Chapter 8

Affine Transformations

Topics :

1. Collineations

2. Affine Linear Transformations



Copyright © Claudiu C. Remsing, 2014. All rights reserved.

8.1 Collineations

We now turn to transformations that were first introduced by LEONHARD EULER (1707-1783).

Affine transformations (as collineations)

8.1.1 DEFINITION. An affine transformation (or affinity) is a collineation that preserves parallelness among lines.

So, if \mathcal{L} and \mathcal{M} are parallel lines and α is an affine transformation, then lines $\alpha(\mathcal{L})$ and $\alpha(\mathcal{M})$ are parallel. It is easy to prove the following result.

8.1.2 PROPOSITION. A collineation is an affine transformation and, conversely, an affine transformation is a collineation.

PROOF : An affine transformation is by definition a collineation. If β is any collineation and \mathcal{L} and \mathcal{M} are distinct parallel lines, then $\beta(\mathcal{L})$ and $\beta(\mathcal{M})$ cannot contain a common point $\beta(P)$, as point P would then have to be on both \mathcal{L} and \mathcal{M} . Therefore, every collineation is an affine transformation. \Box

NOTE : Affine transformations and collineations are *exactly* the same thing for the Euclidean plane. The choice between the terms *affine transformation* and *collineation* is sometimes arbitrary and sometimes indicates a choice of emphasis on parallelness of lines or on collinearity of points. Loosely speaking, *affine geometry* is what remains after surrendering the ability to measure length (isometries) and surrendering the ability to measure angles (similarities), but maintaining the incidence structure of lines and points (collineations).

8.1.3 EXAMPLE. Similarities preserve parallelness and hence are affine transformations. In particular, isometries are also affine transformations.

8.1.4 EXAMPLE. The mapping

$$\alpha : \mathbb{E}^2 \to \mathbb{E}^2, \quad (x, y) \mapsto (2x, y)$$

is an affine transformation that is not a similarity.

NOTE : The word *symmetry* brings to mind such general ideas as balance, agreement, order, and harmony. We have been exceedingly conservative in our use of the word *symmetry*; for us, symmetries are restricted to isometries. With a broader mathematical usage of the term, we would certainly be saying that the similarities are the symmetries of similarity geometry and that the collineations are the symmetries of affine geometry. In the most broad usage, the group of all transformations on a structure that preserves the essence of that structure constitutes the symmetries (also called the automorphisms) of the structure.

A collineation preserves collinearity of points. We wish to show that, conversely, a transformation such that the image of every three collinear points are themselves collinear must be a collineation.

8.1.5 PROPOSITION. A transformation such that the images of every three collinear points are themselves collinear is an affine transformation.

PROOF : We suppose α is a transformation that preserves collinearity and aim to show $\alpha(\mathcal{L})$ is a line whenever \mathcal{L} is a line. Let A and B be distinct points on \mathcal{L} , and let \mathcal{M} be the line through $\alpha(A)$ and $\alpha(B)$. By the definition of α , all the points of $\alpha(\mathcal{L})$ are on \mathcal{M} . However, are all the points of \mathcal{M} on $\alpha(\mathcal{L})$? Suppose C' is a point on \mathcal{M} distinct from $\alpha(A)$ and $\alpha(B)$, and let C be the point such that $\alpha(C) = C'$. To show C must be on \mathcal{L} , we assume C is off \mathcal{L} and then obtain a contradiction. Now the image of all the points of \overrightarrow{AB} , \overrightarrow{BC} , and \overrightarrow{AC} are on \mathcal{M} since collinearity is preserved under α . However, any point P in the plane is on a line containing two distinct points of $\triangle ABC$. Since the images of these two points lie on \mathcal{M} , then the image of P lies on \mathcal{M} . Therefore, the image of every point lies on \mathcal{M} , contradicting the fact that α is an onto mapping. Hence, C must lie on \mathcal{L} , $\mathcal{M} = \alpha(\mathcal{L})$, and α is a collineation, as desired.

Are the affine transformations the same as those transformations for which the images of any three noncollinear points are themselves noncollinear? The answer is "Yes". **8.1.6** PROPOSITION. A transformation is an affine transformation if and only if the images of any three noncollinear points are themselves noncollinear.

PROOF : Suppose α is an affine transformation. Then α^{-1} is an affine transformation and can't take three noncollinear points to three collinear points. Therefore, affine transformation α must take any three noncollinear points to three noncollinear points.

Conversely, suppose β is a transformation such that the images of any three noncollinear points are themselves noncollinear. Assume β is *not* an affine transformation. Then β^{-1} is not an affine transformation. By the contrapositive of the preceding result, then there are three collinear points whose images under β^{-1} are not collinear. Hence, since β is the inverse of β^{-1} , then there are three noncollinear points whose images under β are collinear, contradiction. Therefore, β is an affine transformation.

An affine transformation preserves betweenness

The result above does *not* state that the image of a triangle under an affine transformation is necessarily a triangle, but states only that the images of the vertices of a triangle are themselves vertices of a triangle. We do not know the image of a segment is necessarily a segment. More fundamental, we do not know that an affine transformation necessarily preserves betweenness. It will take some effort to prove this. We begin by showing that *midpoint is actually an affine concept*; that is, an affine transformation carries the midpoint of two given points to the midpoint of their images.

8.1.7 PROPOSITION. If α is an affine transformation and M is the midpoint of points A and B, then $\alpha(M)$ is the midpoint of $\alpha(A)$ and $\alpha(B)$.

PROOF: Suppose A and B are distinct points and α is an affine transformation. Let P be any point off \overrightarrow{AB} . Let Q be the intersection of the line through A that is parallel to \overrightarrow{PB} and the line through B that is parallel to \overrightarrow{PA} . So $\Box APBQ$ is a parallelogram. Let $A' = \alpha(A), B' = \alpha(B), P' = \alpha(P),$ and $Q' = \alpha(Q)$. Since two parallel lines go to two parallel lines under α , then $\Box A'P'B'Q'$ is a parallelogram. (We are *not* claiming that $\alpha(\Box APBQ) =$ $\Box A'P'B'Q'$ but only that A', P', B', Q' are vertices in order of a parallelogram.) Further, M, the intersection of \overrightarrow{AB} and \overrightarrow{PQ} , must go to M', the intersection of $\overrightarrow{A'B'}$ and $\overrightarrow{P'Q'}$. However, since the diagonals of a parallelogram bisect each other, then M is the midpoint of A and B while M' is the midpoint of A' and B'. Hence, α preserves midpoints. \Box

8.1.8 PROPOSITION. If α is an affine transformation, the n + 1 points P_0, P_1, P_2 ,

..., P_n divide the segment $\overline{P_0P_n}$ into *n* congruent segments $\overline{P_{i-1}P_i}$, and $P'_i = \alpha(P_i)$, then the n+1 points $P'_0, P'_1, P'_2, \ldots, P'_n$ divide the segment $\overline{P'_0P'_n}$ into *n* congruent segments $\overline{P'_{i-1}P'_i}$.

PROOF: Suppose α is an affine transformation and the n+1 points $P_0, P_1, P_2, \ldots, P_n$ divide the segment $\overline{P_0P_n}$ into n congruent segments $\overline{P_{i-1}P_i}$. Let $P'_i = \alpha(P_i)$. Since $P_0P_1 = P_1P_2, P_1P_2 = P_2P_3, \ldots$, then P_1 is the midpoint of P_0 and P_2 , point P_2 is the midpoint of P_1 and P_3 , etc. Hence, P'_1 is the midpoint of P'_0 and P'_2 , point P'_2 is the midpoint of P'_1 and P'_3 , etc. So the images $P'_0, P'_1, P'_2, \ldots, P'_n$ divide the segment $\overline{P'_0P'_n}$ into n congruent segments $\overline{P'_{i-1}P'_i}$.

It follows from this last result that P between A and B implies $\alpha(P)$ between $\alpha(A)$ and $\alpha(B)$ provided that <u>AP/PB</u> is rational.

NOTE : It would have to be a very strange collineation that allowed betweenness not to be preserved in general although preserving midpoints. Early geometers avoided such a "monster transformation" simply by incorporating the preservation of betweenness within the definition of an affine transformation. In 1880 GASTON DAR-BOUX (1842-1917) showed that the "monster transformation" does not exist. Thus the following result holds (but the proof wil be omitted).

8.1.9 THEOREM. If α is an affine transformation and point P is between points A and B, then point $\alpha(P)$ is between $\alpha(A)$ and $\alpha(B)$.

As an immediate consequence of THEOREM 8.1.9, we know that an affine transformation preserves all those geometric entities whose definition goes back only to the definition of betweenness. Thus, an affine transformation preserves segments, rays, triangles, quadrilaterals, halfplanes, interiors of triangles, etc. In particular, the following result holds :

8.1.10 PROPOSITION. If A', B', C' are the respective images of three noncollinear points A, B, C under affine transformation α , then

$$\alpha(\overline{AB}) = \overline{A'B'}$$
 and $\alpha(\triangle ABC) = \triangle A'B'C'$.

8.1.11 PROPOSITION. An affine transformation fixing two points on a line fixes that line pointwise.

PROOF : Suppose affine transformation α fixes two points A and B. Assume there is a point C on \overrightarrow{AB} such that $C' \neq C$ with $C' = \alpha(C)$. Without loss of generality, we may assume C is on AB^{\rightarrow} . As an intermediate step, we shall show C is between two fixed points A and D. Let $B_0 = B$ and define B_{i+1} so that B_i is the midpoint of A and B_{i+1} for $i = 0, 1, 2 \dots$ Since A and B_0 are given as fixed by α , then each of B_1, B_2, B_3, \dots in turn must be fixed by α since α preserves midpoints. Let $D = B_k$ where k is an integer such that

$$AB_k = 2^k AB > AC \,.$$

Then C lies between fixed points A and D. So \overline{AD} is then fixed and both C and C' lie in \overline{AD} . Now, let n be an integer large enough so that nCC' > AD. Let $P_0 = A$, $P_n = D$, and the n+1 points P_0, P_1, \ldots, P_n divide the segment \overline{AD} into n congruent segments $\overline{P_{i-1}P_i}$. Each of the points P_i is fixed by α by PROPOSITION 8.1.7. So each $\overline{AP_i}$ and $\overline{P_iD}$ is fixed by α . However, integer n was chosen large enough so that for some integer j point P_j is between C and C'. So C and C' are in different fixed segments $\overline{AP_j}$ and $\overline{P_jD}$, contradiction. Therefore, $\alpha(C) = C$ for all points on \overleftarrow{AB} , as desired. \Box **8.1.12** COROLLARY. An affine transformation fixing three noncollinear points must be the identity. Given $\triangle ABC$ and $\triangle DEF$, there is at most one affine transformation α such that $\alpha(A) = D$, $\alpha(B) = E$, and $\alpha(C) = F$.

NOTE : In the next section we shall see that there is also at least one affine transformation α as described in the corollary above. Thus an affine transformation is completely determined once the images of any three noncollinear points are known.

8.2 Affine Linear Transformations

We start by making an "ad hoc" definition.

8.2.1 DEFINITION. An affine linear transformation is any mapping

$$\alpha : \mathbb{E}^2 \to \mathbb{E}^2$$
, $(x, y) \mapsto (ax + by + h, cx + dy + k)$ where $ad - bc \neq 0$

The number ad - bc is called the **determinant** of α .

An affine linear transformation is actually a transformation since a given (x, y) obviously determines a unique (x', y') and, conversely, a given (x', y') determines a unique (x, y) precisely because the determinant is nonzero. As we might expect, affine linear transformations are related to affine transformations.

Exercise 126 If $P = (p_1, p_2)$, $Q = (q_1, q_2)$, and $R = (r_1, r_2)$ are vertices of a triangle, show that the area of $\triangle PQR$ is

$$\frac{1}{2}|(q_1-p_1)(r_2-p_2)-(q_2-p_2)(r_1-p_1)|.$$

(Hence the area of a triangle with vertices (0,0), (a,b), (c,d) is half the absolute value of ad - bc.)

8.2.2 PROPOSITION. An affine linear transformation is an affine transformation and, conversely, an affine transformation is an affine linear transformation.

PROOF: Let α be an affine linear transformation and suppose line \mathcal{L} has equation px + qy + r = 0. Since p and q are not both zero, then ap + cq and bp + dq are not both zero. So there is a line \mathcal{M} with equation

$$(ap+cq)x + (bp+dq)y + r + hp + kq = 0.$$

Line \mathcal{M} is introduced because each of the following implies the next, where $\alpha((x,y)) = (x',y')$:

- (1) (x', y') is on line \mathcal{L} .
- (2) px' + qy' + r = 0.
- (3) p(ax + by + h) + q(cx + dy + k) + r = 0.
- (4) (ap + cq)x + (bp + dq)y + r + hp + kq = 0.
- (5) (x, y) is on line \mathcal{M} .

We have shown that α^{-1} is a transformation that takes any line \mathcal{L} to some line \mathcal{M} . So α^{-1} is a collineation. Hence, α is itself a collineation.

Conversely, suppose α is an affine transformation. Let

 $\alpha((0,0)) = (p_1, p_2) = P, \quad \alpha((1,0)) = (q_1, q_2) = Q, \text{ and } \alpha((0,1)) = (r_1, r_2) = R.$

Since (0,0), (1,0), (0,1) are noncollinear, then P, Q, R are noncollinear. Hence the mapping β with equations

$$\begin{cases} x' = (q_1 - p_1)x + (r_1 - p_1)y + p_1 \\ \\ y' = (q_2 - p_2)x + (r_2 - p_2)y + p_2 \end{cases}$$

is an affine linear transformation, since the absolute value of its determinant is twice the area of $\triangle PQR$ and therefore nonzero (see **Exercise 126**). Further,

 $\beta((0,0)) = \alpha((0,0)), \quad \beta((1,0)) = \alpha((1,0)), \quad \text{and} \quad \beta((0,1)) = \alpha((0,1)).$

Therefore (COROLLARY 8.1.11), we have $\alpha = \beta$. So α is an affine linear transformation.

NOTE : Chosing the term *affine linear transformation* over its equivalents *collineation* and *affine transformation* can emphasize a coordinate viewpoint.

Given $\triangle ABC$ and $\triangle DEF$, we know that there is at most one affine transformation α such that $\alpha(A) = D$, $\alpha(B) = E$, and $\alpha(C) = F$. We can now show that there is at least one such transformation α .

8.2.3 PROPOSITION. Given $\triangle ABC$ and $\triangle DEF$, there is a unique affine transformation α such that

$$\alpha(A) = D, \quad \alpha(B) = E, \quad and \quad \alpha(C) = F.$$

PROOF : Given $\triangle ABC$ and $\triangle DEF$, we know (COROLLARY 8.1.12) there is at most one affine transformation α such that $\alpha(A) = D$, $\alpha(B) = E$ and $\alpha(C) = F$. We now show there is at least one such affine transformation α . From the preceding paragraph, we see how to find the equations for an affine linear transformation β_1 such that

$$\beta_1((0,0)) = A$$
, $\beta_1((1,0)) = B$, and $\beta_1((0,1)) = C$.

Repeating the process, we see there is an affine linear transformation β_2 such that

$$\beta_2((0,0)) = D, \quad \beta_2((1,0)) = E, \text{ and } \beta_2((0,1)) = F.$$

The transformation $\beta_2 \beta_1^{-1}$ is the desired affine transformation α that takes points A, B, C to points D, E, F, respectively.

Matrix representation

Let $\alpha : \mathbb{E}^2 \to \mathbb{E}^2$ be a transformation given by

$$(x,y)\mapsto (ax+by+h,cx+dy+k).$$

(α is an affine linear transformation.)

NOTE : Recall that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

Hence the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ defines a mapping $(x, y) \mapsto (ax + by, cx + dy)$. Indeed, we write the pair (x, y) as a column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ (in fact, we identify points with geometric vectors) and so we get

$$(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = (ax + by, cx + dy).$$

This mapping is *linear* (i.e. preserves the vector structure of \mathbb{E}^2) and is invertible if (and only if) the matrix is invertible.

When the coefficients h and k vanish, α is linear and hence admits a matrix representation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We say that the (invertible) matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ represents the (linear) transformation α . In order to extend this representation to the general case, of affine linear transformations, we need to accomodate translations.

Exercise 127

(a) Verify that
$$\begin{bmatrix}
1 & 0 & 0 \\
h & a & b \\
k & c & d
\end{bmatrix}
\begin{bmatrix}
1 \\
x \\
y
\end{bmatrix} =
\begin{bmatrix}
1 \\
ax + by + h \\
cx + dy + k
\end{bmatrix}.$$
(b) Show that the matrix
$$\begin{bmatrix}
1 & 0 & 0 \\
h & a & b \\
k & c & d
\end{bmatrix}$$
is invertible if and only if $ad - bc \neq 0$, and then find its inverse.

If we "redefine" the concept of point – and write the pair (x, y) as a column matrix $\begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$ (this identification is more than just a "clever" notation) – then

we have

$$(x,y) = \begin{bmatrix} 1\\x\\y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0\\h & a & b\\k & c & d \end{bmatrix} \begin{bmatrix} 1\\x\\y \end{bmatrix} = \begin{bmatrix} 1\\ax+by+h\\cx+dy+k \end{bmatrix} = (ax+by+h, cx+dy+k).$$

We see that the 3×3 matrix

$$[\alpha] = \begin{bmatrix} 1 & 0 & 0 \\ h & a & b \\ k & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & A \end{bmatrix}$$

(where $\mathbf{v} = \begin{bmatrix} h \\ k \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$) represents the transformation
 $\alpha : \mathbb{E}^2 \to \mathbb{E}^2$, $(x, y) \mapsto (ax + by + h, cx + dy + k)$.

Exercise 128 Use matrix representation to show that the set of all linear affine transformations forms a group. (This group consists of all collineations, and is usually denoted by \mathfrak{Aff} .)

8.2.4 EXAMPLE. The *identity transformation* ι is represented by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus $[\iota] = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}$.

8.2.5 EXAMPLE. Consider the point P = (h, k) and let $\mathbf{v} = \begin{bmatrix} h \\ k \end{bmatrix}$. The translation $\tau = \tau_{O,P}$ is represented by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ k & 0 & 1 \end{bmatrix}$. Thus

$$[\tau] = \begin{bmatrix} 1 & 0 \\ \mathbf{v} & I \end{bmatrix}$$

8.2.6 EXAMPLE. Again, consider the point P = (h, k). The halfturn $\sigma = \sigma_P$ is represented by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2h & -1 & 0 \\ 2k & 0 & -1 \end{bmatrix}$. Thus $[\sigma] = \begin{bmatrix} 1 & 0 \\ 2\mathbf{v} & -I \end{bmatrix}.$

Exercise 129 Let P = (h, k) be a point. Determine the matrix which represents the dilation $\delta_{P,r}$ (of ratio $r \neq 0$) and hence verify the relations :

- (a) $\delta_{P,-r} = \sigma_P \delta_{P,r}$.
- (b) $\delta_{P,1} = \iota$.
- (c) $\delta_{P,-1} = \sigma_P$.
- (d) $\delta_{P,s}\delta_{P,r} = \delta_{P,rs} \quad (r, s \neq 0).$

Strains and shears

Some specific, basic affine transformations are introduced next.

8.2.7 DEFINITION. For number $k \neq 0$, the affine transformation

$$\varepsilon_{\mathcal{X},k}: (x,y) \mapsto (x,ky)$$

is called a strain of ratio k about the x-axis.

8.2.8 DEFINITION. For number $k \neq 0$, the affine transformation

$$\varepsilon_{\mathcal{Y},k}: (x,y) \mapsto (kx,y)$$

is called a strain of ratio k about the y-axis.

For fixed k, the product of the two affine transformations above is the familiar dilation about the origin $(x, y) \mapsto (kx, ky)$. Thus

$$\varepsilon_{\mathcal{X},k}\varepsilon_{\mathcal{Y},k}=\delta_{O,k}.$$

NOTE : The concept of a strain of ratio k about a given line \mathcal{L} can be defined analogously. However, one can prove that any dilation is the product of two strains about perpendicular lines.

8.2.9 EXAMPLE. The strain with equations

$$\begin{cases} x' = 2x \\ y' = y \end{cases}$$

fixes the y-axis pointwise and stretches out the plane away from and perpendicular to the y-axis.

NOTE : As with similarity theory, the terminology here is *not* standardized. Each of the following words has been used for a strain or for a strain with positive ratio : enlargement, expansion, lengthening, stretch, compression.

8.2.10 DEFINITION. For number $k \neq 0$, the affine transformation

 $\zeta_{\mathcal{X},k}: (x,y) \mapsto (x+ky,y)$

is called a shear along the *x*-axis.

Here the x-axis is fixed pointwise and every point is moved "horizontally" a directed distance proportional to its directed distance from the x-axis. We shall see below that a shear has the property of preserving area.

8.2.11 DEFINITION. An affine transformation that preserves area is said to be **equiaffine**.

8.2.12 PROPOSITION. An affine transformation is the product of a shear, a strain, and a similarity.

PROOF : We can see that the general affine linear transformation with equations

$$\begin{cases} x' = ax + by + h \\ & \text{with} \quad ad - bc \neq 0 \\ y' = cx + dy + k \end{cases}$$