Chapter 5

Manifolds

Topics :

1. Manifolds: Definition and Examples
2. Smooth Functions and Mappings
3. The Tangent and Cotangent Spaces
4. Smooth Submanifolds
5. Vector Fields
6. Differential Forms

5.1 Manifolds: Definition and Examples

Submanifolds (in fact, immersed submanifolds) of Euclidean space $\mathbb{R}^m$ are a generalization of the concept of regular curve in the Euclidean 3-space $\mathbb{R}^3$. The major defect of the definition of a submanifold is its dependence of $\mathbb{R}^m$. Indeed, the natural idea of an ℓ-dimensional smooth submanifold is of *a set which is ℓ-dimensional* (in a certain sense) *and to which the differential calculus of $\mathbb{R}^m$ can be applied*; the unnecessary presence of $\mathbb{R}^m$ is simply an imposition of our physical nature.

NOTE : In his monograph on surface theory, published in 1827, CARL F. GAUSS (1777-1855) developed the geometry on a surface (based on its fundamental form); the necessity of an *abstract* idea of surface – that is, without involving the ambient space – was already clear to him. This was generalized by BERNHARD RIEMANN (1826-1866) to m-dimensions in his inaugural lecture (Habilitationschrift) at Göttingen, "On the Hypotheses which lie at the Foundation of Geometry" (1854), marking the birth of modern (differential) geometry. However, it was nearly a century before such an idea attained the definite form that we shall present here.

The concept of *manifold* is one of the most sophisticated basic concepts in mathematics.

Definition (of a manifold) and examples

Let $\mathbb{R}^m$ denote the Euclidean m-space *in the broad sense* (i.e., the vector space $\mathbb{R}^m$ equipped with its *canonical topology* and natural *differentiable structure*).

Let M be a set.

5.1.1 DEFINITION. A (coordinate) **chart** on M is a pair (U, ϕ), where $U \subseteq M$ and $\phi : U \to \mathbb{R}^m$ is a one-to-one mapping *onto* an open subset $\phi(U)$ of $\mathbb{R}^m$.

One often writes $\phi(p) = (\phi_1(p), \dots, \phi_n(p))$, viewing this as the coordinate m-tuple of the point $p \in U$. The functions $\phi_i : U \to \mathbb{R}$, $i = 1, 2, \dots, m$ are called the *coordinate functions* associated with the chart (U, ϕ).

Note : A chart is also called a (local) *coordinate system* (on M).

Relative to such a *coordinatization*, one can do calculus in the region U of M. The problem is that the point p will generally belong to infinitely many different coordinate charts and calculus in one of these coordinatizations about p might not agree with calculus in another. One needs the coordinate systems to be *smoothly compatible* in the following sense.

5.1.2 Definition. Two charts (U, ϕ) and (V, ψ) on M are said to be C^∞*-related* if either $U \cap V = \emptyset$ or

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is a smooth diffeomeorphism (between open subsets in $\mathbb{R}^m$).

We think of $\psi \circ \phi^{-1}$ as a smooth change of coordinates (on $\phi(U \cap V)$). Thus, on $U \cap V$, functions are smooth relative to one coordinate system if and only if they are smooth relative to the other. Indeed, differential calculus carried out in $U \cap V$ via the coordinates of $\phi(U \cap V)$ is equivalent to the calculus carried out via the coordinates of $\psi(U \cap V)$. (The explicit formulas will, of course, change from the one coordinate system to the other.) Furthermore, piecing together these local calculi produces a global calculus on M. The concept that allows us to make these remarks precise is that of a *smooth atlas.*

5.1.3 Definition. A (smooth) **atlas** on M is a family $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathfrak{A}}$ of charts (on M) such that

(AT1) $M = \bigcup_{\alpha \in \mathfrak{A}} U_\alpha$;

(AT2) (U_α, ϕ_α) is C^∞-related to (U_β, ϕ_β) for every $\alpha, \beta \in \mathfrak{A}$.

Two atlases $\mathcal{A}$ and $\mathcal{A}'$ on M are *compatible* provided their union $\mathcal{A} \cup \mathcal{A}'$ is also an atlas on M. Compatibility is an equivalence relation (on the set of all atlases on M). *Each atlas on M is equivalent to a unique maximal atlas on M.* Thus we arrive at the definition of a manifold.

5.1.4 DEFINITION. A maximal atlas $\mathcal{A}$ on M is called a **smooth structure** on M (also called a *differentiable structure* or a C^∞ structure). An n-**dimensional smooth** (or differentiable or C^∞) **manifold** is a pair $(M, \mathcal{A})$ (i.e. a set equipped with a smooth structure).

By a typical abuse of notation, we usually write M for the smooth manifold, the presence of the differentiable structure $\mathcal{A}$ being understood. An **admissible chart** on (the smooth manifold) M is any chart belonging to any (smooth) atlas in the differentiable structure of M.

NOTE : (1) We often refer to m-dimensional smooth manifolds simply as *m-manifolds.*

(2) In practice one defines a manifold M by means of a single (smooth) atlas (not necessarily maximal) on M which completely determines the differentiable structure.

We will now define on a manifold M a *canonical topology*, one that only depends on the differentiable structure.

NOTE : One could also have started from a topological space M and required that the domains U_α of the charts be open sets in M and that the mappings $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ be homeomorphisms.

5.1.5 PROPOSITION. *Let M be a (smooth) m-manifold. The collection of unions of domains of admissible charts on M forms a topology (called the canonical topology) on M.*

PROOF : Let $\mathcal{O}$ be the set thus defined. Clearly, $M \in \mathcal{O}$ and we have to show that $\mathcal{O}$ satisfies the two axioms for a topology :

(O1) Every union of elements of $\mathcal{O}$ is an element of $\mathcal{O}$.

(O2) Every finite intersection of elements of $\mathcal{O}$ is an element of $\mathcal{O}$.

Clearly (O1) is satisfied, since a set is in $\mathcal{O}$ if and only if it is a union of domains of charts. To show (O2), we just have to consider the intersection of two elements of $\mathcal{O}$. Let them be $A = \cup_{\alpha \in \mathfrak{A}_1} U_\alpha$ and $B = \cup_{\beta \in \mathfrak{A}_2} U_\beta$; then

$$A \cap B = \bigcup_{(\alpha,\beta) \in \mathfrak{A}_1 \times \mathfrak{A}_2} (U_\alpha \cap U_\beta).$$

We have to show that each intersection $U_\alpha \cap U_\beta$ can be taken as the domain of a chart *compatible* with the differentiable structure (i.e. an admissible chart on M). Let (U_α, ϕ_α) be an admissible chart on M and set $\psi := \phi_\alpha|_{U_\alpha \cap U_\beta}$; we claim that $(U_\alpha \cap U_\beta, \psi)$ is the desired admissible chart. Clearly $\psi(U_\alpha \cap U_\beta) = \phi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^m$. If (U, ϕ) is any admissible chart, the composition $\phi \circ \phi_\alpha^{-1}$ is a (smooth) diffeomorphism between (the open sets) $\phi_\alpha(U \cap U_\alpha)$ and $\phi(U \cap U_\alpha)$, so

$$\phi \circ \psi^{-1} = \phi \circ \phi_\alpha^{-1}|_{\phi_\alpha(U_\alpha \cap U_\beta \cap U)}$$

is a (smooth) diffeomorphism between $\psi(U \cap (U_\alpha \cap U_\beta))$ and $\phi(U \cap (U_\alpha \cap U_\beta))$. Similarly, $\psi \circ \phi^{-1}$ is a (smooth) diffeomorphism between $\phi(U \cap (U_\alpha \cap U_\beta))$ and $\psi(U \cap (U_\alpha \cap U_\beta))$. This proves compatibility.

□

Note : Sometimes it is desirable to characterize the open sets in the canonical topology of M in terms of a single atlas. One can prove that *given an atlas* $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathfrak{A}}$ *on an* m*-manifold* M*, a subset* $U \subseteq M$ *is open if and only if the set* $\phi_\alpha(U \cap U_\alpha) \subseteq \mathbb{R}^m$ *is open for every chart* $(U_\alpha, \phi_\alpha) \in \mathcal{A}$. This result provides another way of defining the (canonical) topology of a manifold : for every chart (U, ϕ) on an m-manifold M, considered with its canonical topology, the mapping $\phi : U \to \phi(U) \subseteq \mathbb{R}^m$ is a homeomorphism.

The canonical topology of a manifold can be quite strange. In particular, it can happen that one (or both) of the following conditions (axioms) *not* be satisfied :

(A) *Hausdorff Axiom :* Given two distinct points of M, there exist (open) neighborhoods of these points that do not intersect.

(B) *Countable Basis Axiom :* M can be covered by a countable number of coordinate neighborhoods (i.e. domains of admissible charts on M). We say then that M *has a countable basis* (or that M is second countable).

Note : Axiom (A) is essential for the uniqueness of limits of convergent sequences whereas Axiom (B) is essential for the existence of a (smooth) *partition of unity*, an almost indispensabil tool for the study of certain questions on manifolds. A topological space which is locally compact (each point has at least one compact neighborhood), Hausdorff, and has a countable basis (of open sets) is *paracompact*, and hence admits a partition of the unity. For example, a partition of unity is required for piecing together global functions and structures out of local ones, and conversely for representing global structures as locally finite sums of local ones. The following result holds : A (smooth) manifold M has a (smooth) partition of unity if and only if every connected component of M is Hausdorff and has a countable basis.

For all practical purposes, we shall be interested in *only* (smooth) manifolds that satisfy Axiom (A) and Axiom (B). Henceforth, we shall refer to such objects, simply, as *manifolds*.

NOTE : (1) Manifolds are locally Euclidean spaces (A Hausdorff topological space is said to be *locally Euclidean* of dimension m if each point p has an open neighborhood homeomorphic to an open set of $\mathbb{R}^m$). Second countable locally Euclidean spaces are known as *topological manifolds*. A topological manifold is *smoothable* provides it can be given a smooth structure. For $m = 1, 2, 3$ it is known that all topological m-manifolds are smoothable. The first dimension in which there exist nonsmoothable manifolds is $m = 4$.

(2) Manifolds are paracompact spaces (A Hausdorff space is called *paracompact* if every open cover has a locally finite subcover). Moreover, manifolds are metrizable spaces (A topological space is called *metrizable* if there exists a metric such that its associated metric topology coincides with the space topology; any metrizable space is paracompact).

(3) Any m-manifold admits a *finite* atlas consisting of $m+1$ (not necessarily connected) charts. This is a consequence of *topological dimension theory*.

(4) A manifold is connected if and only if it is path-connected. (A path-connected topological space is connected, but the converse is *not* true in general.)

(5) A natural question in the theory of (differentiable) manifolds is to know whether a given manifold can be *immersed* (or even *embedded*) into some Euclidean space. A fundamental result in this direction is the famous theorem of HASSLER WHITNEY (1907-1989) which states the following : *Any m-manifold can be immersed in* $\mathbb{R}^{2m}$ *and embedded in* $\mathbb{R}^{2m+1}$ (in fact, the theorem can be improved, for $m \geq 2$, to $\mathbb{R}^{2m-1}$

and $\mathbb{R}^{2m}$, respectively).

(6) A set M may have more than one inequivalent smooth structure. For instance, the spheres from dimension 7 on have finitely many. A most surprising result is that *on* $\mathbb{R}^4$ *there are uncountably many pairwise inequivalent (exotic) smooth structures.*

We give now some preliminary examples of manifolds.

5.1.6 Example. (*Euclidean space*) The standard smooth structure on the Euclidean m-space $\mathbb{E}^m$ is obtained by taking the atlas consisting of a single (global) chart $(\mathbb{E}^m, \iota)$, where $\iota : \mathbb{E}^m \to \mathbb{R}^m$ is the identity mapping. (Many examples will make it abundantly clear that manifolds in general can *not* be covered by a single coordinate system nor are there preferred coordinates.)

Note : It is common practice to identify $\mathbb{E}^m$ and $\mathbb{R}^m$; however, we DO NOT follow this custom. It is often better in thinking of the Euclidean space $\mathbb{E}^m$ as a "flat" Riemannian manifold (i.e. a "geometrical" model for classical geometry, *without* coordinates; a Riemannian manifold is a manifold equipped with an additional "geometrical" structure, called a *Riemannian metric*) and of the Cartesian space $\mathbb{R}^m$ as a normed vector space (i.e. an "algebraic" model for classical geometry, *with* coordinates). The additive group of $\mathbb{R}^m$, also denoted by $\mathbb{R}^m$, is a matrix group. This group is isomorphic to (and customarily identified with) the group of all translations on the Euclidean space $\mathbb{E}^m$.

5.1.7 Example. Let V be an m-dimensional vector space (over $\mathbb{R}$). Then V has a natural manifold structure. Indeed, if $\{v_1, \dots, v_m\}$ is a basis in V, then the correspondence

$$\phi : p = p_1 v_1 + \cdots + p_m v_m \mapsto (p_1, \dots, p_m)$$

is a bijection (between V and the open set $\mathbb{R}^m$). The pair (V, ϕ) is a (global) chart on V and hence uniquely determines a smooth structure on V. This smooth structure is independent of the choice of the basis, since different bases give C^∞-related charts. (In fact, the change of coordinates is given simply by an $m \times m$ invertible matrix.)

5.1.8 Example. (*Open submanifolds*) An open subset U of a manifold M is itself a manifold. Indeed, if $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathfrak{A}}$ is the (maximal) atlas of admissible charts on M, then the family of charts (atlas)

$$\mathcal{A}_U = \{(U \cap U_\alpha,\, \phi_\alpha|\, U \cap U_\alpha) \,|\, (U_\alpha, \phi_\alpha) \in \mathcal{A}\}$$

defines a smooth structure on U. Unless otherwise stated, open subsets of manifolds will always be given this natural (induced) smooth structure.

More generally, any ℓ-dimensional smooth submanifold of some Euclidean space $\mathbb{E}^m$ is a (smooth) ℓ-manifold.

$\diamond$ **Exercise 254** Let S be a (non-empty) subset of the Euclidean space $\mathbb{E}^m$ and assume that S satisfies the *ℓ-submanifold property* (i.e. S is an *ell*-dimensional submanifold of $\mathbb{E}^m$). Show that S is naturally endowed with a smooth structure, hence it *is* an *ell*-manifold.

5.1.9 Example. (*The general linear group*) The general linear group $\mathsf{GL}\,(n, \mathbb{R})$ is an open subset of the manifold $\mathbb{E}^{n^2}$ (we may identify $\mathbb{R}^{n \times n}$ with the Cartesian space $\mathbb{R}^{n^2}$). Hence $\mathsf{GL}\,(n, \mathbb{R})$ is a manifold.

5.1.10 Example. (*The sphere*) The *n-sphere* is the set

$$\mathbb{S}^n := \left\{x \in \mathbb{E}^{n+1} \,|\, x_1^2 + \cdots + x_{n+1}^2 = 1\right\}.$$

(We have seen that $\mathbb{S}^n$ is an n-dimensional smooth submanifold of $\mathbb{E}^{n+1}$.) Let $p_N = (0, \ldots, 0, 1)$ be the north pole and $p_S = (0, \ldots, 0, -1)$ the south pole of $\mathbb{S}^n$. Define the mapping $\phi_1 : U_1 := \mathbb{S}^n \setminus \{p_N\} \to \mathbb{R}^n$ that takes the point $p = (x_1, \ldots, x_{n+1})$ in U_1 into the intersection of the hyperplane $x_{n+1} = 0$ with the line that passes through p and p_N. This mapping is the so-called *stereographic projection* from the north pole. In a similar manner one defines the stereographic projection $\phi_{-1} : U_{-1} := \mathbb{S}^n \setminus \{p_S\} \to \mathbb{R}^n$ from the south pole.

$\diamond$ **Exercise 255** Show that the stereographic projections (ϕ_1 and ϕ_{-1}) are given by

$$\phi_{\pm 1}(x_1, \ldots, x_{n+1}) = \left(\frac{x_1}{1 \mp x_{n+1}}, \cdots, \frac{x_n}{1 \mp x_{n+1}}\right).$$

Clearly, the stereographic projections are one-to-one and hence the pairs (U_1, ϕ_1) and (U_{-1}, ϕ_{-1}) are charts on $\mathbb{S}^n$. The domains (coordinate neighborhoods) of these two charts cover $\mathbb{S}^n$ and is not difficult to see that they are C^∞-related (and hence form a smooth atlas on the sphere). Indeed, the change of coordinates

$$y_i = \frac{x_i}{1 - x_{n+1}} \longleftrightarrow y_i' = \frac{x_i}{1 + x_{n+1}} \quad (i = 1, 2, \ldots, n)$$

is given by

$$y_i' = \frac{y_i}{y_1^2 + \cdots + y_n^2}$$

(here we use the fact that $x_1^2 + \cdots + x_{n+1}^2 = 1$). Therefore, the n-sphere $\mathbb{S}^n$ is an n-manifold.

5.1.11 Example. (*Product manifolds*) Let M and N be manifolds (of dimension m and n, respectively). Suppose that $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathfrak{A}}$ and $\mathcal{B} = \{(V_\beta, \psi_\beta)\}_{\beta \in \mathfrak{B}}$ are the maximal atlases on M and N, respectively.

◇ **Exercise 256** Show that the family (of charts)

$$\{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta) \,|\, (U_\alpha, \phi_\alpha) \in \mathcal{A}, \;\; (V_\beta, \psi_\beta) \in \mathcal{B}\}$$

where $\phi_\alpha \times \psi_\beta(p, q) := (\phi_\alpha(p), \psi_\beta(q)) \in \mathbb{R}^m \times \mathbb{R}^n$, is a smooth atlas on $M \times N$ (which determines a smooth structure).

With this smooth structure $M \times N$ is an $(m+n)$-manifold, called the *product manifold* of M and N. An important example is the *torus* $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, the product of two *circles*. More generally, the k-dimensional torus $\mathbb{T}^k = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is a k-manifold obtained as a Cartesian product.

5.2 Smooth Functions and Mappings

On a topological space the concept of continuity has meaning; in an analogous way, on a manifold we may define the concept of *smooth* (also called differentiable or C^∞) function. Let M be an m-manifold.

5.2.1 Definition. A function $f : M \to \mathbb{R}$ is said to be **smooth** if for any point $p \in M$ there is an admissible chart (U, ϕ) on M such that $p \in U$ and the composite function

$$f \circ \phi^{-1} : \phi(U) \subseteq \mathbb{R}^m \to \mathbb{R}$$

is smooth.

Clearly, a smooth function is continuous. The set of all smooth functions on M will be denoted by $C^\infty(M)$. It is a consequence of the definition that if $f \in C^\infty(M)$ and $W \subseteq M$ is an open set, then $f|_W$ is smooth (on the manifold W).

Note : The definition only requires us to be able to find *some* chart about each point $p \in M$, but the following result assures us that *all* admissible charts will then work : *The function* $f : M \to \mathbb{R}$ *is smooth if and only if* $f \circ \phi^{-1}$ *is smooth for every admissible chart* (U, ϕ) *on* M.

We think of $f \circ \phi^{-1}$ as a *formula* for $f|_U$ relative to the coordinate system (U, ϕ). For $x \in U$, with coordinates $\phi(x) = (x_1, \dots, x_m)$, we can write

$$\begin{aligned} y &= f(x) \\ &= f \circ \phi^{-1}(\phi(x)) \\ &= f \circ \phi^{-1}(x_1, \dots, x_m). \end{aligned}$$

We shall refer to $f \circ \phi^{-1}$ as the *local representation* of f with respect to (U, ϕ).

5.2.2 Example. Among the smooth functions on M are the coordinate functions of an admissible chart (U, ϕ). Indeed, for each $i = 1, 2, \dots, m$, the local representation of $\phi_i = \mathrm{pr}_i \circ \phi$ is given by

$$\begin{aligned} y &= \phi_i(x) \\ &= \phi_i \circ \phi^{-1}(x_1, \dots, x_m) \\ &= \mathrm{pr}_i \circ \phi \circ \phi^{-1}(x_1, \dots, x_m) \\ &= \mathrm{pr}_i(x_1, \dots, x_m) \\ &= x_i \end{aligned}$$

which is clearly smooth (see also **Exercise 120**).

Just as in the case of (the manifold) $\mathbb{E}^n$ we proceed from definition of smooth function to definition of smooth mapping. Suppose that M and N are manifolds.

5.2.3 Definition. A mapping $F : M \to N$ is said to be **smooth** if for any point $p \in M$ there is an admissible chart (U, ϕ) on M with $p \in U$ and an admissible chart (V, ψ) on N with $F(p) \in V$ such that $F(U) \subseteq V$ and the composite mapping

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth.

Smooth mappings are continuous; their restrictions to open subsets are also smooth. The set of all smooth mappings from M into N will be denoted by $C^\infty(M, N)$.

Note : A smooth mapping is a more general notion than smooth function, the latter being a mapping (from a manifold M) into $N = \mathbb{R}$, which is, of course, the same as (the manifold) $\mathbb{E}^1$.

The local representation of F with respect to (U, ϕ) and (V, ψ) is given by

$$y_i = \psi \circ F \circ \phi^{-1}(x_1, \ldots, x_m), \qquad i = 1, 2, \ldots, n.$$

$\diamond$ **Exercise 257** Prove that a mapping $F : M \to N$ is smooth if and only if for any smooth function $f : N \to \mathbb{R}$, the function $f \circ F$ is smooth (on M). (We write $F^* f$ for the function $f \circ F$, and shall refer to $F^* f$ as the *pull-back* of f under F.)

An open interval J of $\mathbb{R}$ is an open submanifold of $\mathbb{R}$ (in fact, the Euclidean 1-space $\mathbb{E}^1$) and hence is a manifold. Then *a curve $\sigma : J \to N$ is smooth if and only if for any smooth function f on N, (the pull-back of f under σ) $\sigma^* f : J \to \mathbb{R}$ is a smooth function.*

$\diamond$ **Exercise 258** Let M and N be manifolds. Prove that the *canonical projections*

$$\mathrm{pr}_M : M \times N \to M \quad \text{and} \quad \mathrm{pr}_N : M \times N \to N$$

are smooth mappings (between manifolds).

$\diamond$ **Exercise 259** Let M, N, and P be manifolds. Prove that if $F : M \to N$ and $G : N \to P$ are smooth mappings, then $G \circ F : M \to P$ is also smooth.

$\diamond$ **Exercise 260** Let M be a manifold. Show that the set $C^\infty(M)$ of all smooth functions on M is an *algebra* (over $\mathbb{R}$) under the natural operations of addition, scalar multiplication, and product.

5.3 The Tangent and Cotangent Spaces

The tangent space

There are several alternative ways in which we can define *tangent vectors* (and hence tangent spaces) to a manifold, independent of any embedding in some Euclidean space.

NOTE : The whole reason for introducing tangent vectors is to produce linear approximations to nonlinear problems.

An intuitive (and very useful) way to define tangent vectors is as equivalence classes of curves. (Roughly speaking, two curves are equivalent if they have the same velocity vector at some point.)

Let M be an m-manifold and let $\mathbf{C}(p)$ denote the set of all smooth curves $\sigma : (-\varepsilon, \varepsilon) \to M$ such that $\sigma(0) = p$. Elements (curves) α and β in $\mathbf{C}(p)$ are said to be *infinitesimally equivalent* at p and we write $\alpha \sim_p \beta$ if

$$\frac{d}{dt}\phi(\alpha(t))\bigg|_{t=0} = \frac{d}{dt}\phi(\beta(t))\bigg|_{t=0}$$

for any admissible chart (U, ϕ) on M.

$\diamond$ **Exercise 261** Show that if (U, ϕ) and (V, ψ) are two admissible charts at p (i.e. such that $p \in U \cap V$), then

$$\frac{d}{dt}\phi \circ \alpha(t)\bigg|_{t=0} = \frac{d}{dt}\phi \circ \beta(t)\bigg|_{t=0}$$

if and only if

$$\frac{d}{dt}\psi \circ \alpha(t)\bigg|_{t=0} = \frac{d}{dt}\psi \circ \beta(t)\bigg|_{t=0}.$$

(The infinitesimal equivalence is well defined.)

It is easy to check that $\sim_p$ is an equivalence relation on the set $\mathbf{C}(p)$. The infinitesimal equivalence class of α in $\mathbf{C}(p)$ is denoted by $[\alpha]_p$ and is called an *infinitesimal curve* at p. An infinitesimal curve at p is also called a **tangent vector** to M at p.

5.3.1 DEFINITION. The (quotient) set $T_pM := \mathbf{C}(p)/_{\sim_p}$ of all infinitesimal curves at p is called the **tangent space** to M at p.

Let (U, ϕ) be any admissible chart on M such that $p \in U$. The mapping

$$\bar{\phi} : T_pM \to \mathbb{R}^m, \quad [\alpha]_p \mapsto \left. \frac{d}{dt}\phi(\alpha(t)) \right|_{t=0}$$

is one-to-one and onto $\mathbb{R}^m$. In fact, for any $v \in \mathbb{R}^m$, $\alpha(t) := \phi^{-1}(\phi(p) + tv)$ is a curve such that $\bar{\phi}([\alpha]_p) = v$. We define the vector structure on T_pM so that $\bar{\phi}$ becomes a linear isomorphism. That is, for $[\alpha]_p, [\beta]_p \in T_pM$ and $a \in \mathbb{R}$,

$$\begin{aligned} [\alpha]_p + [\beta]_p \; &:= \; \bar{\phi}^{-1}\left(\bar{\phi}([\alpha]_p) + \bar{\phi}([\beta]_p)\right) \\ a[\alpha]_p \; &:= \; \bar{\phi}^{-1}\left(a\bar{\phi}([\alpha]_p)\right). \end{aligned}$$

Under the forgoing addition and scalar multiplication, *the tangent space T_pM is an m-dimensional vector space over $\mathbb{R}$.*

NOTE : The linear structure of T_pM is *canonical* in the sense that it is independent of the choice of (local) coordinates. Indeed, let (U, ϕ) and (V, ψ) be two admissible charts at p. Let $\bar{\phi}([\alpha]_p) = v$ and let $\bar{\psi}([\alpha]_p) = w$. It follows that

$$w = \left. \frac{d}{dt}\psi \circ \alpha(t) \right|_{t=0} = \left. \frac{d}{dt}\psi \circ \phi^{-1} \circ \phi \circ \alpha(t) \right|_{t=0}.$$

Therefore the coordinates of v and w transform according to the following formula :

$$w_i = \frac{\partial y_i}{\partial x_1}v_1 + \cdots + \frac{\partial y_i}{\partial x_m}v_m$$

where $y_i = y_i(x_1, \ldots, x_m)$, $i = 1, 2, \ldots, m$ denote the coordinate functions of the mapping $\psi \circ \phi^{-1}$. Hence the vector structure on T_pM is independent of the particular chart (used to define it).

The cotangent space

Let M be an m-dimensional manifold and let $\mathbf{F}(p)$ denote the set of all smooth functions f, defined in some (open) neighborhood of $p \in M$, that satisfy $f(p) = 0$. $\mathbf{F}(p)$ *will have a natural vector space structure* (in fact, associative algebra with unity) *provided that functions that agree on a common domain are regarded as equal.* (The domains of elements of $\mathbf{F}(p)$ need not be the same.)

NOTE : Actually, an element of (the algebra) $\mathbf{F}(p)$ is a certain set (equivalence class) of smooth functions, commonly referred to as a *function germ* at p, which is conveniently identified with any one of its representatives.

Elements (function germs) f and g in $\mathbf{F}(p)$ are said to be *equivalent* (at p) and we write $f \approx_p g$ if

$$D\left(f \circ \phi^{-1}\right)(\phi(p)) = D\left(g \circ \phi^{-1}\right)(\phi(p))$$

for any admissible chart (U, ϕ) on M.

NOTE : We shall write, by a slight abuse of notation,

$$f \circ \phi^{-1}(x_1, \ldots, x_m) = f(x_1, \ldots, x_m) \quad \text{and} \quad D(f \circ \phi^{-1}) = \frac{\partial f}{\partial x} := \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_m} \end{bmatrix}.$$

Again, it is easy to check that $\approx_p$ is an equivalence relation on the set $\mathbf{F}(p)$. The equivalence class of f in $\mathbf{F}(p)$ is denoted by $[f]_p$ and is called a **tangent covector** to M at p.

5.3.2 DEFINITION. The (quotient) set $T_p^* M := \mathbf{F}(p)/_{\approx_p}$ is called the **cotangent space** to M at p.

◇ **Exercise 262** Let $f, \bar{f}, g, \bar{g} \in \mathbf{F}(p)$ and $a \in \mathbb{R}$. Show that

(a) If $f \approx_p \bar{f}$ and $g \approx_p \bar{g}$, then $f + g \approx_p \bar{f} + \bar{g}$.

(b) If $f \approx_p \bar{f}$, then $af \approx_p a\bar{f}$.

That is, for $[f]_p, [g]_p \in T_p^* M$ and $a \in \mathbb{R}$, the following operations

$$\begin{aligned} [f]_p + [g]_p &:= [f+g]_p \\ a[f]_p &:= [af]_p \end{aligned}$$

are well-defined. Under the foregoing addition and scalar multiplication, *the cotangent space* T_p^*M *is a real vector space.*

For each admissible chart (U, ϕ) on M such that $p \in U$, the mapping

$$\underline{\phi} : T_p^*M \to (\mathbb{R}^m)^*, \quad [f]_p \mapsto D\left(f \circ \phi^{-1}\right) \in \mathbb{R}^{1\times m}$$

is a linear *isomorphism.* For each i, the (smooth) function

$$f_i : U \to \mathbb{R}, \quad x \mapsto f_i(x) := \phi_i(x) - \phi_i(p)$$

is an element of $\mathbf{F}(p)$ and $\underline{\phi}\,([f_i]_p) = \begin{bmatrix}\delta_{i1} & \cdots & \delta_{im}\end{bmatrix} \in \mathbb{R}^{1\times m}$. So $[f_1]_p, \cdots, [f_m]_p$ form a basis for (the vector space) T_p^*M.

Note : The linear structure of T_p^*M is *canonical.* Indeed, let (U, ϕ) and (V, ψ) be two admissible charts at p which produce their own bases $[f_1]_p, \dots, [f_m]_p$ and $[g_1]_p, \dots, [g_m]_p$, respectively. Let $[f]_p$ be an arbitrary element of T_p^*M. Then

$$\begin{aligned} [f]_p &= v_1[f_1]_p + \cdots + v_m[f_m]_p \\ &= w_1[g_1]_p + \cdots + w_m[g_m]_p. \end{aligned}$$

It follows that the coordinates $(w_1, \dots, w_m)$ are related to the coordinates $(v_1, \dots, v_m)$ via the following formula

$$v_i = \frac{\partial y_1}{\partial x_i} w_1 + \cdots + \frac{\partial y_m}{\partial x_i} w_m$$

where $y_i = y_i(x_1, \dots, x_m)$, $i = 1, 2, \dots, m$ denote the coordinate functions of the mapping $\psi \circ \phi^{-1}$. Hence the vector structure of T_p^*M is independent of the particular choice of admissible chart.

We shall show now the *duality* between the elements of T_pM and those of T_p^*M. For any $f \in \mathbf{F}(p)$ and any $\sigma \in \mathbf{C}(p)$, consider the *pairing*

$$\langle [f]_p,\, [\sigma]_p \rangle := \left.\frac{d}{dt} f \circ \sigma \right|_{t=0}.$$

Because $f \circ \sigma = f \circ \phi^{-1} \circ \phi \circ \sigma$, it follows that the foregoing pairing is well defined and is *bilinear.* More explicitly,

$$\langle [f]_p,\, [\sigma]_p \rangle = \frac{\partial f}{\partial x_1}\frac{d\sigma_1}{dt} + \cdots + \frac{\partial f}{\partial x_m}\frac{d\sigma_m}{dt}$$

with

$$D\left(f \circ \phi^{-1}\right) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_m} \end{bmatrix} \quad \text{and} \quad \frac{d}{dt}\phi \circ \sigma(t)\Big|_{t=0} = \begin{bmatrix} \frac{d\sigma_1}{dt} \\ \vdots \\ \frac{d\sigma_m}{dt} \end{bmatrix}.$$

Therefore, *each element of* T_p^*M *is a linear functional on* T_pM, and hence

$$T_p^*M = (T_pM)^*.$$

NOTE : It is useful to think of tangent vectors as objects that act (linearly) on functions and produce *directional derivatives*. Let M be a smooth manifold and let $\mathbf{F}(p)$ be the *algebra* of function germs at $p \in M$. A *linear functional* $X_p : \mathbf{F}(p) \to \mathbb{R}$ is called a **derivation** at p if (for every $f, g \in \mathbf{F}(p)$)

$$X_p(f \cdot g) = f(p) \cdot X_p(g) + g(p) \cdot X_p(f) \qquad \text{(Leibniz rule)}.$$

If $f = 1$ (i.e. $f(x) = 1$ for all $x \in M$), then $X_p(f) = 2X_p(f)$, and therefore $X_p(f) = 0$. Thus any derivation of a constant function is zero. It is easy to check that the set of all derivations at p is in fact a vector space (over $\mathbb{R}$). Moreover, this vector space is *isomorphic* to (the tangent space) T_pM. (In general, for manifolds that are *not* smooth, the space of derivations is an infinite dimensional vector space and so cannot be isomorphic to T_pM.)

For each $[\alpha]_p \in T_pM$ and $f \in \mathbf{F}(p)$, let

$$\langle f, [\alpha]_p \rangle = \frac{d}{dt} f \circ \alpha(t)\Big|_{t=0}.$$

Such action is well defined, for if $\alpha \sim_p \bar{\alpha}$, then

$$\begin{aligned} \frac{d}{dt} f \circ \bar{\alpha}(t)\Big|_{t=0} &= \frac{d}{dt} f \circ \phi^{-1} \circ \phi \circ \bar{\alpha}(t)\Big|_{t=0} \\ &= \frac{d}{dt} (f \circ \phi^{-1}) \circ \phi \circ \alpha(t)\Big|_{t=0} \\ &= \frac{d}{dt} f \circ \alpha(t)\Big|_{t=0}. \end{aligned}$$

$[\alpha]_p$ acts linearly on $\mathbf{F}(p)$ and it follows that such an operation is a *derivation*. Let $D_{[\alpha]_p}$ denote the derivation (at p) induced by the foregoing pairing. It can be shown that *for each derivation* X_p *at* p, *there exists an element (infinitesimal curve)* $[\alpha]_p$

in T_pM *such that* $X_p = D_{[\alpha]_p}$. (Given a fixed admissible chart (U, ϕ) at p, consider the curves

$$\alpha_i : t \mapsto \alpha_i(t) := \phi^{-1}\left(\phi(p) + te_i\right), \qquad i = 1, 2, \dots, m.$$

Then $[\alpha_1], \cdots, [\alpha_m]_p$ form a basis for T_pM and $X_p = a_1 D_{[\alpha_1]_p} + \cdots + a_m D_{[\alpha_m]_p}$ for some numbers $a_1, \dots, a_m$.)

Following the usual practice, we shall write $\left.\frac{\partial}{\partial x_i}\right|_p$ for $D_{[\alpha_i]_p}$. Then $\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_m}\right|_p$ is a basis for the (vector space of) derivations at p, and each derivation is an expression of the form

$$a_1 \left.\frac{\partial}{\partial x_1}\right|_p + \cdots + a_m \left.\frac{\partial}{\partial x_m}\right|_p.$$

We shall find it convenient to use two notations for the tangent vectors at p, each of which is suggestive in its own way. If we think of T_pM as the set (vector space) of equivalence classes of curves at p, then we shall denote its elements by

$$\left.\frac{d\alpha}{dt}\right|_{t=0}$$

and if we think of T_pM as the (vector) space of derivations at p, then we shall denote its elements as

$$a_1 \left.\frac{\partial}{\partial x_1}\right|_p + \cdots + a_m \left.\frac{\partial}{\partial x_m}\right|_p$$

the meaning being that

$$\left.\frac{d\alpha}{dt}\right|_{t=0} = a_1 \left.\frac{\partial}{\partial x_1}\right|_p + \cdots + a_m \left.\frac{\partial}{\partial x_m}\right|_p \qquad \iff \qquad D_{[\alpha]_p} = a_1 D_{[\alpha_1]_p} + \cdots + a_m D_{[\alpha_m]_p}.$$

We shall adopt a similar convention with the elements of (the cotangent space) T_p^*M: $(df)_p$ is the equivalence class of f in T_p^*M, with the understanding that

$$(df)_p \cdot \left.\frac{d\alpha}{dt}\right|_{t=0} = \langle [f]_p, [\alpha]_p \rangle = \left.\frac{d}{dt} f \circ \alpha(t)\right|_{t=0}.$$

In particular, then $(dx_1)_p, \cdots, (dx_m)_p$ denotes the *dual basis* of $\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_m}\right|_p$.

Note : The definition of the tangent space T_pM uses only (the algebra) $\mathbf{F}(p)$, not all M; thus if U is any open subset of M containing p, then T_pU and T_pM are *naturally identified.* Also, recall that $T_p\mathbb{E}^m = \{p\} \times \mathbb{E}^m$ is commonly identified with (the vector space) $\mathbb{R}^m$. We can write, for $U \subseteq \mathbb{E}^m$ (open),

$$T_pU = T_p\mathbb{E}^m = \{p\} \times \mathbb{E}^m = \mathbb{R}^m.$$

◇ **Exercise 263** Let $U \subseteq \mathbb{E}^m$ be open and let $f : U \to \mathbb{R}$ be a smooth function. Compare $Df(p)$ and $(df)_p$ for $p \in U$.

Tangent mappings (differentials)

For every smooth mapping $F : \mathbb{E}^m \to \mathbb{E}^n$ between Euclidean spaces and any point $p \in \mathbb{E}^m$, the derivative of F at p is a linear mapping $DF(p) : T_p\mathbb{E}^m = \mathbb{R}^m \to T_{F(p)}\mathbb{E}^n = \mathbb{R}^n$. Now that we have tangent spaces to manifolds, we are ready to associate analogous (linear) mappings (between tangent spaces) to smooth mappings (between manifolds).

Let M and N be smooth manifolds, and $\Phi : M \to N$ a smooth mapping. We have already mentioned that Φ *pulls back* smooth functions on N into smooth functions on M. However, for smooth curves the situation is different : for any smooth curve σ on M, $\Phi \circ \sigma$ is a smooth curve on N. Thus Φ *pushes forward* curves on M into curves on N. We shall write $\Phi_*\sigma$ for the curve $\Phi \circ \sigma$. Both the push-forward Φ_* and the pull-back Φ^* induce linear mappings between tangent spaces and cotangent spaces, respectively.

5.3.3 Definition. Suppose $\Phi : M \to N$ is a smooth mapping between manifolds and $p \in M$. The **tangent mapping** $\Phi_{*,p} : T_pM \to T_{\Phi(p)}N$ (of Φ at p) is defined by

$$\Phi_{*,p} : [\alpha]_p \mapsto [\Phi_*\alpha]_{\Phi(p)}.$$

◇ **Exercise 264** Show that the tangent mapping $\Phi_{*,p}$ is well-defined and is linear.

It is immediate that if $\Phi : M \to M$ is the identity, then $\Phi_{*,p} : T_pM \to T_pM$ is the identity isomorphism.

$\diamond$ **Exercise 265** Suppose that $\Phi : M \to N$ and $\Psi : N \to P$ are smooth mappings between manifolds and $p \in M$. Verify that

$$(\Psi \circ \Phi)_{*,p} = \Psi_{*,\Phi(p)} \circ \Phi_{*,p}.$$

Note : The linear mapping $\Phi_{*,p} : T_pM \to T_{\Phi(p)}N$ is often called the *differential* of Φ at p. One frequently sees other notations for $\Phi_{*,p}$, for example $(d\Phi)_p, \Phi'(p)$, or $T_p\Phi$. The $*$ is a subscript since the mapping is in the same "direction" as Φ (i.e. from M to N).

Recall from linear algebra that every linear mapping $\Phi = \Phi_* : V \to W$ between vector spaces induces a *dual* (linear) mapping $\Phi^* : W^* \to V^*$ by the prescription

$$\begin{aligned}(\Phi^*\lambda)(v) &= \lambda(\Phi_*(v)) \\ &= \lambda \circ \Phi(v) \quad \text{for } v \in V \text{ and } \lambda \in W^*\end{aligned}$$

(or, if one prefers, $\langle \Phi^*(\lambda), v\rangle = \langle \lambda, \Phi_*(v)\rangle$).

Note : The definition of Φ^* does *not* require the choice of a basis; therefore Φ^* is *naturally* (or canonically) determined by Φ_*. The vector spaces V and V^* have the same dimension, thus they must be isomorphic. There is no natural isomorphism; however, we do have the following property : *There is a natural isomorphism between V and $(V^*)^*$ given by $v \mapsto \langle \cdot, v\rangle$ (i.e. v is mapped to the linear functional on V^* whose value on any $\lambda \in V^*$ is $\lambda(v) = \langle \lambda, v\rangle$).* Observe that the mapping $(v, \lambda) \mapsto \langle \lambda, v\rangle$ is *bilinear* (i.e. linear in each variable separately). This shows that the dual of V^* is V itself, accounts for the name "dual" space, and validates the use of the symmetric notation $\langle \lambda, v\rangle$ in preference to the functional notation $\lambda(v)$.

We make the following definition.

5.3.4 Definition. Suppose $\Phi : M \to N$ is a smooth mapping between manifolds and $p \in M$. The **cotangent mapping** $\Phi_p^* : T^*_{\Phi(p)}N \to T^*_pM$ (of Φ at p) is the dual of the tangent mapping $\Phi_{*,p} : T_pM \to T_{\Phi(p)}N$ (i.e. $\Phi_p^* = (\Phi_{*,p})^*$).

The cotangent mapping $\Phi_p^* : T^*_{\Phi(p)}N \to T^*_pM$ is defined by

$$\Phi_p^* : [f]_{\Phi(p)} \mapsto [\Phi^* f]_p.$$

NOTE : The foregoing mapping (between cotangent spaces) is well-defined and acts like the dual of the tangent mapping (between tangent spaces).

◇ **Exercise 266** Suppose that $\Phi : M \to N$ and $\Psi : N \to P$ are smooth mappings between manifolds and $p \in M$. Verify that

$$(\Psi \circ \Phi)^*_p = \Psi^*_{\Phi(p)} \circ \Phi^*_p.$$

In terms of the admissible charts (U, ϕ) at $p \in M$ and (V, ψ) at $\Phi(p) \in N$, we have the following formulas. Let

$$v = \frac{d}{dt}\phi \circ \alpha(t)\Big|_{t=0}, \qquad w = \frac{d}{dt}\psi \circ \Phi \circ \alpha(t)\Big|_{t=0}$$

and

$$\phi_i(x_1, \ldots, x_m) = \psi_i \circ \Phi \circ \phi^{-1}(x_1, \ldots, x_m), \qquad i = 1, 2, \ldots, n.$$

Then the *local representation* of the tangent mapping $\Phi_{*,p}$ is

$$w_i = \frac{\partial \Phi_i}{\partial x_1} v_1 + \cdots + \frac{\partial \Phi_i}{\partial x_m} v_m, \qquad i = 1, 2, \ldots, n.$$

We can write (in compact form)

$$\begin{aligned} w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} &= \begin{bmatrix} \frac{\partial \Phi}{\partial x_1} & \cdots & \frac{\partial \Phi}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial \Phi}{\partial x_1} & \cdots & \frac{\partial \Phi}{\partial x_m} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \\ &= \frac{\partial \Phi}{\partial x} v. \end{aligned}$$

In order to get an analogous expression for the cotangent mapping, let f be a smooth function on N at $\Phi(p)$, and g its pull-back $\Phi^* f$. Denote

$$g(x_1, \ldots, x_m) = \Phi^* f \circ \phi^{-1}(x_1, \ldots, x_m) \quad \text{and} \quad f(y_1, \ldots, y_n) = f \circ \psi^{-1}(y_1, \ldots, y_n).$$

Then $g(x_1, \ldots, x_m) = f(\Phi_1(x), \ldots, \Phi_n(x))$, and hence

$$\frac{\partial g}{\partial x_i} = \frac{\partial \Phi_1}{\partial x_i}\frac{\partial f}{\partial y_1} + \cdots + \frac{\partial \Phi_n}{\partial x_i}\frac{\partial f}{\partial y_n}, \qquad i = 1, 2, \ldots, m.$$

Likewise, we can write (in compact form)

$$\frac{\partial g}{\partial x} = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial y_1} & \cdots & \frac{\partial f}{\partial y_n} \end{bmatrix} \begin{bmatrix} \frac{\partial \Phi}{\partial x_1} & \cdots & \frac{\partial \Phi}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial \Phi}{\partial x_1} & \cdots & \frac{\partial \Phi}{\partial x_m} \end{bmatrix}$$

$$= \frac{\partial f}{\partial y}\frac{\partial \Phi}{\partial x}.$$

. . . .

The tangent bundle and the cotangent bundle

It is natural to assemble all tangent spaces of a (smooth) manifold together into a new structure – and conceivably, this set should again have a natural manifold structure. (We will omit some of the more technical details of this structure.) As discussed earlier, it is desirable to distinguish between tangent vectors at different points.

Let M be a smooth n-manifold, and consider the *set*

$$TM := \left\{ (p, X_p) \in M \times \bigcup_{p \in M} T_pM \,|\, X_p \in T_pM \right\}$$

which is the (disjoint) union of all tangent spaces to M at all points $p \in M$. Let

$$\pi : TM \to M, \qquad (p, X_p) \mapsto p$$

be the *projection* onto M. The *fibre* over $\in M$ is the preimage $\pi^{-1}(p) = \{p\} \times T_pM$. (Occasionally it is convenient to identify the fibre $\pi^{-1}(p)$ with the tangent space T_pM – technically, this includes a tacit projection onto the second factor.) We call TM the **tangent bundle** of M.

Note : To illustrate the natural manifold structure of tangent bundles, consider the example of $M = \mathbb{S}^1$. The *naive* collection of all *tangent lines* to the (embedded) circle $\mathbb{S}^1 \subseteq \mathbb{E}^2$ is full of intersections. More suitable for our purposes is to embed the circle in $\mathbb{E}^3$ as $\{x \in \mathbb{E}^3 \,|\, x_1^2 + x_2^2 = 1,\ x_3 = 0\}$ and *attach* at every point $p \in \mathbb{S}^1$ a *vertical* line, yielding a cylinder. As a set, this cylinder is in bijection with the (disjoint) collection of all *tangent lines* to the circle (embedded in the plane). It is clear that one can consistently choose an orientation of the lines (and even more a consistent scaling). Intuitively, identify the *naive tangent vector* $((\cos\theta, \sin\theta), (-L\sin\theta, L\cos\theta))$ with the point $(\cos\theta, \sin\theta, L) \in \mathbb{E}^3$.

In complete analogy, we may intuitively think of the tangent bundle $T\mathbb{R}$ of the real line $\mathbb{R}$ as $\mathbb{R}^2$. However, for dimensional reasons it is clear that these two examples are the only tangent bundles amenable to such immediate visualization. How quickly things get complicated becomes clear if one tries to think of $T\mathbb{S}^2$ as a sphere with a (different) planes attached to each of its points. A *vector field* on the sphere simply selects one point on each plane. However, from algebraic topology it is known that *there does not exist any continuous vector field on the sphere that vanish nowhere.* In our picture this means that it is impossible to continuously select one point on each *tangent plane* avoiding the origin (zero vector) in each $T_p\mathbb{S}^2$. Intuitively, $T\mathbb{S}^2$ must be nontrivially twisted (when compared to e.g. $T\mathbb{S}^1$ which is the very tame cylinder) and hence must be very different from the *trivial* Cartesian product $\mathbb{S}^2 \times \mathbb{R}^2$.

The set TM has a canonical (smooth) manifold structure of dimension $2n$.

NOTE : The key idea is that locally, above an admissible chart (U, ϕ), the tangent bundle "looks like" $\mathbb{R}^m \times \mathbb{R}^m = \mathbb{R}^{2m}$. This observation is captured in the concept of *local triviality* (compare the later short note on vector bundles). Thus the topology and geometry of M are captured, in the global structure of the tangent bundle, by how the trivial bundles are pieced together with *twists*.

Starting with a (smooth) atlas on M, we shall find it easy to obtain a candidate (smooth) atlas on TM. This can be done as follows. Let $(U_\alpha, \phi_\alpha) \in \{(U_\alpha, \phi_\alpha)\}_{\alpha\in\mathfrak{A}}$ be an admissible chart on M with $p \in U_\alpha$, and consider the set

$$\begin{aligned} TU_\alpha &:= \pi^{-1}(U_\alpha) \\ &= \text{the (disjoint) union of all } T_xM \text{ with } x \in U_\alpha. \end{aligned}$$

To any element $(p, v) \in TU_\alpha \subseteq TM$, where $v = X_p \in T_pM$, we associate the point

$$\big(\phi_\alpha(p), \bar{\phi}_\alpha(v)\big) \in \phi_\alpha(U_a) \times \mathbb{R}^m \subseteq \mathbb{R}^{2m}$$

where $\bar{\phi}_\alpha : T_pM \to \mathbb{R}^m$ is the linear isomorphism asoociated with (U_α, ϕ_α) at p. The mapping

$$T\phi_\alpha : TU_\alpha \to \mathbb{R}^{2m}, \qquad (p, v) \mapsto \big(\phi_\alpha(p), \bar{\phi}_\alpha(v)\big)$$

is one-to-one and onto an open subset $\phi_\alpha(U_\alpha) \times \mathbb{R}^m$ of $\mathbb{R}^{2m}$. We claim that *the family (of charts)* $\{(TU_\alpha, T\phi_\alpha)\}_{\alpha \in \mathfrak{A}}$ *is a smooth atlas on* TM, determining a smooth structure.

◇ **Exercise 267** Verify the preceding statement.

The induced canonical topology on TM is such that all the coordinate mappings $T\phi_\alpha : TU_\alpha \to T\phi_\alpha(TU_\alpha) \subseteq \mathbb{R}^{2m}$ are homeomorphisms (in fact, it is the weakest topology on TM with this property).

Note : Alternatively, the canonical topology on the tangent bundle TM can be characterized as the strongest topology under which the projection mapping $\pi : TM \to M$ is continuous.

Recall that, in order for TM to qualify as a smooth manifold, we still need that the (canonical) topology is reasonably *nice* – Hausdorff and second countable. (It is easy to see that the topology is Hausdorff; however, the second condition is rather tricky, and we shall skip the details.)

◇ **Exercise 268** Show that (as a mapping between smooth manifolds) the projection mapping $\pi : TM \to M$ is smooth.

5.3.5 Example. If an m-dimensional vector space V is regarded as a (smooth) manifold (see Example 4.2.7), then the tangent bundle TV is *isomorphic* to $V \times V$.

Note : It is often convenient to replace $\phi_\alpha(U_\alpha) \times \mathbb{R}^m$ with $U_\alpha \times \mathbb{R}^m$, identifying $T\phi_\alpha$ with the mapping $v \mapsto (p, \bar{\phi}_\alpha(v))$. (This minor abuse of notation turns out to be a major convenience.) For each $\alpha \in \mathfrak{A}$, we get a commutative diagram

$$\begin{array}{ccc} TU_\alpha & \xrightarrow{T\phi_\alpha} & U_\alpha \times \mathbb{R}^m \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle pr_1} \\ U_\alpha & \xrightarrow[id]{} & U_\alpha \end{array}$$

where pr_1 denotes projection on the first factor and $T\phi_\alpha$ is a diffeomorphism that restricts to be a linear isomorphism $T_pM \to \{p\} \times \mathbb{R}^m$ for every $p \in U_\alpha$. Thus, TM *is "locally" a Cartesian product of* M *and* $\mathbb{R}^m$*, the projection* π *being "locally" the projection of the Cartesian product onto the first factor, and the fiber* $\pi^{-1}(p) = T_pM$ *has a canonical vector space structure, for every* $p \in M$.

Tangent bundles are examples of vector bundles. (Vector bundles play a very important role in manifold theory.)

NOTE : Let M be a smooth m-manifold, E a smooth $(m+k)$-dimensional manifold, and $\pi : E \to M$ a smooth mapping. The triple (E, M, π) is called a **vector bundle** over M (of *fibre dimension* k) if the following properties hold.

(VB1) For each $p \in M$, the *fibre* $E_p := \pi^{-1}(p)$ has the structure of a (real) k-dimensional vector space.

(VB2) For each $p \in M$, there exist an open neighborhood W and a (smooth) diffeomorphism $\zeta : \pi^{-1}(W) \to W \times \mathbb{R}^k$ such the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(W) & \xrightarrow{\zeta} & W \times \mathbb{R}^k \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle pr_1} \\ W & \xrightarrow[id]{} & W \end{array}$$

(Any such pair $(\pi^{-1}(W), \zeta)$ is called a (vector) *bundle chart* on (E, M, π).)

(VB3) For each $p \in W$, the restriction

$$\zeta_p = \zeta|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$$

is a linear isomorphism.

We call E the *total space*, M the *base space*, and π the *bundle projection.* We shall denote a vector bundle (over M), simply, $\pi : E \to M$. An obvious example of a vector bundle is given by $pr_1 : M \times \mathbb{R}^k \to M$. Here $(M \times \mathbb{R}^k, id)$ is a global bundle chart and the vector bundle is said to be *trivial.*

Given two vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ over M, a (vector) *bundle isomorphism* is a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ M & \xrightarrow[id]{} & M \end{array}$$

such that φ is a (smooth) diffeomorphism, and carries E_{1p} isomorphically (as a vector space) onto E_{2p}, for every $p \in M$.

So far, the only real eaxamples of vector bundles that we have seen are the tangent bundles and trivial bundles. The following is the least complicated example of a nontrivial vector bundle. We give an example of a "line" bundle (i.e. of fibre dimension 1) over the circle, known as the *Möbius bundle*. On $\mathbb{R} \times \mathbb{R}$, define the equivalence relation $(s,t) \sim (s+n, (-1)^n t)$, $n \in \mathbb{Z}$. Observe that $t \mapsto (-1)^n t$ is a linear automorphism of $\mathbb{R}$. The projection $(s,t) \mapsto s$ passes to a well defined mapping $\pi : (\mathbb{R} \times \mathbb{R})/_{\sim} \to \mathbb{R}/\mathbb{Z} = \mathbb{S}^1$. It should be clear, intuitively, that this is a vector bundle over $\mathbb{S}^1$ of fibre dimension 1, but a rigorous proof of this involves checking many details.

Beyond vector bundles are *fibre bundles* in which the fibres need not necessarily be vector spaces. Arguably the most important such fibre bundle is the *principal bundle* in which each fibre is a copy of the general linear group $\mathsf{GL}\,(k, \mathbb{R})$. (Differential geometry may be described as *the study of a connection on a principal bundle.*)

In complete analogy to the tangent bundle we assemble all cotangent spaces T_p^*M into the **cotangent bundle**, denoted T^*M. It is a vector bundle (of fibre dimension m) over the (smooth) m-manifold M with bundle projection again denoted by π. The set (total space)

$$T^*M := \left\{ (p, \omega_p) \in M \times \bigcup_{\in M} T_p^*M \,|\, \omega_p \in T_p^*M \right\}$$

has a (smooth) manifold structure of dimension $2m$, given by the (smooth) atlas $\{(T^*U_\alpha, T^*\phi_\alpha)\}_{\alpha \in \mathfrak{A}}$ where $T^*U_\alpha = \pi^{-1}(U_\alpha) \subseteq T^*M$ and

$$T^*\phi_\alpha : (p, \omega) \mapsto (\phi_\alpha(p), \underline{\phi}_\alpha(\omega)) \in \phi_\alpha(U_\alpha) \times (\mathbb{R}^m)^* \subseteq \mathbb{R}^{2m}$$

($\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \mathfrak{A}}$ is an atlas on M).

5.3.6 Example. If an m-dimensional vector space V is regarded as a (smooth) manifold, then the cotangent bundle T^*V is *isomorphic* to $V \times V^*$.

Note : One can show that the tangent and cotangent bundles are isomorphic, but not canonically. We *do not* identify these (vector) bundles.

5.4 Smooth Submanifolds

. . . .

5.5 Vector Fields

Vector fields and flows

Let M be a manifold and let TM be the corresponding tangent bundle.

5.5.1 DEFINITION. A **vector field** X (on M) is a mapping from M into TM such that for each $p \in M$ the natural projection $\pi : TM \to M$ projects $X(p)$ to p (i.e. the compositon $\pi \circ X$ is the identity on M).

Rather than considering arbitrary such mappings, our interest is primarily in those that vary smoothly. (In topological considerations continuity may suffice.) Since a vector field is defined as a mapping between (smooth) manifolds M and TM, we already have a notion of smoothness : We say that X is a *smooth* vector field provided $X : M \to TM$ is a smooth mapping. We shall write $\mathfrak{X}(M)$ for the set of all smooth vector fields on M.

NOTE : A *section* of the vector bundle $\pi : E \to M$ is a smooth mapping $s : M \to E$ such that $\pi \circ s = id_M$. The set of all such sections is denoted by $\Gamma^\infty(E)$. It is easy to verify that *the set* $\Gamma^\infty(E)$ *is a* $C^\infty(M)$*-module* under the natural (pointwise) operations of addition and (function) multiplication.

Thus, $\mathfrak{X}(M) = \Gamma^\infty(TM)$. In complete analogy (to the tangent bundle), the $C^\infty(M)$-module of all smooth *covector fields* on M is

$$A^1(M) := \Gamma^\infty(T^*M).$$

Covector fields are also (and more commonly) called **differential 1-forms**. If $\omega \in A^1(M)$, then $\omega : M \to T^*M$ is writen as

$$p \mapsto \omega_p \in T_p^*M.$$

NOTE : Having defined these objects in an "intrinsic" way, let us now examine their meaning in a more intuitive way. It is well known in physics that the position of a particle is a scalar-like quantity, and its velocity is a vector quantity. Therefore, if $t \mapsto p(t)$ is a curve that describes the position, then the velocity $\frac{dp}{dt}$ is a different object, since it is a vector. These two objects "live" in different spaces. The manifolds formalism clarifies this issue, and it provides a natural point of view from which differential equations (systems) should be studied.

If $\dot{x} = F(x)$ is a differential equation in $\mathbb{R}^m$, then F *cannot* be viewed as a mapping from $\mathbb{R}^m$ into $\mathbb{R}^m$. Rather, it must be viewed as a mapping form $\mathbb{R}^m$ (in fact, $\mathbb{E}^m$) into the tangent bundle of $\mathbb{R}^m$, since $F(x(t))$ is equal to the tangent vector of the curve $x(\cdot)$ at $x(t)$. For equations in $\mathbb{R}^m$, it is easy to confuse mappings and vector fields (in much the same way as it is to confuse vectors with their duals). It is only on arbitrary manifolds that the genuine differences of these objects become apparent.

Each vector field $X \in \mathfrak{X}(M)$ in some admissible chart (U, ϕ) becomes an expression of the form

$$X_1(x_1, \dots, x_m)\frac{\partial}{\partial x_1} + X_2(x_1, \dots, x_m)\frac{\partial}{\partial x_2} + \cdots + X_m(x_1, \dots, x_m)\frac{\partial}{\partial x_m}\cdot$$

The (smooth) functions $X_1, \dots, X_m$ are called the *coordinate functions* of the vector field X. (Strictly speaking, X should be expressed in terms of $2m$ coordinates; however, because the first m coordinates contain redundant information, they are suppressed.)

Let $\alpha : J = (a, b) \to M$ be a smooth curve on the manifold M. Then the *tangent vector* to α at $t \in I$ is given by

$$\dot{\alpha}(t) := \alpha_{*,t}\left(\frac{d}{dt}\right) \in T_{\alpha(t)}M.$$

(Thus $\dot{\alpha} : J \to TM$ is a smooth curve in TM, commonly referred to as the *lift* of α.) Let X be a smooth vector field on M. A smooth curve $\alpha : J \to M$ is an *integral curve* of X provided the tangent vector to α at each $t \in J$ equals the value of X at $\alpha(t)$ (i.e. $\dot{\alpha}(t) = X(\alpha(t))$ for each $t \in J$). Thus the accompanying diagram

$$\begin{tikzcd} & TM \\ I \arrow[ur, "\dot{\alpha}"] \arrow[r, "\alpha"'] & M \arrow[u, "X"'] \end{tikzcd}$$

is commutative. (The lift $\dot{\alpha}$ of α coincides with $X \circ \alpha$.)

Let (U, ϕ) be an admissible chart on M and let $\alpha : J \to U \subseteq M$ be a smooth curve as before.

$\diamond$ **Exercise 269** Verify that (for $t \in J$)

$$\dot{\alpha}(t) = \frac{dx_1}{dt}(t)\, \frac{\partial}{\partial x_1}\bigg|_{\alpha(t)} + \cdots + \frac{dx_m}{dt}(t)\, \frac{\partial}{\partial x_m}\bigg|_{\alpha(t)}$$

where $x_i = \phi_i \circ \alpha, \quad i = 1, 2, \ldots, m.$

Then $\dot{\alpha} = X \circ \alpha$ yields

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_m), \qquad i = 1, 2, \ldots, m.$$

This system of differential equations admits solution curves in the open set $\phi(U)$. That is, through each point x_0 in $\phi(U)$ there exists a solution curve $x(\cdot) : J_0 \to \phi(U) \subseteq \mathbb{R}^m$ that passes through x_0 at $t = 0$ (i.e. $x(0) = x_0$). Any two such solutions curves agree for values of t for which they are both defined. It follows from the theory of differential equations that *for each* x_0 *there exist a maximum open interval* J_{max} *(that contains* 0 *) and a unique solution curve* $x(\cdot) : J_{max} \to \mathbb{R}^m$ *such that* $x(0) = x_0$. We shall refer to such a solution curve as *the* solution curve through x_0.

Any solution curve $x(\cdot)$ in $\phi(U)$ defines an integral curve

$$t \mapsto p(t) = \phi^{-1}(x_1(t), \ldots, x_m(t))$$

on M.

Note : Consider another admissible chart (V, ψ) on M such that $p(t_0) \in U \cap V$ for some t_0. We denote by $(y_1, \ldots, y_m)$ the coordinates on V, and by $Y_1, \ldots, Y_m$ the coordinate functions of X relative to (V, ψ). The curve $t \mapsto y(t) = \psi \circ p(t)$ is a (smooth) curve in $\psi(V)$ defined in some neighborhood of t_0. Furthermore, $y(t) = \psi \circ \phi^{-1}(x(t))$ and

$$\begin{aligned} \frac{dy_i}{dt} &= \frac{\partial y_i}{\partial x_1}(x(t))\frac{dx_1}{dt} + \cdots + \frac{\partial y_i}{\partial x_m}(x(t))\frac{dx_m}{dt} \\ &= \frac{\partial y_i}{\partial x_1}(x(t))X_1(x(t)) + \cdots + \frac{\partial y_i}{\partial x_m}(x(t))X_m(x(t)). \end{aligned}$$

Because $(Y_1, \ldots, Y_m)$ and $(X_1, \ldots, X_m)$ are the coordinates of the same tangent vector $X(p)$, they are related through

$$Y_i(y) = \frac{\partial y_i}{\partial x_1} X_1(x) + \cdots + \frac{\partial y_i}{\partial x_m} X_m(x), \quad i = 1, 2, \ldots, m.$$

Therefore, $y(\cdot)$ is a solution curve of the system of differential equations

$$\frac{dy_i}{dt} = Y_i(y_1, \ldots, y_m), \qquad i = 1, 2, \ldots, m.$$

Let $\bar{y}(\cdot)$ be the solution curve of this differential system in $\psi(V)$ that passes through $y_0 = \psi \circ p(t_0)$ at $t = t_0$, and denote $\bar{p}(t) = \psi^{-1} \circ \bar{y}(t)$. It then follows that the two integral curves $p(\cdot)$ and $\bar{p}(\cdot)$ on M agree at all values of t for which they are both defined.

5.5.2 DEFINITION. We say that an integral curve $\gamma = \gamma_p$ of $X \in \mathfrak{X}(M)$ is *the* **integral curve** through $p \in M$ provided $\gamma_p(0) = p$ and the domain $J_p \subseteq \mathbb{R}$ of γ_p is maximal.

That is, if α is any integral curve of X that satisfies $\alpha(0) = p$, then its domain can be extended to J_p so that $\alpha(t) = \gamma_p(t)$ for all t.

A (smooth) vector field X is called *complete* if the integral curves γ_p through each point $p \in M$ are defined for *all* values of $t \in \mathbb{R}$. In such case, X is said to define a *flow* $\Phi = \Phi^X$ on M.

NOTE : A *flow* on M is a smooth mapping $\Phi : \mathbb{R} \times M \to M$ such that (for all $t_1, t_2 \in \mathbb{R}$ and all $p \in M$)

(FL1) $\Phi(0, p) = p$.

(FL2) $\Phi(t_1 + t_2, p) = \Phi(t_1, \Phi(t_2, p))$.

(If we fix p and let t vary, we get a smooth curve $\Phi(\cdot, p)$ in M; thus as t varies each point of M moves smoothly inside M, and various points move in a coherent fashion, so that we can form a mental picture of them "flowing" through M, each point along its individual path.) For each $t \in \mathbb{R}$, the (smooth) mapping

$$\varphi_t : M \to M, \qquad p \mapsto \Phi(t, p)$$

is a smooth diffeomorphism of M. We have $\varphi_0 = id_M$ and (for all $t_1, t_2 \in \mathbb{R}$)

$$\varphi_{t_1 + t_2} = \varphi_{t_1} \circ \varphi_{t_2}.$$

Hence the collection $\{\varphi_t \mid t \in \mathbb{R}\}$ forms a group under the composition of mappings. Such a group is called a *one-parameter group of diffeomorphisms* of M (or a smooth *action* of $\mathbb{R}$ on M) and is denoted by $\{\varphi_t\}$ or, simply, by φ_t.

The flow Φ^X (generated by the complete vector field X) is defined by

$$\Phi^X(t,p) := \gamma_p(t).$$

We shall also use $\exp tX$ to denote the mapping (diffeomorphism) $\varphi_t = \varphi_t^X$. (Each notation is fairly standard, and each has different merits, depending on the context.)

Each (smooth) flow Φ on M is generated by a vector field X, called the *infinitesimal generator* of Φ. The relation between X and Φ is given by

$$X(p) = \left.\frac{d}{dt}\Phi(t,p)\right|_{t=0}.$$

($X(p) \in T_pM$ is the value of the lift of $\Phi(\cdot,p) : \mathbb{R} \to M$ at $t=0$.) Therefore, *there is a one-to-one correspondence between complete vector fields and flows.*

NOTE : The *support* of a vector field X is the closure of the set $\{p \in M \mid X(p) \neq 0\}$. It can be shown that *every vector field with compact support on M is complete.* So on a compact manifold M, each vector field is complete. If M is not compact and of dimension ≥ 2 the set of complete vector fields is not even a vector space as the following example (on $\mathbb{E}^2$) shows : the vector fields

$$X = x_2\frac{\partial}{\partial x_1} \quad \text{and} \quad Y = \frac{x_1^2}{2}\frac{\partial}{\partial x_2}$$

are complete, but $X + Y$ is not.

$\diamond$ **Exercise 270** Show that the (smooth) vector field

$$X = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}$$

is complete (on $\mathbb{E}^2$). Is the vector field

$$Y = e^{-x_1}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$$

complete ?

◇ **Exercise 271** Consider the (smooth) vector field on $\mathbb{E}^3$ defined by

$$X = x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} \cdot$$

Find the integral curve γ of X so that $\gamma(0) = (-1, 1, 1)$.

We have seen that not every vector field is complete. If this is the case, then $X \in \mathfrak{X}(M)$ generates (only) a *local flow* on M.

5.5.3 Example. Let $M = \mathbb{E}^2$ and let (the flow) $\Phi : \mathbb{R} \times M \to M$ be defined by

$$(t, (x_1, x_2)) \mapsto (x_1 + t, x_2).$$

Then the infinitesimal generator is $X = \frac{\partial}{\partial x_1} \cdot$ Suppose now that we remove the origin $(0, 0)$ from $\mathbb{E}^2$; let $M_0 = \mathbb{E}^2 \setminus \{(0, 0)\}$. For most points (the diffeomorphism) φ_t is defined as before; however, we cannot obtain an action of $\mathbb{R}$ on M_0 by restriction of Φ to $\mathbb{R} \times M_0$ since points of the (closed) set

$$\{(t, (x_1, 0)) \,|\, x_1 + t = 0\} = \Phi^{-1}((0, 0)) \subseteq \mathbb{R} \times M$$

are mapped by Φ to the origin. On the other hand, let $W \subseteq \mathbb{R} \times M_0$ be the open set defined by

$$W = \left(\bigcup_{x_2 \neq 0} \mathbb{R} \times \{(x_1, x_2)\} \right) \cup \{(t, (x_1, 0)) \,|\, x_1(x_1 + t) > 0\}.$$

Then $\Phi = \Phi|_W$ maps W onto M_0 and preserves many of the features of Φ which we have used. For example, let $p = (x_1, x_2) \in M_0$. Then

- $(0, p) \in W$ and $\Phi(0, p) = p$
- $\Phi(t_1, \Phi(t_2, p)) = \Phi(t_1 + t_2, p)$

if all terms are defined, and the infinitesimal generator is again $X = \frac{\partial}{\partial x_1} \cdot$ Finally, we have *orbits* $t \mapsto \Phi(t, p)$, which are the lines $x_2 =$ constant (as before) when $p = (x_1, x_2)$, $x_2 \neq 0$, and for $p = (x_1, 0)$ the portion of the x_1-axis minus the origin which contains p. This curve is not defined for all values of t in the case of the orbit of a point on the x_1-axis.

NOTE : In order to define the local flow of a vector field at $p \in M$, it is first necessary to define the escape times of the integral curve γ_p of X through p. The *positive escape time* $e^+(p)$ is defined to be the supremum of t such that an integral curve passing through p can be defined at t. The *negative escape time* $e^-(p)$ is defined similarly. Let $W := \{(t,p) \,|\, e^-(p) < t, e^+(p)\}$. Then W is an open subset of $\mathbb{R} \times M$ and a neighborhood of $\{0\} \times M$. The *local flow* Φ of X is defined on W and it satisfies the following :

- The mapping $\Phi : W \subseteq \mathbb{R} \times M \to M$ is smooth
- $\Phi(0,p) = p$ for all $p \in W$.
- $\Phi(t_1 + t_2, p) = \Phi(t_1, \Phi(t_2, p))$ whenever each of (t_1, p) and $(t_1, \Phi(t_2, p))$ is contained in W.
- $\frac{d\Phi}{dt}(t,p) = X \circ \Phi(t,p)$.

5.5.4 EXAMPLE. Let $M = \mathbb{E}^m$, and let

$$X : x \mapsto a = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}^m \ \big(= \mathbb{R}^{m \times 1}\big)$$

be a *constant* (or parallel) vector field on M. Then (in the "derivation notation")

$$X(x) = a_1 \left.\frac{\partial}{\partial x_1}\right|_x + \cdots + a_m \left.\frac{\partial}{\partial x_m}\right|_x .$$

The integral curves of X are parallel lines, all in the direction of a. For each t, the mapping (diffeomorphism) $\varphi_t : t \mapsto \Phi(t,x)$ is a translation of x by ta. Hence $\{\varphi_t\}$ is a one-parameter group of translations on $\mathbb{E}^m$.

5.5.5 EXAMPLE. Let $M = \mathbb{E}^m$ and $A \in \mathbb{R}^{m \times m}$. Let

$$X : x = (x_1, \ldots, x_m) \mapsto Ax := \begin{bmatrix} a_{11}x_1 + \cdots + a_{1m}x_m \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mm}x_m \end{bmatrix} \in \mathbb{R}^m \ \big(= \mathbb{R}^{m \times 1}\big)$$

be a *linear* vector field on M. So

$$X(x_1, \ldots, x_m) = X_1(x_1, \ldots, x_m) \left.\frac{\partial}{\partial x_1}\right|_x + \cdots + X_m(x_1, \ldots, x_m) \left.\frac{\partial}{\partial x_m}\right|_x$$

with coordinate functions given by

$$X_i(x_1, \ldots, x_m) = a_{i1}x_1 + \cdots + a_{im}x_m, \qquad i = 1, 2, \ldots, m.$$

Each integral curve of X is of the form $t \mapsto \exp(tA)x$, where $\exp(tA) = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$ (the matrix exponential of tA). Thus $\varphi_t(x) = \exp(tA)x$, and therefore (the one-parameter group of diffeomorphisms) $\{\varphi_t\}$ is a subgroup of the group of all linear transformations on $\mathbb{R}^n$ (i.e. a matrix group). Here are two familiar cases (for $n = 2$):

- $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $\exp(tA) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. (The one-parameter group $\{\varphi_t\}$ is the rotation group $\mathsf{SO}(2)$, and the integral curves are concentric circles centered at the origin.)
- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\exp(tA) = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$. (The one-parameter group $\{\varphi_t\}$ is a subgroup of $\mathsf{SL}(2, \mathbb{R})$, and the integral curves are hyperbolas.)

5.5.6 EXAMPLE. Let $M = \mathbb{E}^3$ and consider the vector field (on M)

$$X : x \mapsto X(x) := Ax + a$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then

$$\begin{aligned} \Phi(t, x) = \varphi_t(x) &= \exp(tA)x + ta \\ &= \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} x + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Integral curves are helices (with centers along the x_3-axis).

5.5.7 EXAMPLE. Let $M = \mathsf{GL}^+(2, \mathbb{R})$. For $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, let X be the vector field on M defined by $p \mapsto Ap$. Then

$$\Phi(t, p) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} p, \quad p \in M.$$

$\varphi_t(p)$ is the matrix multiplication of $p \in M$ by $\exp(tA)$ from the left for each $t \in \mathbb{R}$.

Vector fields as differential operators

Let M be a manifold and let TM be the corresponding tangent bundle. The algebra of smooth functions on M is denoted by $C^\infty(M)$ (see **Execise 224**).

Recall that tangent vectors act on smooth functions and produce directional derivatives. Specifically, if $X_p = \frac{d\alpha}{dt}\big|_{t=0} \in T_pM$ and $f \in C^\infty(M)$, then

$$X_p f = \frac{d}{dt} f \circ \alpha(t)\bigg|_{t=0} \in \mathbb{R}$$

is the directional derivative of f along X_p.

⋄ **Exercise 272** Given a mapping $X : M \to TM$, show that the following statements are logically equivalent :

(a) X is smooth (as a mapping between manifolds). In other words, X is a smooth vector field on M.

(b) For each admissible chart (U, ϕ) on M, the coordinate functions $X_i : U \to \mathbb{R}$ of X are smooth.

(c) For each smooth function $f : M \to \mathbb{R}$, the function $x \mapsto X(x)f$ is also smooth.

Smooth vector fields act as derivations on the space of smooth functions. Indeed, let $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$. Then Xf will denote the *smooth* function

$$x \mapsto (Xf)(x) := X(x)f.$$

The function Xf is often known as the **Lie derivative** of the function f along the vector field X, and is then denoted $\mathfrak{L}_X f$. In local coordinates, if X is of the form

$$X = X_1 \frac{\partial}{\partial x_1} + \cdots + X_m \frac{\partial}{\partial x_m}$$

then

$$\begin{aligned} \mathfrak{L}_X f &= \frac{\partial f}{\partial x_1} X_1 + \cdots + \frac{\partial f}{\partial x_m} X_m \\ &= \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_m} \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_m \end{bmatrix} \\ &= \frac{\partial f}{\partial x} X. \end{aligned}$$

NOTE : One can also define the *Lie derivative* of a function by the formula

$$\mathfrak{L}_X f := \lim_{t \to 0} \frac{\varphi_t^* f - f}{t}$$

where φ_t is the (local) flow of X. (It is then easy to see that $\mathfrak{L}_X f = Xf$.)

◇ **Exercise 273** Given $X \in \mathfrak{X}(M), f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$, verify that

(D1) $X(f + g) = Xf + Xg$;

(D2) $X(\lambda f) = \lambda X f$;

(D3) $X(f \cdot g) = f \cdot Xg + g \cdot Xf$.

This shows that the mapping $f \mapsto Xf$ (i.e. the Lie derivative $\mathfrak{L}_X : C^\infty(M) \to C^\infty(M)$) is linear and satisfies the Leibniz rule, hence is a *derivation* of (the ring) $C^\infty(M)$.

NOTE : Derivations of $C^\infty(M)$ are also called *first order differential operators*. The set $\mathfrak{D}(M)$ of all such derivations is a vector space (over $\mathbb{R}$).

We have a natural *inclusion* ($X \mapsto \mathfrak{L}_X$)

$$\mathfrak{X}(M) \subseteq \mathfrak{D}(M)$$

(every smooth vector field is a derivation). One can prove that *all derivations of $C^\infty(M)$ are smooth vector fields on M* (i.e. the reverse inclusion $\mathfrak{D}(M) \subseteq \mathfrak{X}(M)$ holds).

Note : For this, we need to show that a derivation of $C^\infty(M)$ can be *localized* to a derivation of the algebra $C^\infty(p)$ of function germs at each $p \in M$. (Caution : For $f \in C^\infty(p)$ we do *not* require that $f(p) = 0$. Such elements form a subalgebra $\mathbf{F}(p)$ of $C^\infty(p)$.) This is by no means evident. The "tricky" part is to show that (for $\Delta \in \mathfrak{D}(M)$ and $p \in M$) the mapping

$$\Delta_p : C^\infty(p) \to \mathbb{R}, \quad f \mapsto \Delta(f)(p)$$

is *well-defined* (i.e. depends only on Δ and the function germ $f = \langle f \rangle_p$). Then it follows that

$$\widetilde{\Delta} : p \mapsto \Delta_p \in T_pM$$

is a smooth section of (the tangent bundle) TM, hence a smooth vector field on M.

Henceforth, we shall regard a smooth vector field (on a given manifold) either as a smooth section of the tangent bundle of the manifold or as a derivation of the algebra of smooth functions on that manifold.

The Lie algebra of vector fields

Given a manifold M, the set of all smooth vector fields on M is denoted by $\mathfrak{X}(M)$. It is itself a *vector space* (over $\mathbb{R}$) since any linear combination (with constant coefficients) of two smooth vector fields is also a smooth vector field. More precisely, if $X, Y \in \mathfrak{X}(M)$ and $\lambda, \mu \in \mathbb{R}$, then (for $f \in C^\infty(M)$)

$$\lambda X + \mu Y : f \mapsto (\lambda X + \mu Y)f := \lambda X f + \mu Y f$$

is a derivation of $C^\infty(M)$, hence a smooth vector field on M.

Note : As a vector space, $\mathfrak{X}(M)$ is *not* finite-dimensional. In fact, $\mathfrak{X}(M)$ is more than just a vector space; it is a Lie algebra as we shall see.

Let $X, Y \in \mathfrak{X}(M)$ (viewed as derivations of $C^\infty(M)$). Then, in general, neither YX nor XY is a derivation. However, oddly enough, the operator $YX - XY$ is a derivation (of $C^\infty(M)$).

$\diamond$ **Exercise 274** Given $X, Y \in \mathfrak{X}(M)$, verify that the operator $YX - XY : C^\infty(M) \to C^\infty(M)$ is a derivation, hence is (identified with) a smooth vector field on M.

We make the following definition.

5.5.8 DEFINITION. The smooth vector field $[X, Y] \in \mathfrak{X}(M)$, defined by

$$[X, Y]f := Y(Xf) - X(Yf)$$

is called the **Lie bracket** of X and Y.

It is easy to check that the Lie bracket $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ has the following properties (for $\lambda, \mu \in \mathbb{R}$ and $X, Y, Z \in \mathfrak{X}(M)$) :

(LA1) $[X, Y] = -[Y, X]$;

(LA2) $[X, \lambda Y + \mu Z] = \lambda[X, Y] + \mu[X, Z]$;

(LA3) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$

This means that the real vector space $\mathfrak{X}(M)$ equipped with the Lie bracket $[\cdot, \cdot]$ is a *Lie algebra*.

We may now derive the expression in local coordinates for $[X, Y]$. Let

$$X = X_1 \frac{\partial}{\partial x_1} + \cdots + X_m \frac{\partial}{\partial x_m} \quad \text{and} \quad Y = Y_1 \frac{\partial}{\partial x_1} + \cdots + Y_m \frac{\partial}{\partial x_m}$$

be local representations of X and Y, respectively (in an admissible chart (U, ϕ) of M). Then

$$\begin{aligned} [X, Y]f &= Y(Xf) - X(Yf) \\ &= \sum_{i,j=1}^{m} Y_i \frac{\partial X_j}{\partial x_i} \frac{\partial f}{\partial x_j} - \sum_{i,j=1}^{m} X_i \frac{\partial Y_j}{\partial x_i} \frac{\partial f}{\partial x_j} \\ &= \sum_{j=1}^{m} \left(\sum_{i=1}^{m} Y_i \frac{\partial X_j}{\partial x_i} - X_i \frac{\partial Y_j}{\partial x_i} \right) \frac{\partial f}{\partial x_j} \cdot \end{aligned}$$

Thus

$$\begin{aligned} [X, Y] &= \sum_{j=1}^{m} \left(\sum_{i=1}^{m} Y_i \frac{\partial X_j}{\partial x_i} - X_i \frac{\partial Y_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} \\ &= \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \cdots & \frac{\partial X_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial X_m}{\partial x_1} & \cdots & \frac{\partial X_m}{\partial x_m} \end{bmatrix} \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix} - \begin{bmatrix} \frac{\partial Y_1}{\partial x_1} & \cdots & \frac{\partial Y_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial Y_m}{\partial x_1} & \cdots & \frac{\partial Y_m}{\partial x_m} \end{bmatrix} \begin{bmatrix} X_1 \\ \vdots \\ X_m \end{bmatrix} \\ &= \frac{\partial X}{\partial x} Y - \frac{\partial Y}{\partial x} X. \end{aligned}$$

5.5.9 Example. For *constant* (or parallel) vector fields

$$X : x \mapsto a = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \quad \text{and} \quad Y : x \mapsto b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

on $M = \mathbb{E}^m$, we have $[X, Y] = 0$.

5.5.10 Example. Let $X, x \mapsto Ax$ be a *linear* vector field and $Y, x \mapsto b$ be a *constant* vector field on $M = \mathbb{E}^m$. Then

$$\begin{aligned} X &= (a_{11}x_1 + \cdots + a_{1m}x_m)\frac{\partial}{\partial x_1} + \cdots + (a_{m1}x_1 + \cdots + a_{mm}x_m)\frac{\partial}{\partial x_m} \\ Y &= b_1\frac{\partial}{\partial x_1} + \cdots + b_m\frac{\partial}{\partial x_m} \end{aligned}$$

and so

$$[X, Y] = \frac{\partial X}{\partial x}Y - \frac{\partial Y}{\partial x}X = Ab - 0 = Ab.$$

Therefore $[X, Y]$ is a constant vector field $x \mapsto c$, with $c = Ab$.

5.5.11 Example. If $X, x \mapsto Ax$ and $Y, x \mapsto Bx$ are both *linear* vector fields (on $M = \mathbb{E}^m$), then

$$[X, Y] = \frac{\partial X}{\partial x}Y - \frac{\partial Y}{\partial x}X = ABx - BAx = (AB - BA)x.$$

Therefore $[X, Y]$ is also a linear vector field $x \mapsto Cx$, with $C = [A, B]$ (the *commutator* of the matrices A and B).

We have seen that the set $\mathfrak{X}(M)$ (of all smooth vector fields on M) has a natural structure of Lie algebra. In addition to this structure, $\mathfrak{X}(M)$ admits another algebraic structure : for any $f \in C^\infty(M)$ and any $X \in \mathfrak{X}(M)$,

$$fX : p \mapsto (fX)(p) := f(p)X(p) \in T_pM$$

is a smooth vector field on M. (Caution : do not confuse Xf and fX.) With this operation, $\mathfrak{X}(M)$ becomes a *module* over the ring $C^\infty(M)$.

Note : The Lie bracket $[\cdot,\cdot] : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is *not* $C^\infty(M)$-bilinear. In fact (for $g \in C^\infty(M)$),

$$[X, gY] = g[X, Y] - (Xg)Y.$$

◇ **Exercise 275** Let $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$. Show that

$$[fX, gY] = fg[X, Y] - f(Xg)Y + g(Yf)X.$$

Use this formula to derive the formula for the components of $[X, Y]$ in local coordinates.

Commutativity of vector fields

. . . .

Orbits of vector fields

. . . .

5.6 Differential Forms

. . . .