if and only if $\mathcal{M}=\mathcal{N}$ or $\mathcal{M} \perp \mathcal{N}$.

Proof : For lines $\mathcal{M}$ and $\mathcal{N}$ the following five statements are seen to be equivalent:
(1) $\quad \sigma_{\mathcal{M}} \sigma_{\mathcal{N}}=\sigma_{\mathcal{N}} \sigma_{\mathcal{M}}$.
(2) $\quad \sigma_{\mathcal{N}} \sigma_{\mathcal{M}} \sigma_{\mathcal{N}}=\sigma_{\mathcal{M}}$.
(3) $\quad \sigma_{\sigma_{\mathcal{N}}(\mathcal{M})}=\sigma_{\mathcal{M}}$.
(4) $\quad \sigma_{\mathcal{N}}(\mathcal{M})=\mathcal{M}$.
(5) $\mathcal{M}=\mathcal{N}$ or $\mathcal{M} \perp \mathcal{N}$.

Comparing (1) and (2), we have the answer to our question.
We now consider products of even isometries. We already know that

- The product of two translations is a translation (Proposition 2.1.8).
- The product of two rotations can be a translation in some cases; for example, $\sigma_{B} \sigma_{A}=\tau_{A, B}^{2}$ (Proposition 2.2.3).
- $\rho_{C, s} \rho_{C, r}=\rho_{C, r+s}$ (Exercise 44).
5.1.10 Theorem. (The Angle-addition Theorem) A rotation of $r^{\circ}$ followed by a rotation of $s^{\circ}$ is a rotation of $(r+s)^{\circ}$ unless $(r+s)^{\circ}=0^{\circ}$, in which case the product is a translation.

Proof : Let's consider the product

$$
\rho_{B, s} \rho_{A, r}
$$

of two nonidentity rotations with different centres. With $\mathcal{C}=\overleftrightarrow{A B}$, there is a line $\mathcal{A}$ through $A$ and a line $\mathcal{B}$ through $B$ such that

$$
\rho_{A, r}=\sigma_{\mathcal{C}} \sigma_{\mathcal{A}} \quad \text { and } \quad \rho_{B, s}=\sigma_{\mathcal{B}} \sigma_{\mathcal{C}} .
$$

So

$$
\rho_{B, s} \rho_{A, r}=\sigma_{\mathcal{B}} \sigma_{\mathcal{C}} \sigma_{\mathcal{C}} \sigma_{\mathcal{A}}=\sigma_{\mathcal{B}} \sigma_{\mathcal{A}} .
$$

When $(r+s)^{\circ}=0^{\circ}$, then the lines $\mathcal{A}$ and $\mathcal{B}$ are parallel and our product is a translation. On the other hand, when $(r+s)^{\circ} \neq 0^{\circ}$, then the lines $\mathcal{A}$ and $\mathcal{B}$ intersect at some point $C$ and our product is a rotation. We can see (by The Exterior Angle Theorem) that one directed angle from $\mathcal{A}$ to $\mathcal{B}$ is $\left(\frac{r}{2}+\frac{s}{2}\right)^{\circ}$. Hence, our product $\sigma_{\mathcal{B}} \sigma_{\mathcal{A}}$ is a rotation about $C$ through an angle of $(r+s)^{\circ}$. That is,

$$
\rho_{B, s} \rho_{A, r}=\rho_{C, r+s} .
$$

Note : The Angle-addition Theorem can also be proved by using the equations for the even isometries that will be developed later.

Now, what is the product of a translation and a nonidentity rotation ?

Exercise 74 Prove that
(a) A translation followed by a nonidentity rotation of $r^{\circ}$ is a rotation of $r^{\circ}$.
(b) A nonidentity rotation of $r^{\circ}$ followed by a translation is a rotation of $r^{\circ}$.

### 5.2 Classification of Plane Isometries

We have classified all the even isometries as translations or rotations. An odd isometry is a reflection or a product of three reflections. Only those odd isometries $\sigma_{\mathcal{C}} \sigma_{\mathcal{B}} \sigma_{\mathcal{A}}$, where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are neither concurrent nor have a common perpendicular remain to be considered.

## Glide reflections

We begin with the special case where $\mathcal{A}$ and $\mathcal{B}$ are perpendicular to $\mathcal{C}$. Then $\sigma_{\mathcal{B}} \sigma_{\mathcal{A}}$ is a translation and $\sigma_{\mathcal{C}}$ is, of course, a reflection. We make the following definition.
5.2.1 Definition. If $\mathcal{A}$ and $\mathcal{B}$ are distinct lines perpendicular to line $\mathcal{C}$, then $\sigma_{\mathcal{C}} \sigma_{\mathcal{B}} \sigma_{\mathcal{A}}$ is called a glide reflection with axis $\mathcal{C}$.

We might as well call line $\mathcal{M}$ the axis of $\sigma_{\mathcal{M}}$ as the reflection and the glide reflection then share the property that the midpoint of any point and its image under the isometry lies on the axis.
5.2.2 Proposition. A glide reflection fixes no points. A glide reflection fixes exactly one line, its axis. The midpoint of any point and its image under a glide reflection lies on the axis of the glide reflection.

Proof: Suppose $P$ is any point. Let line $\mathcal{L}$ be the perpendicular from $P$
to $\mathcal{C}$. Then there is a line $\mathcal{M}$ perpendicular to $\mathcal{C}$ such that $\sigma_{\mathcal{B}} \sigma_{\mathcal{A}}=\sigma_{\mathcal{M}} \sigma_{\mathcal{L}}$. If $M$ is the intersection of $\mathcal{M}$ and $\mathcal{C}$, then $P$ and $M$ are distinct points such that

$$
\sigma_{\mathcal{C}} \sigma_{\mathcal{B}} \sigma_{\mathcal{A}}(P)=\sigma_{\mathcal{C}} \sigma_{\mathcal{M}} \sigma_{\mathcal{L}}(P)=\sigma_{\mathcal{C}} \sigma_{\mathcal{M}}(P)=\sigma_{M}(P) \neq P
$$

Since $\sigma_{\mathcal{C}} \sigma_{\mathcal{B}} \sigma_{\mathcal{A}}(P)=\sigma_{M}(P)$ and $M$ is the midpoint of distinct points $P$ and $\sigma_{M}(P)$, we have shown that glide reflection $\sigma_{\mathcal{C}} \sigma_{\mathcal{B}} \sigma_{\mathcal{A}}$ fixes no point but the midpoint of any point $P$ and its image $\sigma_{\mathcal{C}} \sigma_{\mathcal{B}} \sigma_{\mathcal{A}}(P)$ lies on the axis of the glide reflection. So a glide reflection interchanges the halfplanes of its axis. Hence, any line fixed by the glide reflection must intersect the axis at least twice. That is, the glide reflection can fix no line except its axis. The axis of a glide reflection is the unique line fixed by the glide reflection.
5.2.3 Proposition. A glide reflection is the composite of a reflection in some line $\mathcal{A}$ followed by a halfturn about some point off $\mathcal{A}$. A glide reflection is the composite of a halfturn about some point $A$ followed by a reflection in some line off $A$. Conversely, if point $P$ is off line $\mathcal{L}$, then $\sigma_{P} \sigma_{\mathcal{L}}$ and $\sigma_{\mathcal{L}} \sigma_{P}$ are glide reflections with axis the perpendicular from $P$ to $\mathcal{L}$.

Proof: If $\gamma$ is a glide reflection, then there are distinct lines $\mathcal{A}, \mathcal{B}, \mathcal{C}$ such that $\gamma=\sigma_{\mathcal{C}} \sigma_{\mathcal{B}} \sigma_{\mathcal{A}}$, where $\mathcal{A}$ and $\mathcal{B}$ are perpendicular to $\mathcal{C}$, say at points $A$ and $B$, respectively. Now

$$
\sigma_{A}=\sigma_{\mathcal{A}} \sigma_{\mathcal{C}}=\sigma_{\mathcal{C}} \sigma_{\mathcal{A}} \quad \text { and } \quad \sigma_{B}=\sigma_{\mathcal{B}} \sigma_{\mathcal{C}}=\sigma_{\mathcal{C}} \sigma_{\mathcal{B}}
$$

Hence

$$
\begin{aligned}
\gamma=\sigma_{\mathcal{C}}\left(\sigma_{\mathcal{B}} \sigma_{\mathcal{A}}\right) & =\left(\sigma_{\mathcal{C}} \sigma_{\mathcal{B}}\right) \sigma_{\mathcal{A}}=\sigma_{\mathcal{B}}\left(\sigma_{\mathcal{C}} \sigma_{\mathcal{A}}\right)=\left(\sigma_{\mathcal{B}} \sigma_{\mathcal{A}}\right) \sigma_{\mathcal{C}} \\
& =\sigma_{B} \sigma_{\mathcal{A}}=\sigma_{\mathcal{B}} \sigma_{A} .
\end{aligned}
$$

The first line of these equations tells us that $\gamma$ is the product of the glide $\sigma_{\mathcal{B}} \sigma_{\mathcal{A}}$ and the reflection $\sigma_{\mathcal{C}}$ in either order. More important, the second line tells us that $\gamma$ is a product $\sigma_{B} \sigma_{\mathcal{A}}$ with $B$ off $\mathcal{A}$ and a product $\sigma_{\mathcal{B}} \sigma_{A}$ with $A$ off $\mathcal{B}$.

We want to show, conversely, that such a product is a glide reflection. Suppose point $P$ is off line $\mathcal{L}$. Let $\mathcal{P}$ be the perpendicular from $P$ to $\mathcal{L}$ and let $\mathcal{M}$ be the perpendicular at $P$ to $\mathcal{P}$. Lines $\mathcal{L}$ and $\mathcal{M}$ are distinct since $P$ is off $\mathcal{L}$. Furthermore,

$$
\sigma_{P} \sigma_{\mathcal{L}}=\sigma_{\mathcal{P}} \sigma_{\mathcal{M}} \sigma_{\mathcal{L}} \quad \text { and } \quad \sigma_{\mathcal{L}} \sigma_{P}=\sigma_{\mathcal{L}} \sigma_{\mathcal{P}} \sigma_{\mathcal{M}}=\sigma_{\mathcal{P}} \sigma_{\mathcal{L}} \sigma_{\mathcal{M}}
$$

Therefore, the products $\sigma_{P} \sigma_{\mathcal{L}}$ and $\sigma_{\mathcal{L}} \sigma_{P}$ are glide reflections by the definition of a glide reflection.
5.2.4 Corollary. The set of all glide reflections has the inverse property.

Note : The set of all glide reflections does not have the closure property because the product of two glide reflections (= odd isometries) must be an even isometry.
5.2.5 Proposition. Lines $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are neither concurrent nor have a common perpendicular if and only if $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is a glide reflection.

Proof : $(\Leftarrow)$ If $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is a glide reflection, then $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is not a reflection and the lines $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ cannot be either concurrent or parallel.
$(\Rightarrow)$ Suppose $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are any lines that are neither concurrent nor have a common perpendicular. We wish to prove that $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is a glide reflection.

First, we consider the case lines $\mathcal{P}$ and $\mathcal{Q}$ intersect at some point $Q$. Then $Q$ is off $\mathcal{R}$ as the lines are not concurrent. Let $P$ be the foot of the perpendicular from $Q$ to $\mathcal{R}$, and let $\mathcal{M}$ be the line through $P$ and $Q$. There is a line $\mathcal{L}$ through $Q$ such that $\sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}=\sigma_{\mathcal{M}} \sigma_{\mathcal{L}}$. Since $\mathcal{P} \neq \mathcal{Q}$, then $\mathcal{L} \neq \mathcal{M}$ and $P$ is off $\mathcal{L}$. Hence,

$$
\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}=\sigma_{\mathcal{R}} \sigma_{\mathcal{M}} \sigma_{\mathcal{L}}=\sigma_{P} \sigma_{\mathcal{L}}
$$

with $P$ off $\mathcal{L}$. Therefore, $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is a glide reflection by Proposition 5.2.3.

There remains the case $\mathcal{P} \| \mathcal{Q}$. In this case, lines $\mathcal{R}$ and $\mathcal{Q}$ must intersect as otherwise $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ have a common perpendicular. Then, by what we just
proved, there is some point $P$ off some line $\mathcal{L}$ such that $\sigma_{\mathcal{P}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{R}}=\sigma_{P} \sigma_{\mathcal{L}}$. Hence,

$$
\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}=\left(\sigma_{\mathcal{P}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{R}}\right)^{-1}=\left(\sigma_{P} \sigma_{\mathcal{L}}\right)^{-1}=\sigma_{\mathcal{L}} \sigma_{P}
$$

with point $P$ off line $\mathcal{L}$. Therefore, again we have $\sigma_{\mathcal{R}} \sigma_{\mathcal{Q}} \sigma_{\mathcal{P}}$ is a glide reflection.

An immediate corollary of this result is that a product of three reflections is a reflection or a glide reflection. Thus, we have a classification of odd isometries.
5.2.6 Proposition. An odd isometry is either a reflection or a glide reflection.

We finally have

### 5.2.7 Theorem. (The Classification Theorem for Plane Isometries)

Each nonidentity isometry is exactly one of the following : translation, rotation, reflection or a glide reflection.

Exercise 75 Prove that
(a) A translation that fixes line $\mathcal{C}$ commutes with a glide reflection with axis $\mathcal{C}$.
(b) The square of a glide reflection is a nonidentity translation.
5.2.8 Proposition. If $\gamma$ is a glide reflection with axis $\mathcal{C}$ and $\alpha$ is an isometry, then $\alpha \gamma \alpha^{-1}$ is a glide reflection with axis $\alpha(\mathcal{C})$.

Proof: We have $\gamma^{2} \neq \iota$. So, a glide reflection is not an involution. Since $\alpha \gamma \alpha^{-1}$ is an odd isometry that fixes line $\alpha(\mathcal{C})$ but is not an involution, then $\alpha \gamma \alpha^{-1}$ has to be a glide reflection with axis $\alpha(\mathcal{C})$.

### 5.3 Equations for Isometries

The equations for a general translation were incorporated in the definition of a translation. Equations for a reflection were determined in Proposition 3.1.3. We now turn to rotations.
5.3.1 Proposition. Rotation $\rho_{O, r}$ about the origin has equations

$$
\left\{\begin{aligned}
x^{\prime} & =(\cos r) x-(\sin r) y \\
y^{\prime} & =(\sin r) x+(\cos r) y
\end{aligned}\right.
$$

Proof : Let

$$
\rho_{O, r}=\sigma_{\mathcal{M}} \sigma_{\mathcal{L}}
$$

where $\mathcal{L}$ is the $x$-axis. Then one directed angle from $\mathcal{L}$ to $\mathcal{M}$ has directed measure $\frac{r}{2}$. From the definition of trigonometric functions we know that $\left(\cos \frac{r}{2}, \sin \frac{r}{2}\right)$ is a point on $\mathcal{M}$. So line $\mathcal{M}$ has equation

$$
\left(\sin \frac{r}{2}\right) x-\left(\cos \frac{r}{2}\right) y=0
$$

Hence $\sigma_{\mathcal{M}}$ has equations:

$$
\begin{aligned}
x^{\prime} & =x-\frac{2 \sin \frac{r}{2}\left(\left(\sin \frac{r}{2}\right) x-\left(\cos \frac{r}{2}\right) y\right)}{\sin ^{2} \frac{r}{2}+\cos ^{2} \frac{r}{2}} \\
& =\left(1-2 \sin ^{2} \frac{r}{2}\right) x+\left(2 \sin \frac{r}{2} \cos \frac{r}{2}\right) y \\
& =(\cos r) x+(\sin r) y \\
y^{\prime} & =y+\frac{2 \cos \frac{r}{2}\left(\left(\sin \frac{r}{2}\right) x-\left(\cos \frac{r}{2}\right) y\right)}{\sin ^{2} \frac{r}{2}+\cos ^{2} \frac{r}{2}} \\
& =\left(2 \sin \frac{r}{2} \cos \frac{r}{2}\right) x-\left(1-2 \cos ^{2} \frac{r}{2}\right) y \\
& =(\sin r) x-(\cos r) y
\end{aligned}
$$

Since $\sigma_{\mathcal{L}}$ has equations

$$
\left\{\begin{array}{l}
x^{\prime}=x \\
y^{\prime}=-y
\end{array}\right.
$$

then the rotation $\rho_{O, r}=\sigma_{\mathcal{M}} \sigma_{\mathcal{L}}$ has the equations

$$
\left\{\begin{array}{l}
x^{\prime}=(\cos r) x-(\sin r) y \\
y^{\prime}=(\sin r) x+(\cos r) y .
\end{array}\right.
$$

5.3.2 Proposition. The general equations for an even isometry are

$$
\left\{\begin{array}{ll}
x^{\prime}=a x-b y+h \\
y^{\prime}=b x+a y+k &
\end{array} \quad \text { with } \quad a^{2}+b^{2}=1\right.
$$

and, conversely, such equations are those of an even isometry.
Proof : Let $C=(u, v)$. Since

$$
\rho_{C, r}=\tau_{O, C} \rho_{O, r} \tau_{C, O} \quad \text { (by Proposition 5.1.8) }
$$

the equations for rotation $\rho_{C, r}$ about the point $C=(u, v)$ are easily obtained by composing three sets of equations. The rotation has equations

$$
\left\{\begin{array}{l}
x^{\prime}=(\cos r)(x-u)-(\sin r)(y-v)+u \\
y^{\prime}=(\sin r)(x-u)+(\cos r)(y-v)+v
\end{array}\right.
$$

These equations for the rotation $\rho_{C, r}$ have the form

$$
\left\{\begin{array}{l}
x^{\prime}=(\cos r) x-(\sin r) y+h \\
y^{\prime}=(\sin r) x+(\cos r) y+k
\end{array}\right.
$$

which, conversely, are the equations of a rotation unless $r^{\circ}=0^{\circ}$. Indeed, given $h, k$, and $r$, there are unique solutions for $u$ and $v$ given by

$$
\begin{aligned}
& h=u(1-\cos r)+v \sin r \\
& k=u(-\sin r)+v(1-\cos r)
\end{aligned}
$$

unless $r^{\circ}=0^{\circ}$. In case $r^{\circ}=0^{\circ}$, the equations above are those of a general translation.

Since the even isometries are the translations and the rotations, setting

$$
a=\cos r \quad \text { and } \quad b=\sin r
$$

we have the general equations for an even isometry :

$$
\left\{\begin{array}{l}
x^{\prime}=a x-b y+h \\
y^{\prime}=b x+a y+k
\end{array} \quad \text { with } \quad a^{2}+b^{2}=1 .\right.
$$

5.3.3 Proposition. The general equations for an isometry (on the plane) are

$$
\left\{\begin{array}{l}
x^{\prime}=a x-b y+h \\
y^{\prime}= \pm(b x+a y)+k
\end{array} \quad \text { with } \quad a^{2}+b^{2}=1\right.
$$

and, conversely, such equations are those of an isometry.
Proof: If $\alpha$ is an odd isometry and $\mathcal{L}$ any line, then $\alpha$ is the product of even isometry $\sigma_{\mathcal{L}} \alpha$ followed by $\sigma_{\mathcal{L}}$. Taking $\mathcal{L}$ as the $x$-axis, we have any odd isometry is the product of an even isometry followed by the reflection in the $x$-axis. This observation, together with Proposition 5.3.2, gives the desired result, where the positive sign applies when isometry is even and negative sign applies when isometry is odd.

### 5.4 Exercises

## Exercise 76 TRUE or FALSE ?

(a) An even isometry that fixes two points is the identity.
(b) The set of rotations generates $\mathfrak{I s o m}{ }^{+}$.
(c) An odd isometry is a product of three reflections.
(d) An even isometry is a product of four reflections.
(e) If $\rho_{\alpha(C), r}=\rho_{C, r}$ for isometry $\alpha$, then $\alpha$ fixes $C$.
(f) $\rho_{B, r} \rho_{A,-r}$ is the translation that takes $A$ to $\rho_{B, r}(A)$.

Exercise 77 PROVE or DISPROVE: Given $\tau_{A, B}$ and nonidentity rotation $\rho_{C, r}$, there is a rotation $\rho_{D, s}$ such that $\tau_{A, B}=\rho_{D, s} \rho_{C, r}$.

Exercise 78 Show that if $\rho_{1}, \rho_{2}, \rho_{2} \rho_{1}$, and $\rho_{2}^{-1} \rho_{1}$ are rotations, then the centres of $\rho_{1}, \rho_{2} \rho_{1}$, and $\rho_{2}^{-1} \rho_{1}$ are collinear.

Exercise 79 Show that translation $\tau$ commutes with $\sigma_{\mathcal{C}}$ if and only if $\tau$ fixes $\mathcal{C}$. Also, that $\tau$ commutes with a glide reflection with axis $\mathcal{C}$ if and only if $\tau$ fixes $\mathcal{C}$.

## Exercise 80 TRUE or FALSE ?

(a) Every isometry is a product of two involutions.
(b) An isometry that does not fix a point is a glide reflection.
(c) If $\gamma=\sigma_{\mathcal{L}} \sigma_{P}$, then $\gamma$ is a glide reflection with axis the line through $P$ that is perpendicular to $\mathcal{L}$.
(d) If $\gamma$ is a glide reflection with axis $\mathcal{C}$ and $P$ is a point on $\mathcal{C}$, then there are unique lines $\mathcal{L}$ and $\mathcal{M}$ such that $\gamma=\sigma_{\mathcal{M}} \sigma_{P}=\sigma_{P} \sigma_{\mathcal{L}}$.
(e) If $\sigma_{\mathcal{C}} \sigma_{B} \sigma_{\mathcal{A}}$ fixes line $\mathcal{L}$, then $\sigma_{\mathcal{C}} \sigma_{B} \sigma_{\mathcal{A}}$ is a glide reflection with axis $\mathcal{L}$.
(f) If $\sigma_{C} \sigma_{\mathcal{B}} \sigma_{A}$ fixes line $\mathcal{L}$, then $\sigma_{C} \sigma_{\mathcal{B}} \sigma_{A}$ is a glide reflection with axis $\mathcal{L}$.

Exercise 81 PROVE or DISPROVE : If point $M$ is on the axis of glide reflection $\gamma$, then there is a point $P$ such that $M$ is the midpoint of $P$ and $\gamma(P)$.

Exercise 82 PROVE or DISPROVE : Every glide reflection is a product of three reflections in the three lines containing the sides of some triangle.

Exercise 83 Which isometries are dilatations?

Exercise 84 Prove that if $\tau$ is a translation, then there is a glide reflection $\gamma$ such that $\tau=\gamma^{2}$.

Exercise 85 What are the equations for each of the rotations $\rho_{O, 90}, \rho_{O, 180}$, and $\rho_{O, 270}$ ?

Exercise 86 If

$$
\left\{\begin{array}{l}
x^{\prime}=a x+b y+h \\
y^{\prime}=b x-a y+k
\end{array} \quad \text { with } \quad a^{2}+b^{2}=1\right.
$$

are the equations for isometry $\alpha$, show that $\alpha$ is a reflection if and only if $a h+b k+h=$ 0 and $a k-b h-k=0$.

## Exercise 87 TRUE or FALSE?

(a) $x^{\prime}=-x+6$ and $y^{\prime}=-y-7$ are equations for a rotation.
(b) $x^{\prime}=p x-q y+r$ and $y^{\prime}=q x+p y+s$ are equations for an even isometry.
(c) $x^{\prime}=-p x-q y-r$ and $y^{\prime}=q x-p y-s$ are equations for an even isometry if $p^{2}+q^{2}=1$.
(d) $x^{\prime}=-a x+b y+h$ and $y^{\prime}=b x+a y+k$ are equations for an odd isometry if $a^{2}+b^{2}=1$.
(e) $x^{\prime}= \pm a x-b y+h$ and $y^{\prime}= \pm b x+a y+k$ are equations for an isometry if $a^{2}+b^{2}=1$.
(f) If $\mathcal{M}$ is any line, then every odd isometry is the product of $\sigma_{\mathcal{M}}$ followed by an even isometry.
(g) If $\mathcal{M}$ is any line, then every odd isometry is the product of an even isometry followed by $\sigma_{\mathcal{M}}$.

## Exercise 88 If

$$
\left\{\begin{array}{l}
x^{\prime}=-\frac{\sqrt{3}}{2} x-\frac{1}{2} y+1 \\
y^{\prime}=\frac{1}{2} x-\frac{\sqrt{3}}{2} y-\frac{1}{2}
\end{array}\right.
$$

are equations for $\rho_{P, r}$, then find $P$ and $r$.

## Exercise 89 If

$$
\left\{\begin{array}{l}
x^{\prime}=(\cos r) x-(\sin r) y+h \\
y^{\prime}=(\sin r) x+(\cos r) y+k
\end{array}\right.
$$

are equations for nonidentity rotation $\rho_{C, r}$, then find $C$.

## Exercise 90 If

$$
\left\{\begin{array}{l}
x^{\prime}=a x+b y+h \\
y^{\prime}=b x-a y+k
\end{array} \quad \text { with } \quad a^{2}+b^{2}=1 .\right.
$$

are equations for $\sigma_{\mathcal{L}}$, then find $\mathcal{L}$.

DISCUSSION : The idea of doing geometry in terms of numbers and equations caught on after the publication of Descartes' La Géométrie in 1637. However, the idea that numbers and equations are geometric objects arose much later. In fact, the idea had no solid foundation until 1858, when the set $\mathbb{R}$ of real numbers was first given a clear definition, by Richard Dedekind (1831-1916). Dedekind's definition explains in particular the continuity of $\mathbb{R}$ which enables it to serve as a model for the line.

Once one has this model for the line it is relatively straightforward to model the plane by $\mathbb{R}^{2}$ and to verify EUcLID's axioms. This was first done in detail by David Hilbert (1862-1943), thus subordinating geometry to the number concept after 2000 years of independence. It should be mentioned, however, that any construction of $\mathbb{R}$ from the natural numbers $0,1,2, \cdots$ involves infinite sets. Thus, a "point" is a much subtle object than naïve intuition suggests.

The idea of interpreting points as numbers has been unexpectedly fruitful. Of course, we expect $\mathbb{R}$ to behave like a line because $\mathbb{R}$ was constructed with that purpose in mind, and it is no surprise that + and $\times$ have a geometric meaning on the line ( + as a translation, $\times$ as a dilation). It may also not be a surprise that + has a meaning in $\mathbb{R}^{2}$ (translation $=$ vector addition) and so does multiplication by a real number (dilation again). But it is surely an unexpected bonus when multiplication by a complex number turns out to be geometrically meaningful.

After all, this multiplication is forced on us by algebra - by the demand that $i^{2}=-1$ and that the field laws hold - yet when $a+i b \in \mathbb{C}$ is interpreted as $(a, b) \in \mathbb{R}^{2}$, multiplication by a complex number is simply the product of a dilation and a rotation. In particular, we have the miraculous fact that multiplication by $e^{i r}$ is rotation through $r$. And this is just the beginning of the interplay between complex numbers and angles, leading to many applications of complex numbers, and particularly complex functions, in geometry.

In terms of the correspondence between vectors, points, and complex numbers we can set up a "dictionary" between geometry and complex numbers, as follows :

| Vector, $\vec{v}=\left[\begin{array}{l}x \\ y\end{array}\right] \quad$ (or point, $\left.P=(x, y)\right)$ | Complex number, $z=x+i y$ |
| :--- | :--- |
| Length of a vector, $\\|\vec{v}\\|$ | Modulus, $\|z\|$ |
| Distance between two points, $P_{1} P_{2}$ | Modulus of the difference, $\left\|z_{1}-z_{2}\right\|$ |
| Dot product, $\vec{v}_{1} \bullet \vec{v}_{2}$ | Real part of product, $\operatorname{Re}\left(\bar{z}_{1} z_{2}\right)$ |
| Collinear points, $P_{1}-P_{2}-P_{3}$ | Vanishing imaginary part, $\frac{z_{2}-z_{1}}{z_{3}-z_{1}} \in \mathbb{R}$ |

Oriented angle between $\vec{\imath}=\left[\begin{array}{l}1 \\ 0\end{array}\right]$ and $\vec{v} \quad$ Argument, $\arg (z) \in(-\pi, \pi]$

| Orientation, $\vec{v} \mapsto \vec{v}^{\perp}$ | Multiplication by $i, z \mapsto i z$ |
| :--- | :--- |
| Translation, $\vec{v} \mapsto \vec{v}+\vec{a}$ | Addition, $z \mapsto z+w$ |

Rotation, $\vec{v} \mapsto \rho_{r}(\vec{v}) \quad$ Multiplication by $e^{i r}, z \mapsto e^{i r} z$

Reflection in $x$-axis, Complex conjugation, $z \mapsto \bar{z}$.

Using this dictionary we could translate all that we have done so far into the language of complex numbers.

The meaning of the word geometry changes with time and with the speaker.

