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Lecture Notes CCR

Groups and Geometry

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'Beauty is truth, truth beauty' – that is all
Ye know on earth, and all ye need to know.

John Keats

Imagination is more important than knowledge.

Albert Einstein

Do not just pay attention to the words;
Instead pay attention to meaning behind the words.
But, do not just pay attention to meanings behind the words;
Instead pay attention to your deep experience of those meanings.

Tenzin Gyatso, The 14th Dalai Lama

Where there is matter, there is geometry.

Johannes Kepler

Geometry is the art of good reasoning from poorly drawn figures.

Anonymous

What is a geometry ? The question [...] *is metamathematical rather than mathematical, and consequently mathematicians of unquestioned competence may (and, indeed, do) differ in the answer they give to it. Even among those mathematicians called geometers there is no generally accepted definition of the term. It has been observed that the abstract, postulational method that has permeated nearly all parts of modern mathematics makes it difficult, if not meaningless, to mark with precision the boundary of that mathematical domain which should be called geometry. To some, geometry is not so much a subject as it is a point of view – a way of loking at a subject – so that geometry is the mathematics that a geometer does ! To others, geometry is a language that provides a very useful and suggestive means of discussing almost every part of mathematics (just as, in former days, French was the language of diplomacy); and there are, doubtless, some mathematicians who find such a query without any real significance and who, consequently, will disdain to vouchsafe any answer at all to it.*

Leonard Blumenthal

> Geometry is a uniquely favorable environment for young students to learn the spirit of pure mathematics and exercise their intuition.
>
> John D. Smith

> Any objective definition of geometry would probably include the whole of mathematics.
>
> J.H.C. Whitehead

> Meaning is important in mathematics and geometry is an important source of that meaning.
>
> David Hilbert

What is Geometry ?

The Greek word for **geometry**, $\gamma\epsilon\omega\mu\epsilon\tau\rho\iota\alpha$, which means *measurement of the earth*, was used by the historian Herodotus (c.484-c.425 b.c.), who wrote that in ancient Egypt people used geometry to restore their land after the inundation of the Nile. Thus the theoretical use of figures for practical purposes goes back to pre-Greek antiquity. Tradition holds that Thales of Milet (c.639-c.546 b.c.) knew some properties of congruent triangles and used them for indirect measurement, and that Pythagoras (572-492 b.c.) and his school had the idea of systematizing this knowledge by means of *proofs*. Thus the Greeks made two vital contributions to geometry : they made geometry *abstract* and *deductive*. Starting from unquestionable premisses (or *axioms*), and basic laws of thought, they would reason and prove their way towards previously unguessed knowledge. This whole process was codified by EUCLID (C.300 B.C.) in his book, the *Elements*, the most successful scientific textbook ever written. In this work, we can see the entire mathematical knowledge of the time presented as a *logical system*.

Geometry – in today's usage – **means the branch of mathematics dealing with spatial figures**. Within mathematics, it plays a significant role. Geometry consists of a variety of intelectual structures, closely related to each other and to the original experiences of *space* and *motion*. A brief historical account of the subsequent development of this "science of space" from its Greek roots through modern times is given now.

In ancient Greece, however, all of mathematics was regarded as geometry. Algebra was introduced in Europe from the Middle East toward the end of the Middle Ages and was further developed during the Renaissance. In the 17th and 18th centuries, with the development of analysis, geometry achieved parity with algebra and analysis.

As RENÉ DESCARTES (1596-1650) pointed out, however, figures and numbers are closely related. Geometric figures can be treated algebraically (or analytically) by means of *coordinates*; conversely, algebraic facts can be expressed geometrically. **Analytic geometry** was developed in the 18th century, especially by LEONHARD EULER (1707-1783), who for the first time established a complete algebraic theory of curves of the second order. Previously, these curves had been studied by APOLLONIUS OF PERGA (262-C.200 B.C.) as *conic sections*.

The idea of DESCARTES was fundamental to the development of analysis in the 18th century. Toward the end of that century, analysis was again applied to geometry. GASPARD MONGE (1746-1818) can be regarded as a foreruner of **differential geometry**. CARL GAUSS (1777-1855) founded the *theory of surfaces* by introducing concepts of the geometry of surfaces. The influence that differential-geometric investigations of curves and surfaces have exerted upon branches of mathematics, physics, and engineering has been profound.

However, we cannot say that the analytic method is always the best manner of dealing with geometric problems. The method of treating figures directly without using coordinates is called **synthetic geometry**. In this vein, a new field called **projective geometry** was created by GÉRARD DESARGUES (1593-1662) and BLAISE PASCAL (1623-1662) in the 17th century. It was further developed in the 19th century.

On the other hand, the *axiom of parallels* in EUCLID's *Elements* has been an object of criticism since ancient times. In the 19th century, by denying the a priori validity of Euclidean geometry, JÁNOS BOLYAI (1802-1860) and NIKOLAI LOBACHEVSKY (1793-1856) formulated **non-Euclidean geometry**.

In analytic geometry, physical spaces and planes, as we know them, are represented as 3-dimensional or 2-dimensional Euclidean spaces. It is easy to generalize these spaces to n-dimensional Euclidean space. The geometry of this new space is called the **n-dimensional Euclidean geometry**. We obtain **n-dimensional projective** and **non-Euclidean geometries** similarly. FELIX KLEIN (1849-1925) proposed systematizing all these geometries in group-theoretic terms : he called a "space" a set $\mathbf{S}$ on which a group $\mathfrak{G}$ operates and a "geometry" the study of properties of $\mathbf{S}$ invariant under the operations of $\mathfrak{G}$. KLEIN's idea not only synthetized the geometries known at that time, but also became a guiding principle for the development of new geometries.

BERNHARD RIEMANN (1826-1866) initiated another direction of geometric research when he investigated n-dimensional *manifolds* and, in particular, Riemannian manifolds and their geometries. Some aspects of **Riemannian geometry** fall outside of geometry in the sense of KLEIN. It was a starting point for the broad field of **modern differential geometry**, that is, the geometry of *differentiable manifolds* of various types. It became necessary to establish a theory that reconciled the ideas of KLEIN and RIEMANN; ELIE CARTAN (1869-1951) succeeded in this by introducing the notion of *connec-*

tion.

The reexamination of the system of axioms of Euclid's *Elements* led to David Hilbert's (1862-1943) foundations of geometry and to axiomatic tendency of present day mathematics. The study of algebraic curves, which started with the study of conic sections, developed into **algebraic geometry**. Another branch of geometry is **topology**, which has developed since the end of the 19th century. Its influence on the whole of mathematics today is considerable.

Geometry has now permeated all branches of mathematics, and it is sometimes difficult to distinguish it from algebra or analysis. Therefore, **geometry is not just a subdivision or a subject within mathematics, but a means of turning visual images into formal tools for the understanding of other mathematical phenomena.** The importance of geometric intuition, however, has not diminished from antiquity until today.

For accessible, informative materials about Geometry (and Mathematics, in general) – its past, its present and also its future – the following sources are highly recommended :

Expository papers

M. Atiyah – What is geometry ?, *The Mathematical Gazette* **66**(1982), 179-184.

S-S. Chern – From triangles to manifolds, *Amer. Math. Monthly* **86**(1979), 339-349.

S-S. Chern – What is geometry ?, *Amer. Math. Monthly* **97**(1990), 679-686.

R.S. Millman – Kleinian transformation geometry, *Amer. Math. Monthly* **84**(1977), 338-349.

Books

M.J. Greenberg – *Euclidean and Non-Euclidean Geometries. Development and History*, Freeman, 1980.

T.Q. Sibley – *The Geometric Viewpoint. A Survey of Geometries*, Addison-Wesley, 1998.

H. Weyl – *Symmetry*, Princeton University Press, 1952.

I.M. Yaglom – *Felix Klein and Sophus Lie. Evolution of the Ideea of Symmetry in the 19th Century*, Birkhäuser, 1988.

Web sites

The MacTutor History of Mathematics archive [http: //www-history.mcs.st-and.ac.uk]

The Mathematical Atlas [http: //www.math-atlas.org or //www.math.niu.edu/~rusin/known-math/index/mathmap.html]

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