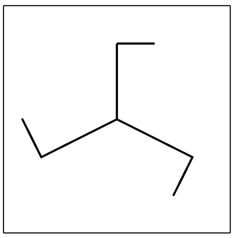
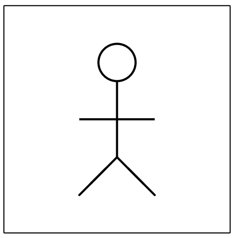
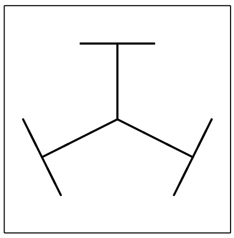
Chapter 6

Symmetry

Topics :

1. Symmetry and Groups
2. The Cyclic and Dihedral Groups
3. Finite Symmetry Groups

6.1 Symmetry and Groups

There is an abundant supply of objects (bodies, organisms, structures, etc.) with *symmetry* in nature. Figures with *symmetry* appear throughout the visual arts. There are also many scientific applications of symmetry (for instance the classification of crystals and quasicrystals in chemistry). Theoretical physics makes heavy use of symmetry. But *what is symmetry* ?

When we say that a geometric figure (shape) is "symmetrical" we mean that we can apply certain isometries, called *symmetry operations*, which leave the whole figure unchanged while permuting its parts.

6.1.1 EXAMPLE. The capital letters **E** and **A** have *bilateral* (or *mirror) symmetry*, the mirror being horizontal for the former, vertical for the latter. (Bilateral symmetry is the symmetry of left and right, which is so noticeable in the structure of higher animals, especially the human body.)

6.1.2 EXAMPLE. The capital letter **N** is left unchanged by a halfturn, which may be regarded as the result of reflecting horizontally and then vertically, or vice versa. (Alternatively, one may prefer to view the halturn as a rotation about the "centre" through an angle of $180°$.) We can say that the capital letter **N** has *rotational symmetry.*

6.1.3 EXAMPLE. Another basic kind of symmetry is *translational symmetry.* Several combinations of these so-called basic symmetries may occur (for instance, bilateral and rotational symmetry, glide symmetry, translational and rotational symmetry, two independent translational symmetries, etc.)

Exercise 91 Find simple geometric figures (patterns) exhibiting each of the foregoing kinds of symmetry.

NOTE : In counting the symmetry operations of a figure, it is usual to include the identity tranformation; any figure has this trivial symmetry.

We make the following definitions. Let **S** be a set of points (in $\mathbb{E}^2$).

6.1.4 Definition. Line $\mathcal{L}$ is a **line of symmetry** (or **symmetry axis**) for $\mathbf{S}$ if

$$\sigma_{\mathcal{L}}(\mathbf{S}) = \mathbf{S}.$$

6.1.5 Definition. Point P is a **point of symmetry** (or **symmetry centre**) for $\mathbf{S}$ if

$$\sigma_P(\mathbf{S}) = \mathbf{S}.$$

Exercise 92 Can a figure have

(a) *exactly* two lines of symmetry ?

(b) *exactly* two points of symmetry ?

Exercise 93 Why can't a (capital) letter of the alphabet (*written in most symmetric form*) have two points of symmetry ?

6.1.6 Definition. Isometry α is a **symmetry** for $\mathbf{S}$ if

$$\alpha(\mathbf{S}) = \mathbf{S}.$$

6.1.7 Example. Find the symmetries of a rectangle $\mathbf{R}=\square ABCD$ that is not a square.

Solution : Without loss of generality, we may assume that

$$A = (h,k)\,,\ B = (-h,k)\,,\ C = (-h,-k)\,, \quad \text{and} \quad D = (h,-k)\,; \quad h,k > 0\,,\ h \neq k.$$

Evidently, the x- and y-axes are lines of symmetry for the rectangle, and the origin is a point of symmetry for the rectangle. Denoting the reflection in the x-axis by σ_x and the reflection in the y-axis by σ_y, we have that σ_x, σ_y, σ_O, and ι are symmetries for $\mathbf{R}$. Note that ι is a symmetry for *any* set of points. Since the image of the rectangle is known once it is known which of A, B, C, D is the image of A, then *these four transformations are the only possible symmetries for* $\mathbf{R}$.

NOTE : The (four) symmetries for a rectangle that is not a square form a group. Traditionally, this group is denoted by $\mathfrak{V}_4$ and is known as **Klein's four-group** (*Vierergruppe* in German).

6.1.8 PROPOSITION. *The set of all symmetries of a set of points forms a group.*

PROOF : Let $\mathbf{S}$ be any set of points. The set of symmetries for $\mathbf{S}$ is not empty as ι is a symmetry for $\mathbf{S}$.

Suppose α and β are symmetries for $\mathbf{S}$. Then

$$\beta\alpha(\mathbf{S}) = \beta(\alpha(\mathbf{S})) = \beta(\mathbf{S}) = \mathbf{S}.$$

So the set of symmetries has the closure property.

If α is a symmetry for $\mathbf{S}$, then α and α^{-1} are transformations and

$$\alpha^{-1}(\mathbf{S}) = \alpha^{-1}(\alpha(\mathbf{S})) = \iota(\mathbf{S}) = \mathbf{S}.$$

So the set of symmetries also has the inverse property.

Hence, the set of all symmetries of the set $\mathbf{S}$ forms a group. □

The group of all symmetries of the set (figure) $\mathbf{S}$ is denoted by $\mathfrak{Sym}\,(\mathbf{S})$ and is called the **symmetry group** of $\mathbf{S}$.

What happens if the set of points is taken to be the set of *all* points, that is the plane $\mathbb{E}^2$? In this special case, the symmetries are exactly the same thing as the isometries. So

$$\mathfrak{Isom} = \mathfrak{Sym}\,(\mathbb{E}^2).$$

In other words, the group $\mathfrak{Isom}$ *is the symmetry group of the Euclidean plane.*

6.1.9 COROLLARY. *The set of all isometries forms a group.*

Examples of symmetry groups

Symmetry groups can be complicated. However, the *discrete* ones can be completely classified and listed (at least for Euclidean geometry).

6.1.10 Example. The symmetry group of the capital letter **E** (or **A**) is the so-called **dihedral group** of order 2, generated by a single reflection and denoted by $\mathfrak{D}_1$.

Note : The Greek origin of the word *dihedral* is almost equivalent to the Latin origin of *bilateral*.

Exercise 94 What is the *symmetry group* of

(a) a *scalene* triangle ?

(b) an isosceles triangle that is not equilateral ?

6.1.11 Example. The symmetry group of the capital letter **N** is likewise of order 2, but in this case the generator is a halfturn and we speak of the **cyclic group** $\mathfrak{C}_2$.

Note : The two groups $\mathfrak{D}_1$ and $\mathfrak{C}_2$ are "abstractly identical" (or *isomorphic*).

Exercise 95 What is the *symmetry group* of

(a) a parallelogram that is not a rhombus ?

(b) a parallelogram that is neither a rectangle nor a rhombus ?

6.2 The Cyclic and Dihedral Groups

Let $\mathfrak{G}$ be a group of isometries (i.e. a subgroup of $\mathfrak{Isom}$). Recall that $\mathfrak{G}$ is said to be **finite** if it consists of a finite number of elements (transformations); otherwise, $\mathfrak{G}$ is said to be **infinite**. The **order** of a (finite) group is the number of elements it contains.

The cyclic groups

Let $\alpha \in \mathfrak{G}$. If every element of $\mathfrak{G}$ is a power of α, then we say that $\mathfrak{G}$ is **cyclic** with *generator* α and denoted by $\langle \alpha \rangle$.

NOTE : The group $\langle \alpha \rangle$ is the smallest subgroup of $\mathfrak{G}$ containing the element (transformation) α. Two possibilities may arise :

(a) All the powers α^k are different. In this case the group $\langle \alpha \rangle$ is infinite and is referred to as an *infinite cyclic group*.

(b) Among the powers of α there are some that coincide. Then there is a positive power of α which is equal to the identity transformation ι. Denote by n the smallest positive exponent satisfying $\alpha^n = \iota$. In this case the group generated by α is

$$\langle \alpha \rangle = \{\iota, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}.$$

Such a (finite) group is a *cyclic group* of order n.

Cyclic groups are Abelian (i.e. commutative).

Let $n \geq 1$ be a positive integer and fix an arbitrary point C in the plane. (Without any loss of generality, we may assume that C is the origin.)

6.2.1 DEFINITION. The **cyclic group** $\mathfrak{C}_n$ is the (finite) group generated by the rotation $\rho = \rho_{C, \frac{360}{n}}$.

This group contains exactly n rotations (about the same centre C). The angles of rotation are multiples of $\frac{360}{n}\cdot$ We have

$$\begin{aligned} \mathfrak{C}_n &= \langle \rho \rangle \\ &= \{\iota, \rho, \rho^2, \ldots, \rho^{n-1}\} \\ &= \left\{\iota, \rho_{C, \frac{360}{n}}, \rho_{C, \frac{360 \cdot 2}{n}}, \ldots, \rho_{C, \frac{360 \cdot (n-1)}{n}}\right\}. \end{aligned}$$

6.2.2 EXAMPLE. The cyclic group $\mathfrak{C}_1$ is the *trivial group* $\{\iota\}$.

6.2.3 EXAMPLE. The cyclic group $\mathfrak{C}_2$ has two elements : the identity transformation ι and the halfturn σ_C. This is the symmetry group of the capital letter **N**.

6.2.4 EXAMPLE. The *swastika* is symmetrical by rotation through any number of right angles; it admits four distinct symmetry operations : rotations through $1, 2, 3,$ or 4 right angles. The last is the identity. The first and the third are inverses of each other, since their product is the identity.

The symmetry group of the swastika is $\mathfrak{C}_4$, the cyclic group of order 4, generated by a rotation ρ of $90°$ (or *quarterturn*).

Exercise 96 Find a figure with symmetry group the cyclic group $\mathfrak{C}_3$.

NOTE : For any positive integer $n \geq 2$, there is polygon having symmetry group $\mathfrak{C}_n$.

The dihedral groups

Again, let $n \geq 1$ be a positive integer and C a fixed point. We are going to extend the cycle groups $\mathfrak{C}_n$ by incorporating appropriate reflections (i.e. bilateral symmetries).

6.2.5 DEFINITION. The **dihedral group** $\mathfrak{D}_n$ is the (finite) group containing the elements (rotations) of $\mathfrak{C}_n$ together with reflections in the n lines through C which devide the plane into $2n$ congruent angular regions.

This group has order $2n$ (i.e. it contains exactly $2n$ elements) : n rotations (about C) and n reflections in lines (passing through C). The angles between the axes of the reflections are multiples of $\frac{180}{n}$.

6.2.6 EXAMPLE. The dihedral group $\mathfrak{D}_1$ has two elements : the identity transformation ι and the reflection $\sigma_{\mathcal{L}}$ in a line $\mathcal{L}$ passing through C (the so-called symmetry axis). This is the symmetry group of the capital letter **A**.

6.2.7 EXAMPLE. The capital letter **H** admits both reflections and rotations as symmetry operations. It has a horizontal mirror (like **E**) and a vertical mirror (like **A**), as well as a center of rotational symmetry (like **N**) where the mirrors intersect. Thus it has four symmetry operations : the identity

ι, the horizontal reflection σ_h, the vertical reflection σ_v, and the halfturn $\sigma_h\sigma_v = \sigma_v\sigma_h$.

The symmetry group of the capital letter **H** is $\mathfrak{D}_2$, the **dihedral** group of order 4, generated by the two reflections σ_h and σ_v. Group $\mathfrak{D}_2$ is the familiar group $\mathfrak{V}_4$ (Klein's four-group).

Exercise 97 Compute the symmetry group of a rectangle that is not a square.

NOTE : Although $\mathfrak{C}_4$ and $\mathfrak{D}_2$ have both order 4, they are *not* isomorphic : they have a different structure, different CAYLEY tables. To see this, it suffices to observe that $\mathfrak{C}_4$ contains two elements of order 4, whereas all the elements of $\mathfrak{D}_2$ (except the identity) are of order 2.

6.2.8 EXAMPLE. Compute the symmetry group of a square.

SOLUTION : We suppose the square is centered at the origin and that one vertex lies on the positive x-axis. We see that the square is fixed by ρ and σ, where

$$\rho = \rho_{O,90} \quad \text{and} \quad \sigma = \sigma_h \,.$$

Observe that

$$\rho^4 = \sigma^2 = \iota \,.$$

Since the symmetries of the square form a group, then the square must be fixed by the four distinct rotations $\rho,\ \rho^2,\ \rho^3,\ \rho^4$ and by the four distinct isometries $\rho\sigma,\ \rho^2\sigma,\ \rho^3\sigma,\ \rho^4\sigma$. Let V_1 and V_2 be adjacent vertices of the square. Under a symmetry, V_1 must go to any one of the four vertices, but then V_2 must go to one of the two vertices adjacent to that one and the images of all remaining vertices are then determined. So *there are at most eight symmetries for the square.* We have listed eight distinct symmetries above. Therefore, there are exactly eight symmetries and we have listed all of them. Isometries ρ and σ generate the entire group.

The symmetry group of the square is

$$\mathfrak{D}_4 = \langle \rho, \sigma \rangle = \{\iota,\ \rho,\ \rho^2,\ \rho^3,\ \sigma,\ \rho\sigma,\ \rho^2\sigma,\ \rho^3\sigma\}$$

the **dihedral** group of order 8. Observe that

$$\sigma\rho = \rho^3\sigma\,, \quad \sigma\rho^2 = \rho^2\sigma\,, \quad \sigma\rho^3 = \rho\sigma.$$

The Cayley table for $\mathfrak{D}_4$ is given below.

$\mathfrak{D}_4$	ι	ρ	ρ^2	ρ^3	σ	$\rho\sigma$	$\rho^2\sigma$	$\rho^3\sigma$
ι	ι	ρ	ρ^2	ρ^3	σ	$\rho\sigma$	$\rho^2\sigma$	$\rho^3\sigma$
ρ	ρ	ρ^2	ρ^3	ι	$\rho\sigma$	$\rho^2\sigma$	$\rho^3\sigma$	σ
ρ^2	ρ^2	ρ^3	ι	ρ	$\rho^2\sigma$	$\rho^3\sigma$	σ	$\rho\sigma$
ρ^3	ρ^3	ι	ρ	ρ^2	$\rho^3\sigma$	σ	$\rho\sigma$	$\rho^2\sigma$
σ	σ	$\rho^3\sigma$	$\rho^2\sigma$	$\rho\sigma$	ι	ρ^3	ρ^2	ρ
$\rho\sigma$	$\rho\sigma$	σ	$\rho^3\sigma$	$\rho^2\sigma$	ρ	ι	ρ^3	ρ^2
$\rho^2\sigma$	$\rho^2\sigma$	$\rho\sigma$	σ	$\rho^3\sigma$	ρ^2	ρ	ι	ρ^3
$\rho^3\sigma$	$\rho^3\sigma$	$\rho^2\sigma$	$\rho\sigma$	σ	ρ^3	ρ^2	ρ	ι

6.2.9 EXAMPLE. Let $n \geq 3$ and consider a regular n-sided polygon centered at the origin. Suppose that one vertex lies on the positive x-axis.

The n-sided polygon is fixed by ρ and σ, where

$$\rho = \rho_{O,\frac{360}{n}} \quad \text{and} \quad \sigma = \sigma_h.$$

(σ_h is the reflection in the x-axis.)

Observe that

$$\rho^n = \sigma^2 = \iota.$$

Since the symmetries of the polygon form a group, then the polygon must be fixed by the n distinct rotations

$$\rho, \rho^2, \ldots, \rho^{n-1}$$

and by the n distinct odd isometries

$$\sigma, \rho\sigma, \rho^2\sigma, \ldots, \rho^{n-1}\sigma.$$

The symmetry group of the n-sided polygon must have at least these $2n$ symmetries. Let V_1 and V_2 be adjacent vertices of the polygon. Under a symmetry, V_1 must go to any one of the n vertices, but then V_2 must go to one of the two vertices adjacent to that one and the images of all remaining vertices are then determined. So *there are at most $2n$ symmetries for the n-sided polygon.* Therefore, there are exactly $2n$ symmetries and we have listed all of them. Isometries ρ and σ generate the entire group.

The symmetry group of the n-sided polygon is

$$\mathfrak{D}_n = \langle \rho, \sigma \rangle = \{\iota, \rho, \rho^2, \ldots, \rho^{n-1}, \sigma, \rho\sigma, \rho^2\sigma, \ldots, \rho^{n-1}\sigma\},$$

the **dihedral** group of order $2n$.

To compute the entire Cayley table, all that is needed are the equations

$$\sigma\rho^k = \rho^{-k}\sigma \quad \text{and} \quad \rho^n = \sigma^2 = \iota.$$

NOTE : The groups $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are, respectively, symmetry groups of an isosceles triangle that is not equilateral and of a rectangle that is not a square. Hence, or any positive integer $n \geq 1$, there is polygon having symmetry group $\mathfrak{D}_n$.

6.3 Finite Symmetry Groups

We want to investigate the possible finite symmetry groups of figures (in the Euclidean plane $\mathbb{E}^2$). So we are led to the study of finite subgroups $\mathfrak{G}$ of the group $\mathfrak{Isom}$ of isometries (on $\mathbb{E}^2$).

The key observation which allows us to describe *all* finite symmetry groups is the following result.

6.3.1 PROPOSITION. *Let $\mathfrak{G}$ be a finite group of iometries. Then there is a point C in the plane which is left fixed by every element of $\mathfrak{G}$.*

PROOF : Let P be any point in the plane, and let $\mathbf{P}$ be the set of points which are images of P under the various elements (isometries) of $\mathfrak{G}$. So each element $P' \in \mathbf{P}$ has the form $P' = \alpha(P)$ for some $\alpha \in \mathfrak{G}$.

Any element of the group $\mathfrak{G}$ will permute $\mathbf{P}$. (In other words, if $P' \in \mathbf{P}$ and $\alpha \in \mathfrak{G}$, then $\alpha(P') \in \mathbf{P}$.)

We list the elements of $\mathbf{P}$ arbitrarily, writing

$$\mathbf{P} = \{P_1, P_2, \ldots, P_n\}.$$

The fixed point we are looking for is the *centre of gravity* of $\mathbf{P}$, namely

$$C = \frac{1}{n}\left(P_1 + P_2 + \cdots + P_n\right).$$

Any element of $\mathfrak{G}$ permutes the set $\{P_1, P_2, \ldots, P_n\}$, hence it sends the centre of gravity to itself. $\square$

Let $\mathfrak{G}$ be a finite symmetry group (hence a finite subgroup of $\mathfrak{Isom}$). Then there is a point C fixed by every element (isometry) of $\mathfrak{G}$, and we may adjust coordinates so that this point is the origin. Also, it follows that $\mathfrak{G}$ cannot contain a nonidentity translation or a glide reflection.

So *the group $\mathfrak{G}$ contains only rotations (about the same point) or reflections.*

Note : The group generated by a nonidentity translation is an infinite subgroup of $\mathfrak{Isom}$. Hence any subgroup of $\mathfrak{Isom}$ which contains either rotations about two different points or a glide reflection is infinite. Indeed, if $\rho_{C,r}$ and $\rho_{D,s}$ are two nonidentity rotations about different centres, then

$$\rho_{C,r}^{-1}\rho_{D,s}^{-1}\rho_{C,r}\rho_{D,s}$$

is a nonidentity translation. Also, the square of any glide reflection is a nonidentity translation.

Leonardo da Vinci (1452-1519), who wanted to determine the possible ways to attach chapels and niches to a central building without destroying the symmetry of the necleus, realized that *all designs (in the plane) with finitely many symmetries have either rotational symmetries and bilateral symmetries or just rotational symmetries.* In other words, the following result holds.

6.3.2 THEOREM. (LEONARDO'S THEOREM) *A finite symmetry group is either a cyclic group* $\mathfrak{C}_n$ *or a dihedral group* $\mathfrak{D}_n$.

PROOF : We shall consider the case $\mathfrak{G}$ contains only rotations and the case $\mathfrak{G}$ contains at least one reflection separately.

Suppose that (the finite group of isometries) $\mathfrak{G}$ contains only rotations. One possibility is that $\mathfrak{G}$ is the *trivial group* $\mathfrak{C}_1 = \{\iota\}$. Otherwise, we suppose $\mathfrak{G}$ contains a nonidentity rotation $\rho = \rho_{C,r}$. Then all the other elements in $\mathfrak{G}$ are rotations about the same centre C.

We note that

$$\rho_{C,-s} \in \mathfrak{G} \iff \rho_{C,s} \in \mathfrak{G}$$

and that all the elements in $\mathfrak{G}$ can be written in the form $\rho_{C,s}$, where $0 \leq s < 360$.

Let $\rho = \rho_{C,s}$, where s has the *minimum* positive value.

If $\rho_{C,t} \in \mathfrak{G}$ with $t > 0$, then $t - ks$ cannot be positive and less than s for any integer k by the minimality of s. So

- $t = ks$ for some integer k
- $\rho_{C,t} = \rho^k$.

In other words, the elements of $\mathfrak{G}$ are precisely the powers of ρ. We conclude that, in this case, $\mathfrak{G}$ is a cyclic group $\mathfrak{C}_n$ for some positive integer n.

Suppose now that (the finite group of isometries) $\mathfrak{G}$ contains at least one reflection. Since the identity transformation ι is an even isometry, since an isometry and its inverse have the same parity, and since the product of two even isometries is an even isometry, it follows that *the subset of all even isometries in* $\mathfrak{G}$ *forms a finite subgroup* $\mathfrak{G}^+$ *of* $\mathfrak{G}$. By the foregoing argument, we see that

$$\mathfrak{G}^+ = \mathfrak{C}_n = \{\iota, \rho, \rho^2, \ldots, \rho^{n-1}\}.$$

So the even isometries in $\mathfrak{G}$ are the n rotations $\iota = \rho^n, \rho, \rho^2, \ldots, \rho^{n-1}$.

Suppose $\mathfrak{G}$ has m reflections. If σ is a reflection in $\mathfrak{G}$, then the n odd isometries

$$\sigma, \rho\sigma, \rho^2\sigma, \dots, \rho^{n-1}\sigma$$

are in $\mathfrak{G}$. So $n \leq m$.

However, the m odd isometries multiplied (on the right) by σ give m distinct even isometries. So $m \leq n$.

Hence $m = n$ and $\mathfrak{G}$ contains the $2n$ elements generated by rotation ρ and reflection σ. We conclude that, in this case, $\mathfrak{G}$ is a dihedral group $\mathfrak{D}_n$ for some positive integer n. □

Recall that an n-sided polygon (regular or not) has at most $2n$ symmetries. Since the symmetry group of a polygon must then be a finite group (of isometries), Leonardo's Theorem has the following immediate corollary.

6.3.3 Corollary. *The symmetry group for a polygon is either a cyclic group or a dihedral group.*

6.4 Exercises

Exercise 98 What is the *symmetry group* of a rhombus that is not a square ?

Exercise 99 TRUE or FALSE ?

(a) If P is a point of symmetry for set $\mathbf{S}$ of points, then P is in $\mathbf{S}$.

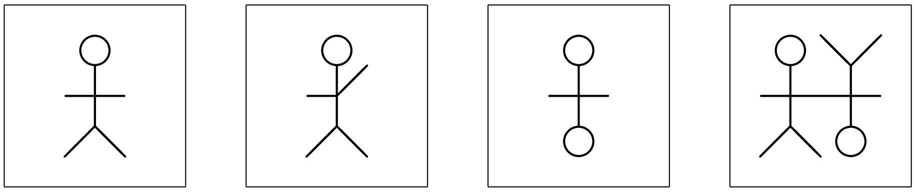
(b) If $\mathcal{L}$ and $\mathcal{M}$ are perpendicular lines, then $\mathcal{L}$ is a line of symmetry for $\mathcal{M}$.

(c) A regular pentagon has a point of symmetry.

(d) The symmetry group of a rectangle has four elements.

Exercise 100 Compute the *symmetry group* of an equilateral triangle.

Exercise 101 Determine the *symmetry groups* of each of the following figures.



Exercise 102 What is the *symmetry group* of the graph of each of the following equations?

(a) $y = x^2$.

(b) $y = x^3$.

(c) $3x^2 + 4y^2 = 12$.

(d) $xy = 1$.

Exercise 103 Arrange the capital letters *written in most symmetric form* into equivalent classes where two letters are in the same class if and only if the two letters have the same symmetries when superimposed in standard orientation.

Exercise 104

(a) Prove that every cyclic group $\mathfrak{C}_n$ is commutative.

(b) Verify that the dihedral groups $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are commutative.

(c) Prove that the groups $\mathfrak{D}_n,\ n \geq 3$ are *not* commutative.

Exercise 105 Find polygons having *symmetry groups* $\mathfrak{C}_3$ and $\mathfrak{C}_4$, respectively.

DISCUSSION : Symmetry appeals to artist and scientist alike; it is intimately associated with an innate human appeciation of pattern. Symmetry is bound up in many of the deepest patterns of Nature, and nowadays it is fundamental to our scientific understanding of the Universe. Conservation principles, such as those for energy and momentum, express a symmetry that (we believe) is possessed by the entire space-time continuum : the laws of physics are the same everywhere. The quantum mechanics of fundamental particles is couched in the mathematical language of symmetries. The symmetries of crystals not only classify their shapes, but determine many of their properties. Many natural forms, from starfish to raindrops, from viruses to galaxies, have striking symmetries.

It took humanity roughly two and a half thousand years to attain a precise formulation of the concept of symmetry, counting from the time when the Greek geometers made the first serious mathematical discoveries about that concept, notably the proof that *there exist exactly five regular solids.* (The five regular solids are the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron.) Only after that lengthy period of gestation was the concept of symmetry something that scientists and mathematicians could *use* rather than just admire.

The understanding that symmetries are best viewed as transformations arose when mathematicians realized that the set of symmetries of an object is not just an arbitrary collection of transformations, but has a beautiful internal structure. The fact that the symmetries of an object form a *group* is a significant one. However, it's such a simple and "obvious" fact that for ages nobody even noticed it; and even when they did, it took mathematicians a while to appreciate just how significant this simple observation really is. It leads to a natural and elegant "algebra" of symmetry, known as *Group Theory.*

In 1952 the distinguished mathematician Hermann Weyl (1885-1955), who was about to retire from the Institute for Advanced Studies at Princeton, gave a series of public lectures on mathematics. His topic, and the title of the book that grew from his talks, was *Symmetry.* It remains one of the classic popularizations of the subject. Some of the Weyl's greatest achievements had been in the deep mathematical setting that underlies the study of symmetry, and his lectures were strongly influenced by his mathematical tastes; but Weyl talked with authority about art and philosophy as well as mathematics and science. You will find in the book discussions of the cyclic groups, dihedral groups, as well as wallpaper groups. Most important, you will find a fascinating treatment in words and pictures of how these purely mathematical abstractions relate to the physical universe and works of art throughout the ages.

Symmetry is a vast subject, significant in art and nature. Mathematics lies at its root, and it would be hard to find a better one on which to demonstrate the working of the mathematical intellect.

Hermann Weyl