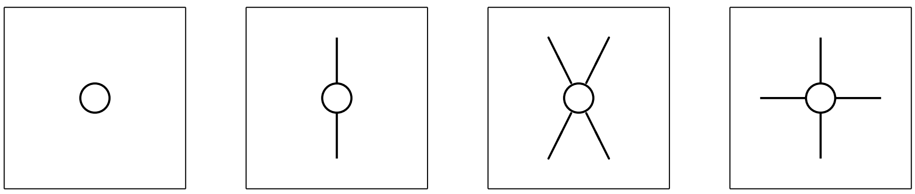
# Chapter 7

# Similarities

**Topics :**

1. Classification of Similarities
2. Equations for Similarities



## 7.1 Classification of Similarities

The image of a triangle as seen through a "magnifying glass" is *similar* to the original triangle. For instance, the transformation $(x, y) \mapsto (2x, 2y)$ is a "magnifying glass" for the Euclidean plane, multiplying all distances by 2. (We shall call such a mapping a *stretch*.)

### Some definitions

We make the following definition.

**7.1.1** Definition. If $C$ is a point and $r > 0$, then a **stretch** (or **homothety**) **of ratio** $r$ **about** $C$ is the transformation that fixes $C$ and otherwise sends point $P$ to point $P'$, where $P'$ is the unique point on $CP^{\rightarrow}$ such that $CP' = rCP$ (or, alternatively, where $P'$ is the unique point on $\overleftrightarrow{CP}$ such that $\underline{CP'} = r\underline{CP}$ ).

We say that the point $C$ is the **centre** and the (positive) factor $r$ is the **magnification ratio** of the stretch. A stretch is also called a **homothetic transformation**.

Note : We allow the identity transformation to be a stretch (of ratio 1 and *any* **centre**). Observe, however, that we allow magnification ratios $r \leq 1$, which is in slight conflict with the everyday meaning of the word "magnification".

**Exercise 106** Verify that the set of all stretches with a given centre $C$ forms a commutative group.

There is nothing to stop us from allowing a *negative* ratio in the definition of a stretch. In this case, point $P$ is taken to a point $P'$ lying on $\overleftrightarrow{CP}$ but on the other side of $C$ from where $P$ is located; that is, $\underline{CP'} = r\underline{CP}$ . Thus such a transformation is the product (in either order) of a stretch about $C$ and a halfturn about the centre. This motivates the following definition.

**7.1.2** Definition. A **dilation about point** $C$ is a stretch about $C$ or else a stretch about $C$ followed by a halfturn about $C$.

Other transformations can be obtained by composing a stretch with any other transformation (e.g. an isometry). Two such special combinations will be given a name.

**7.1.3** Definition. A **stretch reflection** is a nonidentity stretch about some point $C$ followed by the reflection in some line through $C$.

**7.1.4** Definition. A **stretch rotation** is a nonidentity stretch about some point $C$ followed by a nonidentity rotation about $C$.

Any of the above transformations are *shape-preserving : they increase or decrease all lengths in the same ratio but leave shapes unchanged.* We make the following definition.

**7.1.5** Definition. If $r > 0$, then a **similarity** (or **similitude**) **of ratio** $r$ is a transformation $\alpha$ such that

$$P'Q' = rPQ \quad \text{for all points } P \text{ and } Q\,, \quad \text{where } P' = \alpha(P) \text{ and } Q' = \alpha(Q)\,.$$

Since a similarity is a transformation that multiplies all distances by some positive number, then the image of a triangle under a similarity is a triangle.

**Exercise 107** Show that collinear points are mapped onto collinear points by a similarity transformation.

Thus *a similarity is a collineation.* The following proposition is easy to prove (and we shall leave it as an exercise).

**7.1.6** Proposition. *The following results hold.*

*(a) An isometry is a similarity.*

*(b) A similarity with two fixed points is an isometry.*

*(c)* *A similarity with three noncollinear fixed points is the identity.*

*(d)* *A similarity is a collineation that preserves betweenness, midpoints, segments, rays, triangles, angles, angle measure, and perpendicularity.*

*(e)* *The product of a similarity of ratio* $r$ *and a similarity of ratio* $s$ *is a similarity of ratio* $rs$.

*(f)* *The similarities form a group* $\mathfrak{Sim}$ *that contains the group of* $\mathfrak{Isom}$ *of all isometries.*

**Exercise 108** Prove the preceding proposition.

**7.1.7** Proposition. *If* $\triangle ABC \sim \triangle A'B'C'$*, then there is a unique similarity* $\alpha$ *such that*

$$\alpha(A) = A', \quad \alpha(B) = B', \quad \textit{and} \quad \alpha(C) = C'.$$

Proof : Suppose $\triangle ABC \sim \triangle A'B'C'$. Let $\delta$ be the stretch about $A$ such that $\delta(B) = E$ with $AE = A'B'$. With $F = \delta(C)$, then $\triangle AEF \cong \triangle A'B'C'$ by $ASA$. Since there is an isometry $\beta$ such that $\beta(A) = A'$, $\beta(E) = B'$, and $\beta(F) = C'$, then $\beta\delta$ is a similarity taking $A, B, C$ to $A', B', C'$, respectively. If $\alpha$ is a similarity taking $A, B, C$ to $A', B', C'$, respectively, then $\alpha^{-1}(\beta\delta)$ fixes three noncollinear points and must be the identity. Therefore, $\alpha = \beta\delta$.

$\square$

Note : Generalizing from triangles to arbitrary sets of points, we say that (the sets of points) $\mathbf{S}_1$ and $\mathbf{S}_2$ are **similar** provided there is a similarity $\alpha$ such that $\alpha(\mathbf{S}_1) = \mathbf{S}_2$.

### What are the dilatations ?

Recall that a dilatation is a collineation $\alpha$ such that $\mathcal{L} \parallel \alpha(\mathcal{L})$ for every line $\mathcal{L}$ and that the group $\mathfrak{H}$ generated by the halfturns is contained in the group $\mathfrak{D}$ of all dilatations.

**7.1.8** PROPOSITION. *A dilation is a dilatation and a similarity.*

PROOF : Let $\alpha$ be a dilation. First suppose $\alpha$ is a stretch of ratio $r$ about point $C$. Transformation $\alpha$ fixes the lines through $C$. Suppose $P, Q, R$ are three collinear points on a line off $C$ and have images $P', Q', R'$, respectively, under $\alpha$. Since

$$CP' = rCP\,, \quad CQ' = rCQ\,, \quad \text{and} \quad CR' = rCR$$

it follows (from the theory of similar triangles) that $\overleftrightarrow{P'Q'} \parallel \overleftrightarrow{PQ}$, that points $P', Q', R'$ are collinear, and that $P'Q' = rPQ$. Hence, a stretch is a dilatation and a similarity. Since a halfturn is a dilatation and a similarity, then the product of a stretch and a halfturn is both a dilatation and a similarity. $\square$

**7.1.9** PROPOSITION. *If $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$, then there is a unique dilatation $\delta$ such that*

$$\delta(A) = A' \quad \textit{and} \quad \delta(B) = B'.$$

PROOF : Suppose $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$ and there is a dilatation $\delta$ such that $\delta(A) = A'$ and $\delta(B) = B'$. If point $P$ is off $\overleftrightarrow{AB}$, then $\delta(P)$ is uniquely determined as the intersection of the line through $A'$ that is parallel to $\overleftrightarrow{AP}$ and the line through $B'$ that is parallel to $\overleftrightarrow{BP}$. Then, if $Q$ is on $\overleftrightarrow{AB}$, point $\delta(Q)$ is uniquely determined as the intersection of $\overleftrightarrow{A'B'}$ and the line through $\delta(P)$ that is parallel to $\overleftrightarrow{PQ}$. Since the image of each point is uniquely determined by the images of $A$ and $B$, then there is at most one dilatation $\delta$ taking $A$ to $A'$ and $B$ to $B'$. On the other hand, $\tau_{A,A'}$ followed by the dilation about $A'$ that takes $\tau_{A,A'}(B)$ to $B'$ is a dilatation taking $A$ to $A'$ and $B$ to $B'$. $\square$

**7.1.10** PROPOSITION. *If point $A$ is not fixed by dilatation $\delta$, then line $\overleftrightarrow{AA'}$ is fixed by $\delta$, where $A' = \delta(A)$.*

PROOF : If dilatation $\delta$ does not fix point $A$ and if $A' = \delta(A)$, then $\delta(\overleftrightarrow{AA'})$ must be the line through $\delta(A)$ that is parallel to $\overleftrightarrow{AA'}$. $\square$

We can now answer the question "What are the dilatations ?"

**7.1.11** Proposition. *A dilatation is a translation or a dilation.*

Proof : A nonidentity dilatation $\alpha$ must have some nonfixed line $\mathcal{L}$. So $\mathcal{L}$ and $\alpha(\mathcal{L})$ are distinct parallel lines. Any two points $A$ and $B$ on line $\mathcal{L}$ are such that neither $\alpha(A)$ nor $\alpha(B)$ is on $\mathcal{L}$. Let $A' = \alpha(A)$ and $B' = \alpha(B)$. Now $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are distinct parallel lines. If $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$, then $\square AA'B'B$ is a parallelogram, $\tau_{A,A'}(B) = B'$, and (Proposition 7.1.9) dilatation $\alpha$ must be the translation $\tau_{A,A'}$. However, suppose $\overleftrightarrow{AA'} \nparallel \overleftrightarrow{BB'}$. Then the lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ are fixed (Proposition 7.1.10) and must intersect at some fixed point $C$. Since $\overleftrightarrow{AB}$ is not fixed, then $C$ is off both parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ with $C, A', A$ collinear and $C, B', B$ collinear. So $CA'/CA = CB'/CB$. Then there is a dilation $\delta$ about $C$ such that $\delta(A) = A'$ and $\delta(B) = B'$. (If point $C$ is between points $A$ and $A'$, then $\delta$ is a stretch followed by $\sigma_C$ ; otherwise, $\delta$ is simply a stretch about $C$.) By the uniqueness of a dilatation taking $A$ to $A'$ and $B$ to $B'$, the dilatation $\alpha$ must be the translation $\tau_{A,A'}$ or else the dilation $\delta$. $\square$

### The classification theorem

**7.1.12** Proposition. *If $\alpha$ is a similarity and $P$ is any point, then $\alpha = \beta\delta$, where $\delta$ is a stretch about $P$ and $\beta$ is an isometry.*

Proof : A similarity is just a stretch about some point $P$ followed by an isometry. Actually, the point $P$ can be arbitrarily chosen as follows. If $\alpha$ is a similarity of ratio $r$, let $\delta$ be the stretch of ratio $r$ about $P$. Then $\delta^{-1}$ is a stretch of ratio $1/r$. So $\alpha\delta^{-1}$ is an isometry and $\alpha = (\alpha\delta^{-1})\delta$. $\square$

This important result gives us a feeling for the nature of the similarities. We need only one more result on similarities before the Classification Theorem. However, the proof uses a lemma about *directed distance*.

We suppose the lines in the plane are *directed* (in an arbitrary fashion) and $\underline{AB}$ denotes the **directed distance** from $A$ to $B$ on line $\overleftrightarrow{AB}$. For any points $A$ and $B$, we have

$$\underline{AB} = -\underline{BA} \quad \text{and so} \quad \underline{AA} = 0\,.$$

NOTE : For distinct points $A, B, C$ on line $\mathcal{L}$ the number $\underline{AC}/\underline{CB}$ is independent of the choice of positive direction on line $\mathcal{L}$, as changing the positive direction would change the sign of both numerator and denominator and leave the value of the fraction itself unchanged.

**Exercise 109** Show that the function

$$f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{-1\}, \quad x \mapsto \frac{x}{1-x}$$

is a *bijection.*

**7.1.13** LEMMA. *Given the line* $\overleftrightarrow{AB}$, *the function*

$$\widehat{f} : \overleftrightarrow{AB} \setminus \{B\} \to \mathbb{R} \setminus \{-1\}, \quad X \mapsto \underline{AX}/\underline{XB}$$

*is a bijection.*

PROOF : There is a one-to-one correspondence between points $X \in \overleftrightarrow{AB}$ and real numbers $x \in \mathbb{R}$ given by (the equation)

$$\underline{AX} = x\,\underline{AB}.$$

Hence we can identify any point $X \in \overleftrightarrow{AB}$, different from $B$, with its *intrinsic coordinate* $x \in \mathbb{R} \setminus \{1\}$. Then

$$\underline{XB} = \underline{XA} + \underline{AB} = (1-x)\,\underline{AB}$$

and so

$$\begin{aligned} \underline{AX}/\underline{XB} &= \frac{x}{1-x} \\ &= f(x). \end{aligned}$$

It follows that the function

$$X \mapsto \underline{AX}/\underline{XB}$$

is a bijection. $\square$

**7.1.14** Corollary. *If point $P \in \overleftrightarrow{AB}$, different from $B$, then*

$$\underline{AP}/\underline{PB} \neq -1.$$

**7.1.15** Corollary. *If $t \neq -1$, then there exists a unique point $P \in \overleftrightarrow{AB}$, different from $B$, such that*

$$\underline{AP}/\underline{PB} = t.$$

**7.1.16** Corollary. *Point $P \in \overleftrightarrow{AB}$ is between $A$ and $B$ if and only if $\underline{AP}/\underline{PB}$ is positive.*

**7.1.17** Proposition. *A similarity without a fixed point is an isometry.*

Proof : The lemma above will now be used to prove that *a similarity that is not an isometry must have a fixed point.* Suppose $\alpha$ is a similarity that is not an isometry. We may suppose $\alpha$ is not a dilatation. (Why ?) So there is a line $\mathcal{L}$ such that $\mathcal{L}' \not\parallel \mathcal{L}$ where $\mathcal{L}' = \alpha(\mathcal{L})$. Let $\mathcal{L}$ intersect $\mathcal{L}'$ at point $A$. With $A' = \alpha(A)$, then $A'$ is on $\mathcal{L}'$. We suppose $A' \neq A$. Let $\mathcal{M}$ be the line through $A'$ that is parallel to $\mathcal{L}$. With $\mathcal{M}' = \alpha(\mathcal{M})$, then $\mathcal{M}' \parallel \mathcal{L}'$. Let $\mathcal{M}'$ intersect $\mathcal{M}$ at point $B$. With $B' = \alpha(B)$, then $B'$ is on $\mathcal{M}'$ and distinct from $A'$. We suppose $B' \neq B$. So

$$\mathcal{L}' = \overleftrightarrow{AA'}, \quad \mathcal{M}' = \overleftrightarrow{BB'}, \quad \text{and} \quad \overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}.$$

Now $\overleftrightarrow{AB} \not\parallel \overleftrightarrow{A'B'}$ as otherwise $A'B' = AB$ and $\alpha$ is an isometry. So $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect at some point $P$ off both parallel lines $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ with $P, A, B$ collinear and $P, A', B'$ collinear. So $AP/PB = A'P/PB'$. If $\alpha$ has ratio $r$ and $P' = \alpha(P)$, then

$$AP/PB = rAP/rPB = A'P'/P'B'.$$

Hence, $A'P/PB' = A'P'/P'B'$. Point $P$ is between $A'$ and $B'$ if and only if $P$ is between $A$ and $B$ since $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$, but $P$ is between $A$ and $B$ if and only if $P'$ is between $A'$ and $B'$. Hence, $P$ is between $A'$ and $B'$ if and only if $P'$ is between $A'$ and $B'$. Therefore, by LEMMA 7.1.13 (and its corollaries),

$$\underline{A'P}/\underline{PB'} = \underline{A'P'}/\underline{P'B'} \quad \text{and} \quad P = P'.$$

So $\alpha(P) = P$, as desired. □

**7.1.18** THEOREM. (THE CLASSIFICATION THEOREM FOR PLANE SIMILARITIES) *Each nonidentity similarity is exactly one of the following : isometry, stretch, stretch rotation or a stretch reflection.*

PROOF : In order to classify the similarities, suppose $\alpha$ is a similarity that is not an isometry. Then $\alpha$ has some fixed point $C$. So $\alpha = \beta\delta$ where $\delta$ is a stretch about $C$ and where $\beta$ is an isometry. Since $\beta(C) = \alpha\delta^{-1}(C) = C$, then $\beta$ must be one of the identity $\iota$, a rotation $\rho$ about $C$, or a reflection $\sigma_{\mathcal{C}}$ with $C$ on $\mathcal{C}$. Hence, $\alpha$ is one of $\delta$, $\rho\delta$, or $\sigma_{\mathcal{C}}\delta$. We have proved the major part of the result. There remains only the task of verifying the "exactly" in the statement of the classification theorem; this is left as an exercise. □

**Exercise 110** Finish the proof of the Classification Theorem (for similarities).

## 7.2 Equations for Similarities

The following technical result is easy to prove.

**7.2.1** PROPOSITION. *Suppose* $\alpha \in \mathfrak{Sim}$. *Then*

(a) $\alpha\gamma\alpha^{-1} \in \mathfrak{Isom}$ *if* $\gamma \in \mathfrak{Isom}$.

(b) $\alpha\delta\alpha^{-1} \in \mathfrak{D}$ *if* $\delta \in \mathfrak{D}$.

(c) $\alpha\eta\alpha^{-1} \in \mathfrak{H}$ *if* $\eta \in \mathfrak{H}$.

(d) $\alpha\tau\alpha^{-1} \in \mathfrak{T}$ *if* $\tau \in \mathfrak{T}$.

(e) $\alpha\sigma_P\alpha^{-1} = \sigma_{\alpha(P)}$.

(f) $\alpha\sigma_{\mathcal{L}}\alpha^{-1} = \sigma_{\alpha(\mathcal{L})}$.

**Exercise 111** Prove the preceding proposition.

In order to look at the dilatations a little more closely, a notation for the dilation is introduced as follows. If $a > 0$, then $\delta_{P,a}$ is the stretch about $P$ of (magnification) ratio $a$ and dilation $\delta_{P,-a}$ is defined by

$$\delta_{P,-a} := \sigma_P \delta_{P,a}.$$

Multiplying both sides of this last equation by $\sigma_P$ on the left, we have $\sigma_P \delta_{P,-a} = \delta_{P,a}$. So

$$\delta_{P,-r} = \sigma_P \delta_{P,r}\,, \quad r \neq 0.$$

The number $r$ is called the **dilation ratio** of dilation $\delta_{P,r}$. There are two special cases where a dilation is also an isometry :

$$\delta_{P,1} = \iota \quad \text{and} \quad \delta_{P,-1} = \sigma_P.$$

Clearly, the ratio of $\delta_{P,r}$ is the absolute value $|r|$ of the dilation ratio $r$. For example, $\delta_{P,-3}$ has ratio $+3$ but dilation ratio $-3$.

**7.2.2** Proposition. *If $P$ is a point, then*

$$\delta_{P,-r} = \sigma_P \delta_{P,r}\,, \quad \delta_{P,1} = \iota\,, \quad \delta_{P,-1} = \sigma_P\,, \quad \delta_{P,s}\delta_{P,r} = \delta_{P,rs} \;\; (r, s \neq 0).$$

*If $\delta_{P,r}$ is a dilation and $\alpha$ is a similarity, then*

$$\alpha \delta_{P,r} \alpha^{-1} = \delta_{\alpha(P),r}.$$

Proof : From the special case

$$\sigma_P = \delta_{P,r} \sigma_P \delta_{P,r}^{-1} \quad \text{(see Proposition 7.2.1)}$$

it follows

$$\sigma_P \delta_{P,r} = \delta_{P,r} \sigma_P$$

and then

$$\delta_{P,s}\delta_{P,r} = \delta_{P,rs} \quad (r, s \neq 0).$$

Thus,

$$\delta_{P,r}^{-1} = \delta_{P,1/r}\,, \quad r \neq 0.$$

If $\alpha$ is any similarity, then $\alpha\delta_{P,r}\alpha^{-1}$ is a dilatation (PROPOSITION 7.2.1) fixing point $\alpha(P)$ and has ratio $|r|$. Hence,

$$\alpha\delta_{P,r}\alpha^{-1} = \delta_{\alpha(P),s}\,, \quad \text{where } s = \pm r.$$

The question is "Is $r$ the dilation ratio of $\alpha\delta_{P,r}\alpha^{-1}$ ?" With $P' = \alpha(P)$ and $Q' = \alpha(Q)$ for $Q \neq P$, that the answer is "Yes" follows from the equivalence of each of the following :

(1) $r > 0$.

(2) $\delta_{P,r}$ is a stretch.

(3) $\delta_{P,r}(Q)$ is on $PQ^{\rightarrow}$.

(4) $\alpha\delta_{P,r}(Q)$ is on $P'Q'^{\rightarrow}$.

(5) $\alpha\delta_{P,r}\alpha^{-1}(\alpha(Q))$ is on $P'Q'^{\rightarrow}$.

(6) $\delta_{P,s}(Q')$ is on $P'Q'^{\rightarrow}$.

(7) $\delta_{P,s}$ is a stretch.

(8) $s > 0$.

Since $s = \pm r$ and both $r$ and $s$ have both the same sign, then $r = s$, as desired. $\square$

NOTE : If $r \neq 1$, then the nonidentity dilation $\delta_{P,r}$ is said to have **centre** $P$.

### Further results

Further results on the dilatations are more easily obtained by using coordinates.

**7.2.3** Proposition. *If* $P = (u, v)$*, then (the dilation)* $\delta_{P,r}$ *has equations*

$$\begin{cases} x' = rx + (1-r)u \\ \\ y' = ry + (1-r)v. \end{cases}$$

Proof : Let $O = (0,0)$ be the origin of the plane. We clearly have

$$\delta_{O,r}((x,y)) = (rx, ry)\,, \quad r > 0$$

and this same equation must hold for negative $r$ since $\sigma_O((x,y)) = (-x,-y)$. So $\delta_{O,r}$ has equations

$$\begin{cases} x' = rx \\ \\ y' = ry \end{cases}$$

in any case. Now, suppose $P = (u,v)$ and $\delta_{P,r}((x,y)) = (x', y')$. Then, from the equations

$$\delta_{P,r} = \tau_{O,P}\delta_{O,r}\tau_{O,P}^{-1} = \tau_{O,P}\delta_{O,r}\tau_{P,O}$$

we have

$$\delta_{P,r}((x,y)) = (r(x-u)+u, r(y-v)+v) = (x', y').$$

Indeed,

$$(x,y) \mapsto (x-u, y-v) \mapsto (r(x-u), r(y-v)) \mapsto (r(x-u)+u, r(y-v)+v).$$

Hence $\delta_{P,r}$ has equations

$$\begin{cases} x' = rx + (1-r)u \\ \\ y' = ry + (1-r)v. \end{cases}$$

$\square$

This simple result has some interesting corollaries.

**7.2.4** COROLLARY. *Given $\delta_{A,1/r}$ and $\delta_{B,r}$, then for some point $C$*

$$\delta_{B,r}\delta_{A,1/r} = \tau_{A,C}.$$

**7.2.5** COROLLARY. *Given $\delta_{A,r}$ and $\delta_{B,s}$ with $rs \neq 1$, then for some point $C$*

$$\delta_{B,s}\delta_{A,r} = \delta_{C,rs}.$$

**7.2.6** COROLLARY. *Given $\tau_{A,B}$ and $\delta_{A,r}$ with $r \neq 1$, then for some point $C$*

$$\tau_{A,B}\delta_{A,r} = \delta_{B,r}\tau_{A,B} = \delta_{C,r}.$$

**Exercise 112** In each case, work out an explicit expression for the point $C$ (in terms of $A, B, r$, and $s$, as may be the case).

NOTE : Although the coordinate proofs for the corollaries above are easy to give and the content of the equations themselves is easy to understand, the visualization is very hard, if not, in some sense, virtually impossible.

**7.2.7** PROPOSITION. *A similarity (of ratio $r$) has equations of the form*

$$\begin{cases} x' = ax - by + h \\ \\ y' = \pm(bx + ay) + k \end{cases} \qquad \textit{with} \quad r^2 = a^2 + b^2 \neq 0$$

*and, conversely, equations of this form are those of a similarity.*

PROOF : A similarity is a stretch about the origin $O$ followed by an isometry (PROPOSITION 7.1.12). From this fact and the equations for an isometry given by PROPOSITION 5.3.3, it follows that a similarity has equations of the form

$$\begin{cases} x' = (r\cos q)x - (r\sin q)y + h \\ \\ y' = \pm((r\sin q)x + (r\cos q)y) + k \end{cases}$$

and, conversely, equations of this form are those of a similarity. With

$$a = r\cos q \quad \text{and} \quad b = r\sin q$$

we get the desired result. □

**7.2.8** Definition. A similarity $\alpha$ that is a stretch about some point $P$ followed by an even isometry is said to be **direct**.

**7.2.9** Definition. A similarity $\alpha$ that is a stretch about some point $P$ followed by an odd isometry is said to be **opposite**.

From the equations for isometries and similarities it is evident that whether a similarity is direct or opposite is independent of the point $P$ above.

Note : In the equations in Proposition 7.2.7, the positive sign applies to direct similarities and the negative sign applies to opposite similarities.

We have

**7.2.10** Proposition. *Every similarity is either direct or opposite, but not both. The direct similarities form a group. The product of two opposite similarities is direct. The product of a direct similarity and an opposite similarity is an opposite similarity.*

## 7.3 Exercises

**Exercise 113** For what point $P$ does a dilation about $P$ have equations

$$\begin{cases} x' = -2x + 3 \\ \\ y' = -2y - 4 \text{ ?} \end{cases}$$

**Exercise 114** What are the fixed points and fixed lines of a stretch reflection ? What are the fixed points and fixed lines of a stretch rotation ?

**Exercise 115** TRUE or FALSE ?

(a) A similarity that is not an isometry has a fixed point, and a dilatation that is not a translation has a fixed point.

(b) The group of all dilatations is *generated* by the dilations.

(c) $\sigma_P\delta_{P,r} = \delta_{P,r}\sigma_P$ for any point $P$ and nonzero number $r$.

(d) $\delta_{A,r}(B)$ is on $AB^{\rightarrow}$ if $A \neq B$.

(e) There is a unique point $Q$ on $\overleftrightarrow{AB}$ such that $AQ/QB = 7$.

(f) $\alpha\tau_{A,B}\alpha^{-1} = \tau_{\alpha(A),\alpha(B)}$ for any similarity $\alpha$ and points $A$ and $B$.

(g) A dilatation is a similarity.

**Exercise 116** PROVE or DISPROVE : If $\alpha$ is a transformation and $\delta$ is a dilation, then $\alpha\delta\alpha^{-1}$ is a dilatation.

**Exercise 117** PROVE or DISPROVE : If $r > 0$, then a mapping $\alpha$ such that $P'Q' = rPQ$ for all points $P$ and $Q$ with $P' = \alpha(P)$ and $Q' = \alpha(Q)$ is a similarity.

**Exercise 118** Complete each of the following :

(a) If $\delta_{P,3}((x,y)) = (3x+7, 3y-5)$, then $P = \ldots$

(b) If $x' = 3x + 5y + 2$ and $y' = tx - 3y$ are the equations of a similarity, then $t = \ldots$

(c) If $\sigma_P\delta_{P,15} = \delta_{P,x}$, then $x = \ldots$

(d) If $\delta_{C,r}\tau_{A,B} = \tau_{P,Q}\delta_{C,r}$, then $P = \ldots$ and $Q = \ldots$

(e) If $\delta_{B,s}\delta_{A,t} = \delta_{T,t}\delta_{B,s}$, then $T = \ldots$

(f) If $\rho_{A,r}\delta_{A,s} = \delta_{A,s}\rho_{A,x}$, then $x = \ldots$

(g) If $\tau^{-1}_{A,B} = \tau_{A,C}$, then $C = \ldots$

**Exercise 119** PROVE or DISPROVE : Nonidentity dilatations $\alpha$ and $\beta$ commute if and only if $\alpha$ and $\beta$ are translations.

**Exercise 120** If $\alpha((1,2)) = (0,0)$ and $\alpha((3,4)) = (3,4)$, then what is the ratio of similarity $\alpha$ ?

**Exercise 121** If $\alpha((0,0)) = (1,0)$, $\alpha((1,0)) = (2,2)$, and $\alpha((2,2)) = (-1,6)$ for similarity $\alpha$, then find $\alpha((-1,6))$.

**Exercise 122** Show that an involutory similarity is a reflection or a halfturn.

**Exercise 123** PROVE or DISPROVE : There are exactly two dilatations taking circle $A_B$ to circle $C_D$.

**Exercise 124** Show that a nonidentity dilation with centre $P$ commutes with $\sigma_{\mathcal{L}}$ if and only if $P$ is on $\mathcal{L}$.

**Exercise 125** Show that

(a) nonidentity dilations $\delta_{A,a}$ and $\delta_{B,b}$ commute if and only if $A = B$.

(b) dilatations $\delta_{A,a}$ and $\tau_{A,B}$ never commute if $A \neq B$ and $a \neq 1$.

DISCUSSION : The Euclidean plane admits transformations (called *similarities*) which multiply all distances by a constant factor $r \neq 0$. The typical similarity is (the *stretch)* $(x, y) \mapsto (rx, ry)$. Figures related by a similarity are said to be "of the same shape" or "similar". In particular, all triangles with the same angles are similar, as are all squares. The existence of squares of different sizes means, for instance, that $n^2$ unit squares fill a square of side $n$. This leads to the property of Euclidean area that multiplying the lengths in a figure by $r$ multiplies its area by $r^2$.

The Euclidean plane is unique in having this simple dependence of area on length because the sphere and the hyperbolic plane *do not* admit similarities (except with $r = 1$). There the relationships between length and area are more complicated – involving circular and hyperbolic functions, respectively – but the relationship between angles and area is delightfully simple.

This is a benefit of having the triangle's angle sum unequal to $\pi$. One then has a nontrivial angular excess function for triangles $\triangle$ :

$$excess\,(\triangle) = \text{angle sum}\,(\triangle) - \pi,$$

which is proportional to area because it is *additive.* That is, if triangle $\triangle$ splits into triangles $\triangle_1$ and $\triangle_2$, then

$$excess\,(\triangle) = excess\,(\triangle_1) + excess\,(\triangle_2)\,.$$

Euclidean geometry misses out this property because it is too simple – its angular excess function is zero. It can be shown that any continuous, nonzero, additive function must be proportional to area.

The identification of angular excesses with area clearly shows why similar triangles of different sizes cannot exist in the sphere and hyperbolic plane. Triangles with the same angles have the same area, and hence one cannot be larger than the other.

*A branch of mathematics is called* geometry*, because the name seems good on emotional and traditional grounds to a sufficiently large number of competent people.*

O. Veblen and J.H.C. Whitehead