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Lecture Notes CCR

AM3.2 - Linear Control

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The most practical solution is a good theory.

Albert Einstein

Once upon a time, it is said, in the Good Old Days (read nineteenth century), there was only Mathematics, a subject intimately bound up with the ways of Mother Nature. Today this subject has fragmented into two ideological blocs, labelled Pure and Applied. Each views the other with an element of distrust [...] Each side has the conviction of the True Believer in its own moral superiority. The Applied Mathematicians accuse the Pure of a lack of contact with reality; the Pure Mathematicians feel that Applied are altogether too slapdash and have never quite grasped the rules of the game [...] There is without doubt a great difference in attitudes between those who called themselves pure and those who called themselves applied. Halmos reckons that the main difference is that the applied mathematicians are convinced there is no difference, whereas the pure mathematicians know perfectly well there is. A more salient difference is one of intention. ***The applied mathematician wants an answer; the pure mathematician wants to understand the problem.*** *The pure mathematician observes that sometimes the applied one is so keen to answer that he doesn't worry much whether it's the right answer. The applied mathematician observes that when the pure one can't understand a problem he moves on to another one and tries again. Perhaps the true difference is that applied mathematicians devote a lot of thought to the modelling process – devising an effective mathematical model of a natural phenomenon – whereas this step is largely absent from pure mathematicians.*

Ian Stewart

> Scientists use mathematics to build mental universes. They write down mathematical descriptions – models – that capture essential fragments of how they think this world behaves. Then they analyse their consequences. This is called "theory". They test their theories against observations : this is called "experiment". Depending on the result, they may modify the mathematical model and repeat the cycle until theory and experiment agree. Not that it's really that simple; but that's the general gist of it, the essence of the scientific method.
>
> I. Stewart and M. Golubitsky

> Differential equations are the particular dialect of the language of mathematics that most effectively describes how nature works.
>
> J.D. Barrow

What is Mathematical Control Theory ?

Mathematical control theory is the area of *application-oriented mathematics* that deals with the basic principles underlying the analysis and design of *control systems*. To *control* an object means to influence its behaviour so as to achieve a desired goal. In order to implement this influence, control engineers build devices that incorporate various mathematical techniques. These devices range from *Watt's steam engine governor*, designed during the English Industrial Revolution, to the sophisticated microprocessor controllers found in items – such as CD players and automobiles – or in industrial robots and airplane autopilots.

Control theory was originally developed to satisfy the design needs of *ser-*

vomechanisms, under the name of **"automatic control theory"**. The classical theory of automatic control mostly deals with *linear feedback* control systems with single input and single output. Mathematical structures of such systems must be, in principle, described in terms of linear ordinary differential equations (ODEs) with constant coefficients. Hence control engineers use *block diagrams* to describe systems, and operational calculus based on *Laplace transforms* to obtain response characteristics. Thus the input/output relation of a system is described in terms of *transfer functions*. One of the remarkable contributions to classical control theory is *Nyquist's criterion* (after its originator, Harry Nyquist (1889-1976)) for stability testing of linear feedback systems. The test consists of plotting the Nyquist diagram of a transfer function in the *frecquency domain* (complex plane), and differs essentially from the *Routh-Hurwitz stability test* for linear ODEs with constant coefficients.

Control theory became recognized as a *mathematical subject* in the 1960's. Around 1960 three remarkable contributions were made concurently; they are

- **dynamic programming** – Richard E. Bellman (1920-1984)
- **Pontryagin's principle** – Lev S. Pontryagin (1908-1988)
- **linear system theory** – Rudolf E. Kalman (1930).

The first two give rise to mathematical tools to solve *optimal control* problems and to design optimal controllers and regulators. In contrast to the classical theory of control, optimal control problems are formulated in terms of systems of linear *or* nonlinear multivariable ODEs with multiinput (control) variables. Linear system theory derives from the concepts of *controllability* and *observability*. These two concepts are concerned with the interaction between (internal) *states* of a system and its *inputs* and *outputs*.

R.E. Kalman challenged the accepted approach to control theory of that period (limited to the use of Laplace transforms and the frecquency domain) by showing that the basic control problems could be studied effectively through the notion of a *state* of the system that evolves in time according to ODEs

in which controls appear as parameters. Aside from drawing attention to the mathematical content of control problems, KALMAN's work served as a catalyst for further growth of the subject. Liberated from the confines of the frecquency domain and further inspired by the development of computers, automatic control theory became the subject matter of a new science called **systems theory**.

Systems theory grew out of a desire to merge *automata theory*, and *artificial intelligence*, and *discrete* and *continuous control* into a single subject concerned with input/output relations parametrized by the states of the system. The level of generality required to keep these subjects together was well beyond the realm of differential equations, and control theory quickly evolved into *topological dynamical systems*. Systems theory, itself a hybrid of control and automata theory, in its formative period looked to *abstract dynamical systems* and *mathematical logic* for its further growth.

Around 1970 the significance of the *Lie bracket* for problems of control became clear thanks to efforts made by R.M. HERMANN, R.W. BROCKETT, C. LOBRY, H.J. SUSSMANN, V. JURDJEVIC, and others. As a result, *differential geometry* entered into an exciting partnership with control theory, marking the birth of **geometric control theory**.

Present day theoretical research in control theory involves a variety of areas of pure mathematics (e.g. *linear and multilinear algebra, Lie semigroups and Lie groups, algebraic geometry, dynamical systems, complex analysis, functional analysis, calculus of variations, topology, differential geometry, probability theory*, etc.). Concepts and results from these areas find applications in control theory; conversely, questions about control systems give rise to new open problems in mathematics.

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Chapter 1

Introduction

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1. Motivation and Basic Concepts
2. Mathematical Formulation of the Control Problem
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4. Matrix Theory (review)

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A *control system* can be viewed informally as a dynamical object (e.g. ordinary differential equation) containing a parameter (control) which can be manipulated to influence the behaviour of the system so as to achieve a desired goal. In order to implement this influence, engineers build devices that incorporate various mathematical techniques. *Mathematical control theory* is today a well-established branch of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems.

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1.1 Motivation and Basic Concepts

Mathematical control theory is a rapidly growing field which provides theoretical and computational tools for dealing with a variety of problems arising in *electrical* and *aerospace engineering, automatics, robotics, management, economics, applied chemistry, biology, ecology, medicine,* etc. Selected such problems, to mention but a few, are the following : stable performance of motors and machinery, optimal guidance of rockets, optimal exploitation of natural resources, optimal investment or production strategies, regulation of physiological functions, and fight against insects, epidemics.

All these (and many other) problems require a specific approach, the aim being *to compel or control a system to behave in some desired fashion.*

Systems

A **system** is something having parts which is perceived as a single entity.

NOTE : Not everything is a system (for instance, a *point* or the empty set). However, most things can usefully be seen as systems of some kind. A system is, so to speak, a world.

The parts making up a system may be clearly or vaguely defined. The interesting thing about a system is the way the parts are related to each other. For the systems studied in mathematics, the parts and their relations must be so clearly defined that we can single out a particular set of relations as completely characterizing the *state* of the system; then *we identify the system with the collection of all its conceivable states.* It seems to be necessary that the state space be clearly and unambiguously defined. Unfortunately this usually means that the mathematical system is drastically oversimplified in comparison with the *natural system* being modelled.

When attempting to study the behaviour of certain systems, it is convenient to consider the ideal case of an *"isolated system"* – i.e. a number of interacting elements which do not have any interaction with the rest of the

world. In reality, no system is ever completely isolated, but in many cases the interactions with the rest of the world can be neglected in a reasonable approximation. In such "isolated systems" the conditions are simpler, and therefore easier to study.

NOTE : Our Universe is, by definition, an isolated system.

Dynamical and control systems

A **dynamical system** is one which changes in time (in some well defined way); what changes is the state of the system. For such systems, the basic problem is to *predict* the future behaviour. For this purpose the differential equations are exactly tailored. The differential equation itself represents the (physical or otherwise) law governing the evolution of the system; this plus the initial conditions should determine uniquely the future evolution of the system.

NOTE : Philosophically this leads to *determinism*, and is independent of any (human) observer. Modern physics changed this view, to some extent, by making the observer a much more active participant in the possible outcome of future measurements. But even within classical physics, the prediction of future evolution is not the only meaningful problem to be posed. The whole field of engineering and technology deals, to a large extent, with the"inverse" problem : *given a desired future evolution, how should we construct the system?*

One can introduce some way of acting upon a (dynamical) system and influence its evolution (behaviour). We think of this outside action, also called *input* (or *control*), as the result of decisions of a "controller" (possibly human), who may have some definite goal in mind or not. But this last is irrelevant and the important information we need is a rule, within the description of the system, of which inputs are possible and which are not; the possible inputs will then be called "admissible". A simple example of such systems (with inputs) is a *car*, whose motion depends on the input of all the actions by which we drive it. In most cases, it is possible to change the state of the system in any

prescribed fashion by properly choosing the inputs, at least within reasonable limits. In other words, *one may exert influence on the system state by means of intelligent manipulation of its inputs.* This then, in a general sense, constitutes a **control system**.

The control engineer develops the techniques and hardware necessary for the implementation of the control laws to the specific systems in question.

NOTE : The complexity of many systems in present-day world is such that it is often desirable for control to be carried out *automatically*, without direct human intervention. To take a simple example, the room thermostat in a domestic central heating system turns the boiler on and off so as to maintain room temperature at a predetermined level. Nature provides many examples of remarkable self-regulation, such as the way in which body temperature is kept constant despite large variations in external conditions.

Some basic control-theoretic concepts

We summarize some of the main features of a control system.

The **state variables** $x_1, x_2, \ldots, x_m$ describe the condition (or state) of the system, and provide the information which (together with the knowledge of the equations describing the system) enables us to calculate the future behaviour from the knowledge of the **control variables** (or **inputs**) $u_1, u_2, \ldots, u_\ell$. In practice, it is often not possible to determine the values of the state variables directly; instead, only a set of **controlled variables** (or **outputs**) $y_1, y_2, \ldots, y_n$, which depend in some way on the state variables, is measured. In general, *the aim is to make a system perform in some required way by suitably manipulating the inputs*, this being done by some controlling device (or "controller").

If the controller operates according to some pre-set pattern without taking account of the output or state, the system is called **open loop**. If, however, there is *feedback* of information concerning the outputs to controller, which then appropriately modifies its course of action, the system is called **closed loop**.

We assume that our system models have the property that, *given an initial state and any input, the resulting state and output at some specified later time are uniquely determined.*

1.2 Mathematical Formulation of the Control Problem

Roughly speaking, a *control system* is a dynamical system together with a class of "admissible inputs". We whish to make this idea more precise, without striving for full generality.

Control systems

We assume that the *dynamics* of the system, that is, the evolution of the **state vector** $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} \in \mathbb{R}^{m\times 1} = \mathbb{R}^m$ under a given **input** (or **control**) **vector** $u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_\ell(t) \end{bmatrix} \in \mathbb{R}^{\ell\times 1} = \mathbb{R}^\ell$ is determined by a (vector) *ordinary differential equation*

$$\dot{x} = F(t, x, u). \tag{1.1}$$

The *input* vector $u(\cdot)$ is assumed to be an "arbitrary" vector-valued mapping, but some restrictions must be imposed. First of all, its components – the input functions $u_1(\cdot)$, $u_2(\cdot)$, ..., $u_\ell(\cdot)$ – must be *measurable* (think of *piecewise-continuous* functions), since otherwise the differential equation (1.1) wouldn't make sense.

NOTE : To restrict the control to be a *continuous* mapping, would be too much, since in many cases the piecewise-continuous inputs (with some points of discontinuity) are the most interesting controls.

Another restriction of the input vector is the requirement that the values of $u(\cdot)$ belong to a specified set U. (For example, when turning a steering wheel of a car, we are restricted to a maximum turning angle to either side.) Such a restriction is then of the form

$$u(t) \in U \subseteq \mathbb{R}^{\ell}. \tag{1.2}$$

An *admissible input* is therefore a piecewise-continuous (vector-valued) mapping $u(\cdot)$ satisfying (1.2). We denote by $\mathcal{U}$ the set of all these admissible inputs.

Furthermore, it is assumed that the vector-valued mapping $F : J \times \mathbb{R}^m \times U \to \mathbb{R}^m$, $J \subseteq \mathbb{R}$ satisfies certain standard conditions (such as having continuous first order partial derivatives).

NOTE : This assumption guarantees local existence and uniqueness of the solution of (1.1) (subject to *initial condition* $x(t_0) = x_0$) for a given $u(\cdot) \in \mathcal{U}$.

A **control system** is a 4-tuple

$$\Sigma = (M, U, \mathcal{U}, F).$$

In this case, the set $M = \mathbb{R}^m$ is the **state space**, the set $U \subseteq \mathbb{R}^{\ell}$ is the **control set**, $\mathcal{U}$ is the **input class**, and the mapping F is the **dynamics** of Σ. We say that the control system Σ is defined (or described) by the *state equation* (1.1) and write (in classical notation) :

$$\boxed{\Sigma : \quad \dot{x} = F(t, x, u), \qquad x \in M, \;\; u \in U \subseteq \mathbb{R}^{\ell}.}$$

NOTE : (1) In fact, such a system is a *continuous-time, time-varying, finite dimensional, differentiable (nonlinear) control system.*

(2) The state space M carries certain (geometric) "structure". It is natural to assume that M is a *differentiable manifold* (think of an *open* subset of some Euclidean space). The dynamics F is then best viewed as a family of (nonautonomous) vector fields on (the manifold) M, parametrized by controls.

The control system Σ is **linear** if $U = \mathbb{R}^\ell$ and the dynamics $F : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^\ell \to \mathbb{R}^m$ has the form

$$F(t, x, u) = A(t)x + B(t)u$$

where $A(t) \in \mathbb{R}^{m\times m}$ and $B(t) \in \mathbb{R}^{m\times \ell}$ are matrices each of whose entries is a (continuous) function $\mathbb{R} \to \mathbb{R}$; that is, the dynamics F is *linear* in (x, u) for each fixed $t \in \mathbb{R}$.

One distinguishes controls of two types : *open* and *closed loop*. An **open loop control** can be basically an "arbitrary" function $u : [t_0, \infty) \to U$ for which the initial value problem (IVP)

$$\dot{x} = F(t, x, u)\,, \quad x(t_0) = x_0$$

has a well defined solution.

A **closed loop control** can be identified with a mapping $k : M \to U$ (which may depend on $t \geq t_0$) such that the initial value problem (IVP)

$$\dot{x} = F(t, x, k(x(\cdot)))\,, \quad x(t_0) = x_0$$

has a well defined solution. The mapping $k(\cdot)$ is called **feedback**.

One of the main aims of control theory is to find a strategy (input) such that the corresponding output has desired properties. Depending on the properties involved one gets more specific questions. Concepts like *controllability, observability, stabilizability, realization,* as well as *optimality* are fundamental in control theory.

Controllability

One say that a state $x_f \in \mathbb{R}^n$ is **reachable** from x_0 in time T if there exists an open loop control $u(\cdot)$ such that

$$x(0) = x_0 \quad \text{and} \quad x(T) = x_f.$$

If an arbitrary state x_f is reachable from an arbitrary state x_0 in time T, then the control system Σ is called **(completely) controllable**. In several situations one requires a weaker property of transfering an arbitrary state into a given one, in particular the origin. *A formulation of effective characterizations of controllable systems is an important task of control theory.*

Observability

In many situations of practical interest one observes not the state $x(\cdot)$ but its function $t \mapsto h(t, x(t))$, $t \geq t_0$. It is therefore often necessary to investigate the pair of equations (i.e. the *state equation* and the *observation equation*)

$$\begin{cases} \dot{x} = F(t, x, u) \\ \\ y = h(t, x). \end{cases}$$

This is a **control system with outputs**; that is, a 6-tuple

$$\Sigma = (M,\, U,\, \mathcal{U},\, F,\, \mathbb{R}^n,\, h).$$

In this case, $(M,\, U,\, \mathcal{U},\, F)$ is the *underlying* control system and h is the **measurement mapping**. We use the same symbol for the control system with outputs and its underlying control system. The mapping $h = (h_1, h_2, \ldots h_n) : \mathbb{R} \times M \to \mathbb{R}^n$ represents the vector of n *measurements (observations).*

The control system with outputs Σ is **linear** if its underlying system is linear and the measurement mapping $h : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^n$ is linear for each $t \in \mathbb{R}$.

This new system is said to be **(completely) observable** if, knowing a control $u(\cdot)$ and an *observation* $y(\cdot)$, on a given interval $[t_0, T]$, one can determine uniquely the initial condition x_0.

Stabilizability

Another important issue is that of stabilizability. Assume that for some $\bar{x} \in \mathbb{R}^n$ and $\bar{u} \in U$, $F(\bar{x}, \bar{u}) = 0$. A function $k : M \to U$ such that $k(\bar{x}) = \bar{u}$ is called a **stabilizing feedback** if $\bar{x}$ is a *stable equilibrium* for the system

$$\dot{x} = F(t, x, k(x(\cdot))).$$

In the theory of (ordinary) differential equations there exist several methods to determine whether a given equilibrium state is a stable one.

Realization

For a given initial condition $x_0 \in \mathbb{R}^n$, the control system with outputs

$$\begin{cases} \dot{x} = F(t, x, u), \quad x(t_0) = x_0 \\ \\ y = h(t, x) \end{cases}$$

defines a mapping which transforms open loops controls $u(\cdot)$ onto outputs

$$y(t) = h(t, x(t)), \quad t \in [t_0, T].$$

Denote this transformation by $\mathcal{R}$. What are its properties ? What conditions should a transformation $\mathcal{R}$ satisfy to be given by such a control system ? How, among all the possible "realizations" Σ of a transformation $\mathcal{R}$, do we find the simplest one ?

Optimality

Besides the above problems of structural character, in control theory one also asks *optimality questions*. In the so-called *time-optimal problem* one is looking for control which not only transfers a state x_0 onto x_f but does it in the minimal time T. More generally, one is looking for a control $u(\cdot)$ which *minimizes* a *functional* of the form

$$\mathcal{J} := \phi(x(T), T) + \int_{t_0}^{T} L(t, x, u) dt.$$

1.3 Examples

We shall mention several specific models of control systems.

Example 1. (Moving car) Suppose a car is to be driven along a straight road, and let its *distance* from an initial point 0 be $s(t)$ at time t. For simplicity, assume that the car is controlled only by the throttle, producing an *accelerating force* of $u_1(t)$ per unit mass, and by brake wich produces a *retarding force* of $u_2(t)$ per unit mass. Suppose that the only factors of interest are the car's position $x_1(t) := s(t)$ and velocity $x_2(t) := \dot{s}(t)$. Ignoring other forces such as road friction, wind resistance, etc. the equations which describe the state of the car at time t are

$$\begin{cases} \dot{x}_1 = x_2 \\ \\ \dot{x}_2 = u_1 - u_2 \end{cases}$$

or, in matrix form,

$$\dot{x} = Ax + Bu(t)$$

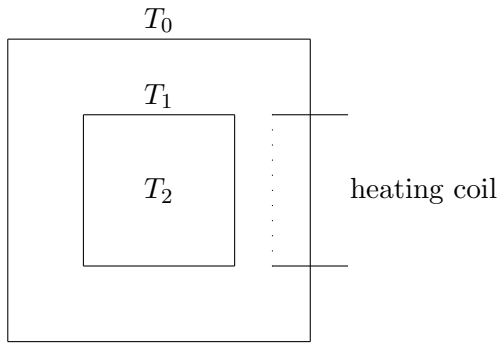
where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

It may be required to start from rest at 0 and reach some fixed point in the least possible time, or perhaps with minimum consumption of fuel. The mathematical problems are firstly to determine whether such objectives are achievable with the selected control variables, and if so, to find appropriate expressions for $u_1(\cdot)$ and $u_2(\cdot)$ as functions of time and/or $x_1(\cdot)$ and $x_2(\cdot)$.

NOTE : The complexity of the model could be increased so as to take into account factors such as engine speed and temperature, vehicle interior temperature, and so on.

Example 2. (Electrically heated oven) Let us consider a simple model of an ellectrically heated oven, which consists of a jacket with a coil directly heating the jacket and of an interior part.



Let T_0 denote the *outside temperature.* We make a simplifying assumption, that at an arbitrary moment $t \geq 0$, *temperature* in the *jacket* and in the *interior part* are uniformly distributed and equal to $T_1(t)$, $T_2(t)$. We assume also that the flow of heat through a surface is proportional to the area of the surface and to the difference of temperature between the separated media. Let $u(t)$ be the *intensity of the heat input* produced by the coil at moment $t \geq 0$. Let moreover a_1, a_2 denote the *area* of exterior and interior surfaces of the jacket, respectively, c_1, c_2 denote *heat capacities* of the jacket and the interior of the oven, respectively, and r_1, r_2 denote *radiation* coefficients of the exterior and interior surfaces of the jacket, respectively. An increase of heat in the jacket is equal to the amount of heat produced by the coil reduced by the amount of heat which entered the interior and exterior of the oven. Therefore, for the interval $[t, t + \Delta t]$, we have the following balance :

$$c_1 \left(T_1(t + \Delta t) - T_1(t)\right) \approx u(t)\Delta t - \left(T_1(t) - T_2(t)\right) a_1 r_1 \Delta t - \left(T_1(t) - T_0\right) a_2 r_2 \Delta t \,.$$

Similarly, an increase of heat in the interior of the oven is equal to the amount of heat radiated by the jacket :

$$c_2 \left(T_2(t + \Delta t) - T_2(t)\right) = \left(T_1(t) - T_2(t)\right) a_1 r_2 \Delta t \,.$$

Dividing the obtained identities by Δt and taking the limit, as $\Delta \to 0$, we

obtain :

$$\begin{cases} c_1\dot{T}_1 = u - (T_1 - T_2)a_1r_1 - (T_1 - T_0)a_2r_2 \quad \text{(for the jacket)} \\ \\ c_2\dot{T}_2 = (T_1 - T_2)a_1r_1 \quad \text{(for the oven interior).} \end{cases}$$

Les us notice that, according to the physical interpretation, $u(t) \geq 0$ for $t \geq 0$. Let the *state variables* be the excesses of temperature over the exterior, that is

$$x_1 := T_1 - T_0 \quad \text{and} \quad x_2 := T_2 - T_0.$$

Then we can write the equations above in matrix form, namely

$$\dot{x} = Ax + Bu(t)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u = \begin{bmatrix} u \end{bmatrix}, \quad A = \begin{bmatrix} -\dfrac{a_1r_1 + a_2r_2}{c_1} & \dfrac{a_1r_1}{c_1} \\ \\ \dfrac{a_1r_1}{c_2} & -\dfrac{a_1r_1}{c_2} \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{1}{c_1} \\ \\ 0 \end{bmatrix}.$$

It is natural to limit the considerations to the case when $x_1(0) \geq 0$ and $x_2(0) \geq 0$. It is physically obvious that if $u(t) \geq 0$ for $t \geq 0$, then also $x_1(t) \geq 0$, $x_2(t) \geq 0$ for $t \geq 0$.

Two interesting aspects to be discussed are firstly whether it is possible to maintain the temperature of the oven interior at any desired level merely by altering u, and secondly, to determine whether the value of T_2 can be determined even if it is not possible to measure it directly.

NOTE : If the desired objective is attainable, then there may well be many different suitable control schemes, and considerations of economy, practicability of application, and so on will then determine how control is actually applied.

Example 3. (Controlled environment) Consider a controlled environment consisting of *rabbits* and *foxes*, the number of each at time t being $x_1(t)$ and

$x_2(t)$, respectively. Suppose that without the presence of foxes the number of rabbits would grow exponentially, but that the rate of growth of rabbit population is reduced by an amount proportional to the number of foxes. Furthermore, suppose that, without rabbits to eat, the fox population would decrease exponentially, but the rate of growth in the number of foxes is increased by an amount proportional to the number of rabbits present. Under these assumptions, the system of equations can be written

$$\begin{cases} \dot{x}_1 = a_1 x_1 - a_2 x_2 \\ \\ \dot{x}_2 = a_3 x_1 - a_4 x_2 \end{cases}$$

where a_1, a_2, a_3 and a_4 are positive constants.

Example 4. (Satellite problem) We shall consider a point mass in an inverse square law force field. The motion of a unit mass is governed by a pair of second order equations in the radius r and the angle θ (polar coordinates). If we assume that the unit mass (say a satellite) has the capability of thrusting in the *radial direction* with the *thrust* $u_1(\cdot)$ and thrusting in the *tangential direction* with *thrust* $u_2(\cdot)$, then we have

$$\begin{cases} \ddot{r} = r\dot{\theta}^2 - \dfrac{k}{r^2} + u_1(t) \\ \\ \ddot{\theta} = -\dfrac{2\dot{\theta}\dot{r}}{r} + \dfrac{1}{r} u_2(t). \end{cases}$$

If $u_1(t) = u_2(t) = 0$, these equations admit the solution

$$r(t) = \sigma \quad (\sigma \text{ constant}) \quad \text{and} \quad \theta(t) = \omega t \quad (\omega \text{ constant})\,; \quad \sigma^3\omega^2 = k\,.$$

That is, *circular orbits are possible.* If we let x_1, x_2, x_3, and x_4 be given by

$$x_1 := r - \sigma, \quad x_2 := \dot{r}, \quad x_3 := \sigma(\theta - \omega t), \quad x_4 := \sigma(\dot{\theta} - \omega)$$

and normalize σ to 1, then it is easy to see that the *linearized equations* of motion about the given solution are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

Example 5. (Market economy) Suppose that the *sales* $S(t)$ of a product are affected by the *amount of advertising* $A(t)$ in such a way that the rate of change of sales decreases by an amount proportional to the advertising applied to the share of the market not already purchasing the product. If the total extent of the market is M, the *state equation* is therefore

$$\dot{S} = -aS + bA(t)\left(1 - \frac{S}{M}\right)$$

subject to

$$S(0) = S_0$$

where a and b are positive constants. In practice, the amount of advertising will be limited (that is, $0 \leq A(t) \leq K$, where K is a constant), and the aim would be to find the advertising schedule (that is, the function $A(t)$ which maximizes the sales over some period of time).

Example 6. (Soft landing) Let us consider a spacecraft of *total mass* M moving vertically with the gas thruster directed toward the landing surface. Let $h(\cdot)$ be the *height* of the spacecraft above the surface, $u(\cdot)$ the *thrust* of its engine produced by the expulsion of gas from the jet. The gas is a product of the combustion of the fuel. The combustion decreases the total mass of the spacecraft, and the thrust u is proportionalto the speed with which the mass decreases. Assuming that there is no atmosphere above the surface and that

g is gravitational acceleration, one arrives at the following equations :

$$\begin{cases} M\ddot{h} &= -gM + u(t) \\ \\ \dot{M} &= -ku(t) \quad (k > 0) \end{cases}$$

with the initial conditions

$$M(0) = M_0\,, \quad h(0) = h_0\,, \quad \dot{h}(0) = h_1.$$

One imposes additional constraints on the control parameter of the type

$$0 \leq u \leq \alpha \quad \text{and} \quad M \geq m,$$

where m is the mass of the spacecraft without fuel. Let us fix $T > 0$. The *soft landing problem* consists of finding a control $u(\cdot)$ such that for the solutions $M(\cdot)$, $h(\cdot)$ of the above equations

$$M(t) \geq m\,, \quad h(t) \geq 0\,, \quad t \in [0, T]\,, \quad \text{and} \quad h(T) = \dot{h}(T) = 0.$$

NOTE : A natural *optimization* question arises when the moment T is not fixed and one is minimizing the landing time.

1.4 Matrix Theory (review)

Matrices and determinants

We write a **matrix** as follows

$$A = \Big[a_{ij}\Big] \quad (i = 1, 2, \ldots, m\,;\ \ j = 1, 2, \ldots, n)$$

where a_{ij} is the element (entry) in its ith row and j^{th} column, and A thus has m **rows** and n **columns**; we use to say that A is an $m \times n$ matrix. We shall denote by $\mathbb{R}^{m\times n}$ the set (vector space) of all $m \times n$ matrices with real entries.

NOTE : (1) It is convenient to *identify* the set (vector space) $\mathbb{R}^m$ of all *m-tuples* of real numbers with the set (vector space) $\mathbb{R}^{m\times 1}$ of all *column m-matrices* (or column m-vectors).

(2) There is a natural one-to-one correspondence between $m\times n$ matrices and linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$: the vector spaces $\mathbb{R}^{m\times n}$ and $L(\mathbb{R}^n, \mathbb{R}^m)$ are *isomorphic*.

The **transpose** A^T of A is obtained by interchanging the rows and columns of $A = \left[a_{ij}\right]$, so $A^T := \left[a_{ji}\right]$ is an $n \times m$ matrix. If λ is a *scalar* (real number), and A and B are matrices of appropriate size, then :

(a) $(A^T)^T = A$.

(b) $(A+B)^T = A^T + B^T$.

(c) $(\lambda A)^T = \lambda A^T$.

(d) $(AB)^T = B^T A^T$.

If A is *invertible*, then A^T is invertible, too and $(A^T)^{-1} = (A^{-1})^T$.

We can write a matrix $A = \left[a_{ij}\right] \in \mathbb{R}^{m\times n}$ in the following forms :

- $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ with $a_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^{m\times 1} \quad (j = 1, 2, \dots, n)$

- $A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$ with $a_i = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \in \mathbb{R}^{1\times n} \quad (i = 1, 2, \dots, m)$.

When $m = n$, A is said to be **square** of order n. We shall write

$$I_n = \left[\delta_{ij}\right] = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

for the **unit matrix** (or **identity matrix**) of order n. Here, δ_{ij} stands for **Kronecker**'s **symbol**; that is,

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The *unit matrix* I_n has all its elements zero except those on the *main diagonal*; any (square) matrix of this form is called a **diagonal matrix**, written

$$\operatorname{diag}\left(a_{11},\, a_{22},\, \ldots,\, a_{nn}\right).$$

Two $n \times n$ matrices A and B related by

$$B = S^{-1}AS$$

are called **similar**. This is a basic *equivalence relation* on matrices.

NOTE : Two matrices are similar if and only if they represent the same linear mapping in different bases. *Any matrix property that is preserved under similarity is a property of the underlying linear mapping.*

If $A = \left[a_{ij}\right] \in \mathbb{R}^{n\times n}$, then its **trace** is the sum of all elements on the main diagonal; that is,

$$\operatorname{tr}(A) := \sum_{i=1}^{n} a_{ii}.$$

The trace operator has some important properties :

(a) $\operatorname{tr}(\lambda A) = \lambda \operatorname{tr}(A)$, where λ is a scalar.

(b) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.

(c) $\operatorname{tr}(I_n) = n$.

(d) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(e) $\operatorname{tr}(A^T) = \operatorname{tr}(A)$.

(f) $\operatorname{tr}(A^T A) \geq 0$.

Exercise 1 Given $A, S \in \mathbb{R}^{n\times n}$ with S invertible, show that

$$\operatorname{tr}\left(SAS^{-1}\right) = \operatorname{tr}(A).$$

That is, *similar matrices have the same trace.*

We recall briefly the main properties of the **determinant** function

$$\det : \mathbb{R}^{n\times n} \to \mathbb{R}, \quad A \mapsto \det(A).$$

These are :

(a) $\det(AB) = \det(A)\cdot\det(B)$.

(b) $\det(I_n) = 1$.

(c) $\det(A) \neq 0$ if and only if A is invertible.

NOTE : There is a *unique* function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ having these three properties. For $A = \left[a_{ij}\right] \in \mathbb{R}^{n\times n}$ we have

$$\det(A) = \sum_{\alpha} \operatorname{sgn}(\alpha)\, a_{1\alpha(1)} a_{2\alpha(2)} \cdots a_{n\alpha(n)}$$

where the sum is taken over the $n!$ *permutations* (on n elements) $\alpha \in S_n$.

Exercise 2 Given $A, S \in \mathbb{R}^{n\times n}$ with S invertible, show that

$$\det\left(SAS^{-1}\right) = \det(A).$$

That is, *similar matrices have the same determinant.*

If $\det(A) = 0$, A is **singular**, otherwise **nonsingular** (or invertible); in the latter case, the **inverse** of A is

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

where $\operatorname{adj}(A) := \left[A_{ij}\right]^T$ is the **adjoint** of A; here,

$$A^i_j := (-1)^{i+j} M_{ij} \quad (\text{ the } \textbf{cofactor} \text{ of } a_{ij}\ )$$

and M_{ij} is the determinant of the submatrix formed by deleting the ith row and the jth column of A.

NOTE : Let $A \in \mathbb{R}^{n\times n}$. Then $\det(A) = 0$ (the matrix A is singular) if and only if $Ax = 0$ for some *nonzero* column n-vector $x \in \mathbb{R}^{n\times 1}$.

Linear dependence and rank

Consider a set of column m-vectors

$$a_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad a_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \quad \ldots, \quad a_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars, then the vector

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = \begin{bmatrix} \alpha_1 a_{11} + \alpha_2 a_{12} + \cdots + \alpha_n a_{1n} \\ \alpha_1 a_{21} + \alpha_2 a_{22} + \cdots + \alpha_n a_{2n} \\ \vdots \\ \alpha_1 a_{m1} + \alpha_2 a_{m2} + \cdots + \alpha_n a_{mn} \end{bmatrix}$$

is called a **linear combination** of $a_1, a_2, \ldots, a_n$. If there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, not all zero, such that

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

then the vectors $a_1, a_2, \ldots, a_n$ are said to be **linearly dependent**; otherwise, they are **linearly independent**.

NOTE : We can equally well consider row n-vectors.

Let $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in \mathbb{R}^{m \times n}$. The **rank** of A, denoted by $\operatorname{rank}(A)$, is defined as the maximum number of linearly independent columns (or rows) of A. Clearly, $\operatorname{rank}(A) \leq \min\{m, n\}$. Consider the **kernel** (or **null-space**)

$$\ker(A) := \{x \in \mathbb{R}^n \mid Ax = 0\} \subseteq \mathbb{R}^n$$

and the **image space** (or **column space**)

$$\operatorname{im}(A) := \{Ax \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

The *dimension* of $\ker(A)$ is termed the **nullity** of A. $\operatorname{im}(A)$ is nothing more than the (vector) space *spanned* by the columns of A; that is,

$$\operatorname{im}(A) = \operatorname{span}\{a_1, a_2, \ldots, a_n\} := \{\alpha_1 a_1 + \alpha_2 a_2 \cdots + \alpha_n a_n \mid \alpha_1, \ldots, \alpha_n \in \mathbb{R}\}$$

where $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$. Hence *the dimension of* $\operatorname{im}(A)$ *is equal to* $\operatorname{rank}(A)$.

NOTE : $\operatorname{im}(A^T)$ is also known as the *row space* of A; it is the (vector) space spanned by the rows of A (i.e. the columns of A^T).

An important result states that (for a matrix $A \in \mathbb{R}^{m \times n}$) :

$$\boxed{\operatorname{rank}(A) + \dim \ker(A) = n.}$$

Rank is invariant under multiplication by a nonsingular matrix. In particular, *rank is invariant under similarity.* However, multiplication by rectangular or singular matrices can alter the rank, and the following formula shows exactly how much alteration occurs.

If $A \in \mathbb{R}^{m \times n}$ *and* $B \in \mathbb{R}^{n \times p}$*, then :*

$$\boxed{\operatorname{rank}(AB) = \operatorname{rank}(B) - \dim \ker(A) \cap \operatorname{im}(B).}$$

Exercise 3 Given $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$, show that :

(a) $\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$.

(b) $\operatorname{rank}(A) + \operatorname{rank}(B) - n \leq \operatorname{rank}(AB)$.

Suppose now that $a_{ij} \quad (i = 1, 2, \ldots, m\,; \;\; j = 1, 2, \ldots, n)$ are the coefficients in a set of m linear *algebraic equations* in n unknowns

$$\sum_{j=1}^{n} a_{ij} x_j = b_i\,, \quad i = 1, 2, \ldots, m.$$

These equations can be written in matrix form as

$$Ax = b$$

where

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Such a linear system (of equations) possesses a solution if and only if

$$\operatorname{rank}(A) = \operatorname{rank} \begin{bmatrix} A & b \end{bmatrix}$$

where $\begin{bmatrix} A & b \end{bmatrix}$ is the $m \times (n+1)$ matrix obtained by appending b to A as an extra column.

Two particular cases should be mentioned :

- When $A \in \mathbb{R}^{n \times n}$, the linear system $Ax = b$ has a unique solution if and only if A is nonsingular.
- When $A \in \mathbb{R}^{m \times n}$, the *homogeneous* linear system $Ax = 0$ has a nonzero solution if and only if $\operatorname{rank}(A) < n$.

NOTE : Similar remarks apply to the set of equations

$$yA = c$$

where

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}, \quad y = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}, \quad c = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}.$$

Eigenvalues and eigenvectors

Let $A = \left[a_{ij}\right] \in \mathbb{R}^{n\times n}$. A *nonzero vector* $w \in \mathbb{R}^{n\times 1}$ is called an **eigenvector** (or **characteristic vector**) of A if there is a *scalar* (real number) λ such that

$$Aw = \lambda w.$$

The scalar λ is called the **eigenvalue** (or **characteristic value**) associated with the eigenvector w. Geometrically, $Aw = \lambda w$ says that under the (linear) mapping $x \mapsto Ax$ the eigenvectors experience only changes in magnitude or sign. The eigenvalue λ is simply the amount of "stretch" or "shrink" to which the eigenvector w is subjected when acted upon by A.

NOTE : The words *eigenvalue* and *eigenvector* are derived from the German word *eigen*, which means "owen by" or "peculiar to".

The set of *distinct* eigenvalues, denoted by $\sigma(A)$, is called the **spectrum** of A. We have

$$\lambda \in \sigma(A) \iff \lambda I_n - A \text{ is singular} \iff \det(\lambda I_n - A) = 0.$$

The set (vector space) of all eigenvectors with eigenvalue λ, together with the zero vector, is called the λ-**eigenspace** of A and is denoted by E_λ. That is,

$$E_\lambda := \ker(\lambda I_n - A).$$

The eigenvectors $w \in E_\lambda$ are found by solving the equation

$$(\lambda I_n - A)\, w = 0.$$

This matrix equation is equivalent to a system of n linear algebraic equations; the solution space is exactly the λ-eigenspace E_λ.

Exercise 4 Let $A \in \mathbb{R}^{n\times n}$. If $\lambda_1, \lambda_2, \ldots, \lambda_r$ are r $(r \le n)$ *distinct* eigenvalues of A with corresponding eigenvectors $w_1, w_2, \ldots, w_r$, show that the vectors $w_1, w_2, \ldots, w_r$ are *linearly independent.*

The **characteristic polynomial** of $A \in \mathbb{R}^{n\times n}$ is

$$\mathrm{char}_A(\lambda) := \det(\lambda I_n - A).$$

The algebraic equation

$$\mathrm{char}_A(\lambda) \equiv \lambda^n + k_1\lambda^{n-1} + \cdots + k_{n-1}\lambda + k_n = 0$$

is called the **characteristic equation** of A.

NOTE : The *degree* of the characteristic polynomial (equation) is n and the leading term is λ^n. The eigenvalues of A are exactly the real roots of $\mathrm{char}_A(\lambda)$.

The *fundamental theorem of algebra* insures that the characteristic polynomial $\mathrm{char}_A(\lambda)$ has n roots, but some roots may be complex numbers, and some roots may be repeated. A complex root of the characteristic polynomial is called a **complex eigenvalue** of A. The complex eigenvalues must occur in conjugate pairs. If λ is a complex eigenvalue of the matrix A, we write $\lambda \in \sigma_{\mathbb{C}}(A)$. Henceforth, we shall refer to both sets $\sigma(A)$ and $\sigma_{\mathbb{C}}(A)$ as the spectrum of A. (In fact, $\sigma_{\mathbb{C}}(A)$ is the spectrum of the "complexification" of (the linear mapping) $A : \xi + i\,\eta \mapsto A\xi + iA\eta$).

An important result is the following : *If the matrix A is symmetric (i.e. $A = A^T$), then all its eigenvalues are real.*

A useful result is the CAYLEY-HAMILTON THEOREM, which states that *every square matrix satisfies its own characteristic equation*; that is, if $A \in \mathbb{R}^{n\times n}$, then :

$$\mathrm{char}_A(A) \equiv A^n + k_1A^{n-1} + \cdots + k_{n-1}A + k_nI_n = O.$$

Let $\lambda \in \sigma_{\mathbb{C}}(A)$.

- The **algebraic multiplicity** m_λ of λ is the number of times it is repeated as a root of the characteristic polynomial.

- The **geometric multiplicity** d_λ of λ is the dimension of the λ-eigenspace E_λ. In other words, d_λ is the maximum number of linearly independent eigenvectors associated with λ.

In general, $d_\lambda \leq m_\lambda$. The following remarkable result holds :

The matrix $A \in \mathbb{R}^{n\times n}$ *is diagonalizable (that is,* A *is similar to a diagonal matrix) if and only if* $d_\lambda = m_\lambda$. If all the eigenvalues of A are real and distinct, then A is diagonalizable. The converse is *not* true.

Exercise 5 Let $A \in \mathbb{R}^{n\times n}$ with complex eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (listed with their algebraic multiplicities). Show that :

(a) $\lambda_1 + \lambda_2 + \cdots + \lambda_n = \mathrm{tr}\,(A)$.

(b) $\lambda_1 \cdot \lambda_2 \cdots \lambda_n = \det\,(A)$.

Quadratic forms

Let $A = \left[a_{ij}\right] \in \mathbb{R}^{n\times n}$ be a *symmetric* matrix. A function $q : \mathbb{R}^n = \mathbb{R}^{n\times 1} \to \mathbb{R}$ defined by

$$\begin{aligned} q(x) &:= x^T A x = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j \\ &= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + \cdots \end{aligned}$$

is called a **quadratic form** (on $\mathbb{R}^n$). Clearly, $q(0) = 0$.

A quadratic form q is said to be :

(a) **positive definite** provided $q(x) > 0$ for all *nonzero* $x \in \mathbb{R}^{n\times 1}$.

(b) **negative definite** provided $q(x) < 0$ for all *nonzero* $x \in \mathbb{R}^{n\times 1}$.

(c) **positive semi-definite** provided $q(x) \geq 0$ for all $x \in \mathbb{R}^{n\times 1}$.

(d) **negative semi-definite** provided $q(x) \leq 0$ for all $x \in \mathbb{R}^{n\times 1}$.

Finally, we call q **indefinite** provided q takes positive as well as negative values.

NOTE : (1) The terms describing the quadratic form q are also applied to the (symmetric) matrix A associated with the form.

(2) The definitions on *definiteness* and *semi-definiteness* can be extended to scalar functions (defined on some $\mathbb{R}^n$) which are not necessarily quadratic.

One simple way of determining the sign properties of a quadratic form is the following:

The quadratic form $q : x \mapsto x^T Ax$ *(or, equivalently, the matrix* A *) is :*

- *positive definite if and only if all the eigenvalues of* A *are positive.*
- *negative definite if and only if all the eigenvalues of* A *are negative.*
- *positive semi-definite if and only if all the eigenvalues of* A *are nonnegative.*
- *negative semi-definite if and only if all the eigenvalues of* A *are nonpositive.*

An alternative approach involves the **principal minors** P_i of A, these being any i^{th} order *minors* whose main diagonal is part of the main diagonal of A. In particular, the **leading principal minors** of A are

$$\Delta_1 := a_{11}\,, \quad \Delta_2 := \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\,, \quad \Delta_3 := \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\,, \quad \text{etc.}$$

The SYLVESTER CONDITIONS state that *the quadratic form* $q : x \mapsto x^T Ax$ *(or, equivalently, the matrix* A *) is :*

- *positive definite if and only if* $\Delta_i > 0\,, \quad i = 1, 2, \ldots, n;$
- *negative definite if and only if* $(-1)^i \Delta_i > 0\,, \quad i = 1, 1, \ldots, n;$
- *positive semi-definite if and only if* $P_i \geq 0$ *for all principal minors*;

- *negative semi-definite if and only if* $(-1)^i P_i \geq 0$ *for all principal minors.*

If q *satisfies none of the above conditions, then it is indefinite.*

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix such that $\operatorname{rank}(A) = r$. Then A is positive semi-definite ($A^T = A \geq 0$) if and only if $A = B^T B$ for some matrix B with $\operatorname{rank}(B) = r$.

The matrix exponential

In order to define the *exponential* of a matrix, we need to discuss the *convergence* of (infinite) sequences and series involving matrices.

Because $\mathbb{R}^{m\times n}$ is a vector space of dimension mn, magnitudes of matrices $A \in \mathbb{R}^{m\times n}$ can be "measured" by employing any *norm* on $\mathbb{R}^{mn}$. One of the simplest *matrix norms* is the following :

$$\|A\| := \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2} = \sqrt{\operatorname{tr}(A^T A)} \quad \text{for } A = \left[a_{ij}\right] \in \mathbb{R}^{m\times n}.$$

For a column matrix (vector) $x \in \mathbb{R}^{m\times 1}$ this gives the **Euclidean norm**

$$\|x\|_e := \sqrt{x_1^2 + x_2^2 + \cdots + x_m^2}.$$

NOTE : (1) Other matrix norms can also be defined. For example, *any* norm $\|\cdot\|_*$ that is defined on $\mathbb{R}^m$ and $\mathbb{R}^n$ *induces* a matrix norm on $\mathbb{R}^{m\times n}$ by setting

$$\|A\|_* := \max_{\|x\|_*=1} \|Ax\|_* \quad \text{for } A \in \mathbb{R}^{m\times n} \text{ and } x \in \mathbb{R}^{n\times 1}.$$

(2) The matrix norm $\|\cdot\|$ and the Euclidean norm $\|\cdot\|_e$ are *compatible* :

$$\|Ax\|_e \leq \|A\|\,\|x\|_e.$$

The matrix norm has all the usual properties of a *norm*; that is (for $A, B \in \mathbb{R}^{m\times n}$ and $\lambda \in \mathbb{R}$) :

(a) $\|A\| \geq 0\,, \quad \text{and} \quad \|A\| = 0 \iff A = 0.$

(b) $\|\lambda A\| = |\lambda|\|A\|.$

(c) $\|A + B\| \leq \|A\| + \|B\|.$

Also, the following two relations hold (provided that the product AB is defined) :

(d) $\|AB\| \leq \|A\|\,\|B\|.$

(e) $\|A^k\| \leq \|A\|^k\,, \quad k = 0, 1, 2, \ldots.$

A *sequence* $(A_k)_{k\in\mathbb{N}}$ of matrices in $\mathbb{R}^{m\times n}$ is said to **converge** to the *limit* $A \in \mathbb{R}^{m\times n}$, denoted by

$$\lim_{k\to\infty} A_k = A,$$

if the sequence $(\|A_k - A\|)_{k\in\mathbb{N}}$ of (positive) real numbers converges to 0; that is, for every $\varepsilon > 0$ there exists an $\nu \in \mathbb{N}$ such that for $k \geq \nu$, $\|A - A_k\| < \varepsilon$.

A necessary and sufficient condition for convergence is that each entry of A_k tends (converges) to the corresponding entry of A as $k \to \infty$.

The (infinite) matrix *series* $\sum_{k\geq 0} A_k$ **converges** provided the sequence of *partial sums* $(S_k)_{k\in\mathbb{N}}$, where $S_k := A_1 + A_2 + \cdots + A_k$, converges to a limit S (as $k \to \infty$); if a limit exists, then it is unique and we shall write $S = \sum_{k=0}^{\infty} A_k$. The series is **absolutely convergent** if the scalar series $\sum_{k\geq 0} \|A_k\|$ is convergent; *an absolutely convergent matrix series is convergent.*

Consider now a matrix $A \in \mathbb{R}^{n\times n}$.

Exercise 6 Show that the matrix power series $\sum_{k\geq 0} \frac{t^k}{k!} A^k$ is *convergent* (in fact, absolutely convergent) for every $t \in \mathbb{R}$.

We define the **matrix exponential** of A by

$$\boxed{\exp(tA) := I_n + tA + \tfrac{t^2}{2!}A^2 + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.}$$

The matrix exponential has a number of important properties.

(a) $\dfrac{d}{dt}(\exp(tA)) = A\exp(tA) = \exp(tA)A.$

(b) $\exp((t+s)A) = \exp(tA)\cdot\exp(sA).$

(c) $\exp(A) = \lim\limits_{k\to\infty}\left(I_n + \dfrac{A}{k}\right)^k.$

(d) $\det(\exp(A)) = e^{\mathrm{tr}\,(A)}.$

From (b) it follows that

$$\exp(tA)\cdot\exp(-tA) = I_n$$

and thus

$$\boxed{\exp(tA)^{-1} = \exp(-tA).}$$

1.5 Exercises

Exercise 7 Suppose $A, S \in \mathbb{R}^{n\times n}$ and S is invertible. Show that

$$(S^{-1}AS)^2 = S^{-1}A^2S.$$

Generalize to $(S^{-1}AS)^n$.

Exercise 8 Set

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad n \geq 2.$$

Write $A = uu^T \in \mathbb{R}^{n\times n}$ and show that A is singular.

Exercise 9 Let $A \in \mathbb{R}^{n\times n}$ and $0 \neq b \in \mathbb{R}^{n\times 1}$ such that

$$A^r b \neq 0\,, \quad A^{r+1}b = 0$$

for some positive integer $r < n$. By considering the equation

$$c_0 b + c_1 Ab + c_2 A^2 b + \cdots + c_r A^r b = 0$$

where the coefficients $c_i \in \mathbb{R}$, deduce that the (column) vectors

$$b,\ Ab,\ A^2b,\ \cdots,\ A^rb$$

are linearly independent.

Exercise 10 Verify that

$$\operatorname{rank}(A^TA) = \operatorname{rank}(A) = \operatorname{rank}(AA^T)$$

for the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{bmatrix} \in \mathbb{R}^{3\times 4}.$$

Exercise 11 Find the characteristic polynomial, the eigenvalues and the corresponding eigenvectors for each given matrix.

(a) $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.

(b) $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

(c) $\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$.

(d) $\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & -1 \\ ab & a+b \end{bmatrix}$.

(f) $\begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$.

(g) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(h) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(i) $\begin{bmatrix} -1 & 3 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

(j) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Exercise 12 Let $A \in \mathbb{R}^{n\times n}$.

(a) How are the eigenvalues of $A - \mu I_n$ related to those of A ?

(b) How are the eigenvalues of μA related to those of A ?

(c) How are the eigenvalues of A^n related to those of A ?

(d) How are the eigenvalues of A^{-1} related to those of A ?

Exercise 13 Consider a matrix $A \in \mathbb{R}^{n\times n}$. Show that :

(a) The characteristic polynomials of A and A^T are the same.

(b) The characteristic polynomials of A and $S^{-1}AS$ are the same.

(c) If $n = 2$, the characteristic polynomial of A can be written as follows

$$\mathrm{char}_A(\lambda) = \lambda^2 - \mathrm{tr}\,(A)\lambda + \det\,(A).$$

Exercise 14 Let the matrix $A \in \mathbb{R}^{n\times n}$ be invertible and w an eigenvector of A with associated eigenvalue λ.

(a) Is w an eigenvector of A^3 ? If so, what is the eigenvalue ?

(b) Is w an eigenvector of A^{-1} ? If so, what is the eigenvalue ?

(c) Is w an eigenvector of $A + 2I_n$? If so, what is the eigenvalue ?

(d) Is w an eigenvector of $7A$? If so, what is the eigenvalue ?

Exercise 15

(a) A **skew-symmetric** matrix $S \in \mathbb{R}^{n\times n}$ is defined by $S^T = -S$. If $q = x^T S x$ show (by considering q^T) that $q = 0$ for all (column) vectors $x \in \mathbb{R}^{n\times 1}$.

(b) Show that any matrix $A \in \mathbb{R}^{n\times n}$ can be written as $A = A_1 + A_2$, where A_1 is symmetric and A_2 is skew-symmetric. Hence (using the result of (a)) deduce that

$$x^T A x = x^T A_1 x \quad \text{for all (column) vectors } x \in \mathbb{R}^{n\times 1}.$$

Exercise 16 Prove that

$$\exp((t+s)A) = \exp(tA)\exp(sA).$$

[Hint: Multiply the series in powers of A formally; the legitimacy of the term-by-term multiplication is assured by the fact that $\exp(tA)$ is absolutly convergent.]

Exercise 17

(a) Show that $\exp(tA)\cdot\exp(tB)$ does not have to be either $\exp(t(A+B))$ or $\exp(tB)\cdot\exp(tA)$ by calculating all three, where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) Suppose that $AB = BA$. Show that

$$\exp(t(A+B)) = \exp(tA)\cdot\exp(tB) = \exp(tB)\cdot\exp(tA).$$

[Hint: Show that if $P(t) = \exp(t(A+B))\cdot\exp(-tA)\cdot\exp(-tB)$, then $\dot{P}(t) = 0$ for all t. Since $P(0) = I_n$, we must have $P(t) = I_n$.]

Exercise 18

(a) Find $\exp(tA)$ if

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

Generalize to $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$ (the *diagonal matrix* with diagonal elements $a_1, a_2, \ldots, a_n$).

(b) Consider the matrix

$$A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}.$$

Show that

$$\exp(tA) = \begin{bmatrix} e^{at} & bte^{at} \\ 0 & e^{at} \end{bmatrix}.$$

(c) A matrix A is **nilpotent** if some power A^k is the zero matrix. Then the matrix exponential $\exp(tA)$ can be calculated easily because the series stops with the power A^{k-1}. That is, we have $A^k = A^{k+1} = \cdots = 0$, so

$$\exp(tA) = I_n + tA + \frac{t^2}{2!}A^2 + \cdots + \frac{t^{k-1}}{(k-1)!}A^{k-1}\,.$$

Find $\exp(tA)$ for

i. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

ii. $A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 0 \end{bmatrix}$.

Exercise 19 Find $\exp(tA)$, where A is given.

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(d) $A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(f) $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

(g) $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

Exercise 20 TRUE or FALSE ? Motivate your answers.

(a) If $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$, then

$$\det(\lambda A) = \lambda \det(A).$$

(b) If $A,\, B \in \mathbb{R}^{n\times n}$ then

$$\det(A+B) = \det(A) + \det(B).$$

(c) If $A \in \mathbb{R}^{n\times n}$ then

$$\det(AA^T) = \det(A^T A).$$

(d) If $A \in \mathbb{R}^{m\times n}$ then

$$\operatorname{rank}(A^T A) = \operatorname{rank}(A) = \operatorname{rank}(AA^T).$$

(e) A matrix $A \in \mathbb{R}^{n\times n}$ is invertible if and only if 0 is *not* an eigenvalue of A.

(f) If $t \in \mathbb{R}$ then

$$\exp\left(t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

(g) If $A,\, B \in \mathbb{R}^{n\times n}$ then

$$\exp(A+B) = \exp(A)\cdot\exp(B).$$

(h) If $A \in \mathbb{R}^{n\times n}$ then

$$\det\,(\exp(A)) \neq 0.$$

Chapter 2

Linear Dynamical Systems

Topics :

1. Solution of Uncontrolled System
2. Solution of Controlled System
3. Time-Varying Systems
4. Relationship between State Space and Classical Forms

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A *control system* (with outputs) $\Sigma = (\mathbb{R}^m, \mathbb{R}^\ell, \mathcal{U}, F, \mathbb{R}^n, h)$ is *linear* if the dynamics F is linear in (x, u), and the measurement function h is linear, for each fixed $t \in \mathbb{R}$. Such a control system is described by (state equation and observation equation)

$$\dot{x} = A(t)x + B(t)u(t) \quad \text{and} \quad y = C(t)x$$

where $A(t) \in \mathbb{R}^{m\times m}$, $B(t) \in \mathbb{R}^{m\times \ell}$, and $C(t) \in \mathbb{R}^{n\times m}$, each of whose entries is a (continuous) function of time. The system is called *time-invariant* if the structure is independent of time. A system that is not necessarily time-invariant is sometimes called, to emphasize the fact, a *time-varying* system. Sets of (scalar) state equations describing a linear (time-invariant) control system are the easiest to manage analytically and numerically, and the first model of a situation is often constructed to be linear for this reason.

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2.1 Solution of Uncontrolled System

To begin with we shall consider dynamical systems (i.e. systems without the presence of control variables). We may also refer to such systems as *uncontrolled* (or unforced) systems.

We discuss methods of finding the *solution* (*state vector*) $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} \in \mathbb{R}^{m\times 1}$ of the (initialized) **linear dynamical system** described by

$$\boxed{\dot{x} = Ax, \quad x(0) = x_0.} \tag{2.1}$$

Here $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in \mathbb{R}^{m\times m}$ ($x \mapsto Ax$ represents the linear dynamics F) and $x_0 \in \mathbb{R}^m$ is the **initial state**.

NOTE : We *identify* the column matrix (or vector) $\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}^{m\times 1}$ with the m-tuple (or point) $(a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$, whenever appropriate. However, we *do not* identify the row matrix (or covector) $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{1\times n}$ with $(a_1, a_2, \ldots, a_n)$ (but rather with the linear *functional* $(x_1, x_2, \cdots, x_n) \mapsto a_1x_1 + a_2x_2 + \cdots + a_nx_n$).

We shall assume that *all the eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *of* A *are distinct.*

NOTE : In fact, in real-life situations, this is not too severe a restriction, since if A does have repeated eigenvalues, very small perturbations in a few of its elements (which will only be known to a certain degree of accuracy) will suffice to separate these equal eigenvalues.

Spectral form

If w_i is an eigenvector corresponding to λ_i, then $w_1, w_2, \ldots, w_m$ are *linearly independent* (see **Exercise 4**), so we can express the solution of (2.1)

as

$$x(t) = c_1(t)w_1 + c_2(t)w_2 + \cdots + c_m(t)w_m \tag{2.2}$$

where $c_i = c_i(t)\,, \quad i = 1, 2, \ldots, m$ are scalar functions of time.

Differentiation of (2.2) and substitution into (2.1) gives :

$$\sum_{i=1}^{m} \dot{c}_i(t)w_i = A\sum_{i=1}^{m} c_i(t)w_i = \sum_{i=1}^{m} c_i(t)\lambda_i w_i.$$

Hence, by the independence of the $w_i \quad (i = 1, 2, \ldots, m)$

$$\dot{c}_i = \lambda_i\, c_i\,, \quad i = 1, 2, \ldots, m$$

and these equations have the solution

$$c_i(t) = c_i(0)\, e^{\lambda_i t}\,, \quad i = 1, 2, \ldots, m$$

giving

$$x(t) = \sum_{i=1}^{m} c_i(0)\, e^{\lambda_i t}\, w_i. \tag{2.3}$$

Let W denote the matrix whose columns are $w_1, w_2, \ldots, w_m$; that is,

$$W = \begin{bmatrix} w_1 & w_2 & \ldots & w_m \end{bmatrix}.$$

We shall denote by $v_1, v_2, \ldots, v_m$ the rows of the matrix W^{-1}; that is,

$$\begin{bmatrix} w_1 & w_2 & \ldots & w_m \end{bmatrix}^{-1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}.$$

Since we have

$$v_i w_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

multiplying (2.3) on the left by v_i and setting $t = 0$ in the resulting expression gives

$$v_i x(0) = c_i(0)\,, \quad i = 1, 2, \ldots, m.$$

Thus the solution of (2.1) is

$$\boxed{x(t) = \sum_{i=1}^{m} (v_i x(0))\, e^{\lambda_i t}\, w_i.} \tag{2.4}$$

Expression (2.4) depends *only* upon the initial condition and the eigenvalues and eigenvectors of A, and for this reason is referred to as the **spectral form solution**.

2.1.1 Example. Find the general solution of the uncontrolled system :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution : The characteristic equation of A is

$$\mid \lambda I_2 - A \mid = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} = 0 \iff \lambda^2 + 3\lambda + 2 = 0$$

which gives

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = -1\,.$$

(i) $\lambda = -2$.

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} 2w_{11} + w_{21} = 0 \\ -2w_{11} - w_{21} = 0 \end{cases}$$

which implies

$$w_{21} = -2w_{11}$$

and thus (we can choose)

$$w_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

(ii) $\lambda = -1$.

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} w_{12} \\ w_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} w_{12} + w_{22} = 0 \\ -2w_{12} - 2w_{22} = 0 \end{cases}$$

which implies

$$w_{22} = -w_{12}$$

and thus (we can choose)

$$w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have

$$W = \begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \Rightarrow W^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix};$$

so

$$v_1 = \begin{bmatrix} -1 & -1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

Finally, we get

$$\begin{aligned} x(t) &= (v_1 x(0))\, e^{-2t}\, w_1 + (v_2 x(0))\, e^{-t}\, w_2 \\ &= (-x_1(0) - x_2(0))\, e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (2x_1(0) + x_2(0))\, e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

or

$$\begin{cases} x_1(t) = -\left(x_1(0) + x_2(0)\right) e^{-2t} + \left(2x_1(0) + x_2(0)\right) e^{-t} \\ \\ x_2(t) = 2\left(x_1(0) + x_2(0)\right) e^{-2t} - \left(2x_1(0) + x_2(0)\right) e^{-t}. \end{cases}$$

Exponential form

We now present a different approach to solving equation (2.1) which avoids the need to calculate the eigenvectors of A.

Recall the definition of the *matrix exponential*

$$\boxed{\exp(tA) := I_m + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots} \tag{2.5}$$

2.1.2 Lemma. *Let* $A \in \mathbb{R}^{m\times m}$. *Then*

$$\frac{d}{dt}(\exp(tA)) = A\exp(tA) = \exp(tA)A.$$

Proof : We have

$$\begin{aligned}
\frac{d}{dt}\exp(tA) &= \lim_{h\to 0}\frac{1}{h}(\exp((t+h)A) - \exp(tA)) \\
&= \lim_{h\to 0}\frac{1}{h}(\exp(tA)\cdot\exp(hA) - \exp(tA)) \\
&= \exp(tA)\lim_{h\to 0}\frac{1}{h}(\exp(hA) - I_m) \\
&= \exp(tA)\lim_{h\to 0}\lim_{k\to\infty}\left(A + \frac{h}{2!}A + \cdots + \frac{h^{k-1}}{k!}A^k\right) \\
&= \exp(tA)A.
\end{aligned}$$

(Two convergent *limit processes* can be interchanged if one of them converges uniformly.) Observe that A commutes with each term of the (absolutely convergent) series for $\exp(tA)$, hence with $\exp(tA)$. This proves the lemma.□

By the preceding lemma, if $x(t) = \exp(tA)x_0$, then

$$\dot{x}(t) = \frac{d}{dt}\exp(tA)x_0 = A\exp(tA)x_0 = Ax(t)$$

for all $t \in \mathbb{R}$. Also,

$$x(0) = I_m x_0 = x_0.$$

Thus $x(t) = \exp(tA)x_0$ is *a* solution of (2.1).

To see that this is the *only* solution, let $x(\cdot)$ be any solution of (2.1) and set

$$y(t) = \exp(-tA)x(t).$$

Then (from the above lemma and the fact that $x(\cdot)$ is a solution of (2.1))

$$\begin{aligned}
\dot{y}(t) &= -A\exp(-tA)x(t) + \exp(-tA)\dot{x}(t) \\
&= -A\exp(-tA)x(t) + \exp(-tA)Ax(t) \\
&= 0
\end{aligned}$$

for all $t \in \mathbb{R}$ since $\exp(-tA)$ and A commute. Thus, $y(t)$ is a constant. Setting $t = 0$ shows that $y(t) = x_0$ and therefore *any solution of (2.1) is given by*

$$x(t) = \exp(tA)y(t) = \exp(tA)x_0.$$

Hence

$$\boxed{x(t) = \exp(tA)\, x_0} \tag{2.6}$$

does represent *the* solution of (2.1).

NOTE : In case the initial condition $x(0) = x_0$ is replaced by the slighty more general one $x(t_0) = x_0$, the solution is often written as

$$x(t) = \Phi(t, t_0)x_0. \tag{2.7}$$

One refers to (the matrix)

$$\boxed{\Phi(t, t_0) := \exp((t - t_0)A)} \tag{2.8}$$

as the **state transition matrix** (since it relates the state at any time t to the state at any other time t_0).

2.1.3 PROPOSITION. *The state transition matrix $\Phi(t, t_0)$ has the following properties :*

(a) $\dfrac{d}{dt}\Phi(t, t_0) = A\Phi(t, t_0)$.

(b) $\Phi(t_0, t_0) = I_m$.

(c) $\Phi(t_0, t) = \Phi^{-1}(t, t_0)$.

(d) $\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$.

PROOF : We have

(a) $\dfrac{d}{dt}\Phi(t, t_0) = \dfrac{d}{dt}\exp((t - t_0)A) = A\exp((t - t_0)A) = A\Phi(t, t_0)$.

(b) $\Phi(t, t) = \exp((t - t)A) = \exp(0) = I_m$.

(c) $\Phi^{-1}(t, t_0) = (\exp((t - t_0)A))^{-1} = \exp(-(t - t_0)A) = \exp((t_0 - t)A) = \Phi(t_0, t)$.

(d) $\Phi(t,t_1)\Phi(t_1,t_0) = \exp((t-t_1)A)\cdot\exp((t_1-t_0)A) = \exp\left((t-t_1+t_1-t_0)A\right) = \exp((t-t_0)A) = \Phi(t,t_0)$.

□

NOTE : The matrix-valued mapping $X(t) = \Phi(t,t_0)$ (or *curve* in $\mathbb{R}^{m\times m}$) is the *unique* solution of the matrix differential equation

$$\dot{X} = AX, \quad X \in \mathbb{R}^{m\times m}$$

subject to the initial condition $X(t_0) = I_m$.

2.1.4 EXAMPLE. (Simple harmonic motion) Consider a unit mass connected to a support through a spring whose spring constant is unity. If z measures the displacement of the mass from equilibrium, then

$$\ddot{z} + z = 0.$$

Letting $x_1 = z$ and $x_2 = \dot{z}$ gives

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

NOTE : The associated transition matrix $\Phi(t,t_0)$ has the form

$$\Phi(t,t_0) = \begin{bmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) \\ \\ \phi_{21}(t,t_0) & \phi_{22}(t,t_0) \end{bmatrix}$$

and therefore satisfies

$$\begin{bmatrix} \dot{\phi}_{11}(t,t_0) & \dot{\phi}_{12}(t,t_0) \\ \\ \dot{\phi}_{21}(t,t_0) & \dot{\phi}_{22}(t,t_0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) \\ \\ \phi_{21}(t,t_0) & \phi_{22}(t,t_0) \end{bmatrix}$$

with the initial condition

$$\Phi(t_0,t_0) = I_m.$$

What is the physical interpretation of $\Phi(t,t_0)$ in this case ? The first column of $\Phi(t,t_0)$ has as its first entry the position as a function of time which results when

the mass is displaced by one unit and released at t_0 with zero velocity. The second entry in the first column is the corresponding velocity. The second column of $\Phi(t, t_0)$ has as its first entry the position as a function of time which results when the mass is started from zero displacement but with unit velocity at $t = t_0$. The second entry in the second column is the corresponding velocity.

The series for computing $\Phi(t, t_0)$ in this case is easily summed because

$$A^k = \begin{cases} A & \text{if } k = 4p+1 \\ -I_2 & \text{if } k = 4p+2 \\ -A & \text{if } k = 4p+3 \\ I_2 & \text{if } k = 4p. \end{cases}$$

A short calculation gives

$$\Phi(t, t_0) = \begin{bmatrix} \cos(t - t_0) & \sin(t - t_0) \\ -\sin(t - t_0) & \cos(t - t_0) \end{bmatrix}.$$

Exercise 21 Work out the preceding computation.

2.1.5 Example. (Satellite problem) In section **1.3** we introduced the equations of a unit mass in an inverse square law force field. These were then *linearized* about a circular orbit to get

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ u_1(t) \\ 0 \\ u_2(t) \end{bmatrix}.$$

The series for computing $\Phi(t,0)$ can be summed to get

$$\Phi(t,0) = \begin{bmatrix} 4-3\cos\omega t & \dfrac{\sin\omega t}{\omega} & 0 & \dfrac{2(1-\cos\omega t)}{\omega} \\ 3\omega\sin\omega t & \cos\omega t & 0 & 2\sin\omega t \\ 6(-\omega t+\sin\omega t) & -\dfrac{2(1-\cos\omega t)}{\omega} & 1 & \dfrac{-3\omega t+4\sin\omega t}{\omega} \\ 6\omega(-1+\cos\omega t) & -2\sin\omega t & 0 & -3+4\cos\omega t \end{bmatrix}.$$

Evaluation of the matrix exponential

Evaluation of $\exp(tA)$, when *all the eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *are distinct*, can be achieved by **Sylvester's formula** which gives

$$\boxed{\exp(tA) = \sum_{k=1}^{m} e^{\lambda_k t}\, Z_k} \tag{2.9}$$

where

$$Z_k = \prod_{\substack{i=1 \\ i\neq k}}^{m} \frac{A-\lambda_i I_m}{\lambda_k-\lambda_i}\,, \quad k=1,2,\ldots,m.$$

NOTE : Since the Z_k $(k=1,2,\ldots,m)$ in (2.9) are constant matrices depending only on A and its eigenvalues, the solution in the form given in (2.9) requires calculation of only the eigenvalues of A.

2.1.6 EXAMPLE. Consider again the uncontrolled system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The solution is

$$x(t) = \exp(tA)\, x_0 = \left(\sum_{k=1}^{2} e^{\lambda_k t} Z_k \right) x_0$$

where

$$Z_k = \prod_{\substack{i=1 \\ i \neq k}}^{2} \frac{A - \lambda_i I_2}{\lambda_k - \lambda_i}, \quad k = 1, 2.$$

We have

$$Z_1 = \frac{A - (-1)I_2}{-2 - (-1)} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}; \quad Z_2 = \frac{A - (-2)I_2}{-1 - (-2)} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} x(t) &= \left(e^{-2t} Z_1 + e^{-t} Z_2\right) x_0 \\ &= \left(e^{-2t} \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} + e^{-t} \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\right) \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \end{aligned}$$

or

$$\begin{cases} x_1(t) = -\left(x_1(0) + x_2(0)\right) e^{-2t} + \left(2x_1(0) + x_2(0)\right) e^{-t} \\ \\ x_2(t) = 2\left(x_1(0) + x_2(0)\right) e^{-2t} - \left(2x_1(0) + x_2(0)\right) e^{-t}. \end{cases}$$

An alternative way of evaluating $\exp(tA)$ (again, when *all the eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *are distinct*), is as follows.

We can write

$$e^{t\lambda} = q(\lambda) \cdot \mathrm{char}_A(\lambda) + r(\lambda) \tag{2.10}$$

where $\deg(r) < m$. Since (2.10) is an identity, we have

$$\exp(tA) \equiv q(A) \cdot \mathrm{char}_A(A) + r(A)$$

which by the **Cayley-Hamilton Theorem** reduces to

$$\exp(tA) \equiv r(A)$$

showing that $\exp(tA)$ *can be represented by a finite sum of powers of* A *of degree not exceeding* $m - 1$. Then m coefficients of $r(\lambda)$ are functions of t obtained from the solution of the system of m linear equations

$$e^{\lambda_i t} = r(\lambda_i), \quad i = 1, 2, \ldots, m.$$

2.1.7 Example. Consider once again the uncontrolled system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Since $m = 2$, the polynomial $r(\lambda)$ can be written

$$r(\lambda) = r_0\lambda + r_1$$

and so we have

$$\begin{cases} e^{-t} = r_1 - r_0 \\ \\ e^{-2t} = r_1 - 2r_0 \end{cases}$$

which gives $r_0 = e^{-t} - e^{-2t}$ and $r_1 = 2e^{-t} - e^{-2t}$. Hence, the solution is

$$\begin{aligned} x(t) &= \exp(tA)\, x_0 = (r_0 A + r_1 I_2) x_0 \\ &= \left((e^{-t} - e^{-2t}) \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + (2e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \end{aligned}$$

or

$$\begin{cases} x_1(t) = -\left(x_1(0) + x_2(0)\right) e^{-2t} + \left(2x_1(0) + x_2(0)\right) e^{-t} \\ \\ x_2(t) = 2\left(x_1(0) + x_2(0)\right) e^{-2t} - \left(2x_1(0) + x_2(0)\right) e^{-t}. \end{cases}$$

2.2 Solution of Controlled System

Consider the (initialized) linear control system, written in *state space form*,

$$\boxed{\dot{x} = Ax + Bu(t)\,, \quad x(0) = x_0} \tag{2.11}$$

where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times \ell}$, and $\ell \le m$.

After multiplication of both sides of (2.11), on the left, by $\exp(-tA)$, the equation can be written

$$\frac{d}{dt}\left(\exp(-tA)x\right) = \exp(-tA)Bu \tag{2.12}$$

which produces

$$x(t) = \exp(tA)\left[x_0 + \int_0^t \exp(-\tau A)Bu(\tau)\,d\tau\right]. \tag{2.13}$$

If the initial condition is $x(t_0) = x_0$, then integration of (2.12) from t_0 to t and use of the definition of Φ gives

$$\boxed{x(t) = \Phi(t,t_0)\left[x_0 + \int_{t_0}^t \Phi(t_0,\tau)Bu(\tau)\,d\tau\right].} \tag{2.14}$$

NOTE : If $u(t)$ is known for $t \geq t_0$, then $x(t)$ can be determined by finding the state transition matrix and carrying out the integration in (2.14).

2.2.1 EXAMPLE. Consider the *equation of motion*

$$\ddot{z} = u(t)$$

of a unit mass moving in a straight line, subject to an external force $u(t)$, $z(t)$ being the displacement from some fixed point. In state space form, taking

$$x_1 = z \quad \text{and} \quad x_2 = \dot{z}$$

as state variables, this becomes

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) = Ax + Bu(t).$$

Since here we have $A^2 = 0$, $\exp(tA) = I_2 + tA$, and so

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\int_0^t \begin{bmatrix} 1 & -\tau \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau)\,d\tau.$$

Solving for $x_1(t)$ leads to

$$z(t) = z(0) + t\dot{z}(0) + \int_o^t (t-\tau)u(\tau)\,d\tau$$

where $\dot{z}(0)$ denotes the initial velocity of the mass.

2.2.2 Example. We are now in a position to express the solution of the linearized equations describing the motion of a satellite in a near circular orbit. We have

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 4-3\cos\omega t & \frac{\sin\omega t}{\omega} & 0 & \frac{2(1-\cos\omega t)}{\omega} \\ 3\omega\sin\omega t & \cos\omega t & 0 & 2\sin\omega t \\ 6(-\omega t+\sin\omega t) & -\frac{2(1-\cos\omega t)}{\omega} & 1 & \frac{-3\omega t+4\sin\omega t}{\omega} \\ 6\omega(-1+\cos\omega t) & -2\sin\omega t & 0 & -3+4\cos\omega t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} +$$

$$+\int_0^t \left(\begin{bmatrix} \frac{\sin\omega(t-\tau)}{\omega} \\ \cos\omega(t-\tau) \\ -\frac{2(1-\cos\omega(t-\tau))}{\omega} \\ -2\sin\omega(t-\tau) \end{bmatrix} u_1(\tau) + \begin{bmatrix} \frac{2(1-\cos\omega(t-\tau))}{\omega} \\ 2\sin\omega(t-\tau) \\ \frac{-3\omega(t-\tau)+4\sin\omega(t-\tau)}{\omega} \\ -3+4\cos\omega(t-\tau) \end{bmatrix} u_2(\tau) \right) d\tau.$$

2.3 Time-varying Systems

Of considerable importance in many applications are linear systems in which the elements of A and B are (continuous) functions of time for $t \geq 0$.

Note : In general, it will *not* be possible to give explicit expressions for solutions and we shall content ourselves with obtaining some general properties.

We first consider the *uncontrolled* case

$$\boxed{\dot{x} = A(t)x\,, \quad x(0) = x_0.} \tag{2.15}$$

2.3.1 Theorem. (Existence and Uniqueness Theorem) *If the matrix-valued mapping*

$$A : [0,\infty) \to \mathbb{R}^{m\times m}, \quad t \mapsto A(t)$$

is continuous, then (2.15) has a unique solution

$$x(t) = X(t)x_0, \qquad t \geq 0$$

where $X(\cdot)$ *is the unique matrix-valued mapping (or curve in* $\mathbb{R}^{m\times m}$*) satisfying*

$$\dot{X} = A(t)X, \quad X(0) = I_m. \tag{2.16}$$

PROOF : We shall use the *method of successive approximations* to establish the existence of a solution of (2.16). In place of (2.16), we consider the *integral equation*

$$X = I_m + \int_0^t A(\tau)X\, d\tau. \tag{2.17}$$

Define the sequence $(X_k)_{k\geq 0}$ of matrices (in fact, of matrix-valued mappings) as follows :

$$\begin{aligned} X_0 &= I_m \\ X_{k+1} &= I_m + \int_0^t A(\tau)X_k\, d\tau\,, \quad k = 0, 1, 2, \ldots \end{aligned}$$

Then we have

$$X_{k+1} - X_k = \int_0^t A(\tau)(X_k - X_{k-1})\, d\tau\,, \quad k = 1, 2, \ldots$$

Let

$$\nu = \max_{0\leq t\leq t_1} \|A(t)\|$$

where

$$\|A(t)\| := \sum_{i,j=1}^{m} |a_{ij}(t)|.$$

NOTE : *Any* matrix norm (on $\mathbb{R}^{m\times m}$) will do.

We have

$$\begin{aligned} \|X_{k+1} - X_k\| &= \|\int_0^t A(\tau)(X_k - X_{k-1})\, d\tau\| \\ &\leq \int_0^t \|A(\tau)\|\|X_k - X_{k-1}\|\, d\tau \\ &\leq \nu \int_0^t \|X_k - X_{k-1}\|\, d\tau \end{aligned}$$

for $0 \le t \le t_1$. Since, in this same interval,

$$\|X_1 - X_0\| \le \int_0^t \|A(\tau)\|\, d\tau \le \nu t$$

we have *inductively*

$$\|X_{k+1} - X_k\| \le M_{k+1} := \frac{\nu^{k+1} t_1^{k+1}}{(k+1)!} \quad \text{for } 0 \le t \le t_1.$$

NOTE : The WEIERSTRASS M-TEST states that *if* $\xi_k : [t_0, t_1] \to \mathbb{R}^{m\times m}$ *are continuous and*

- $\|\xi_k(t)\| \le M_k$ *for every* k
- $\sum_{k=0}^{\infty} M_k < \infty$

then the series $\sum_{k\ge 0} \xi_k(t)$ *converges uniformly and absolutely on the interval* $[t_0, t_1]$.

Hence, the (matrix-valued mapping) series

$$X_0 + \sum_{k\ge 0} (X_{k+1} - X_k)$$

converges uniformly for $0 \le t \le t_1$. Consequently, (X_k) converges *uniformly* and *absolutely* to a matrix-valued mapping $X(\cdot)$, which satisfies (2.17), and thus (2.16).

Since, by assumption, $A(\cdot)$ is continuous for $t \ge 0$, *we must take* t_1 *arbitrarily large*. We thus obtain a solution valid for $t \ge 0$.

It is easy to verify that $x(t) = X(t)x_0$ is a solution of (2.15), satisfying the required initial condition.

Let us now establish uniqueness of this solution. Let Y be another solution of (2.16). Then Y satisfies (2.17), and thus we have the relation

$$X - Y = \int_0^t A(\tau)(X(\tau) - Y(\tau))\, d\tau.$$

Hence

$$\|X - Y\| \le \int_0^t \|A(\tau)\| \|X(\tau) - Y(\tau)\|\, d\tau.$$

Since Y is differentiable, hence continuous, define

$$\nu_1 := \max_{0 \le t \le t_1} \|X(t) - Y(t)\|.$$

We obtain

$$\|X - Y\| \le \nu_1 \int_0^t \|A(\tau)\| \, d\tau, \quad 0 \le t \le t_1.$$

Using this bound, we obtain

$$\begin{aligned} \|X - Y\| &\le \nu_1 \int_0^t \|A(\tau)\| \left(\int_0^\tau \|A(\sigma)\| \, d\sigma \right) d\tau \\ &\le \frac{\nu_1 \left(\int_0^t \|A(\tau)\| \, d\tau \right)^2}{2}. \end{aligned}$$

Iterating, we get

$$\|X - Y\| \le \frac{\nu_1^k \left(\int_0^t \|A(\tau)\| \, d\tau \right)^{k+1}}{(k+1)!}.$$

Letting $k \to \infty$, we see that $\|X - Y\| \le 0$. Hence $X \equiv Y$.

Exercise 22 Show that

$$\lim_{k \to \infty} \frac{\alpha^k}{(k+1)!} = 0 \qquad (\alpha > 0).$$

Having obtained the matrix X, it is easy to see that $x(t) = X(t)x_0$ is a solution of (2.15). Since the uniqueness of solutions of (2.15) is readily established by means of the same argument as above, it is easy to see that $x(t) = X(t)x_0$ is *the* solution.

□

NOTE : We can no longer define a matrix exponential, but there is a result corresponding to the fact that $\exp(tA)$ is nonsingular when A is constant. We can write $x(t) = \Phi(t,0)x_0$, where $\Phi(t,0)$ has the form

$$I_m + \int_0^t A(\tau) \, d\tau + \int_0^t A(\tau_1) \int_0^{\tau_1} A(\tau_2) \, d\tau_2 \, d\tau_1 + \int_0^t A(\tau_1) \int_0^{\tau_1} A(\tau_2) \int_0^{\tau_2} A(\tau_3) \, d\tau_3 \, d\tau_2 \, d\tau_1 \cdots$$

(the PEANO-BAKER SERIES).

Some remarks and corollaries

2.3.2 PROPOSITION. *In the* EXISTENCE AND UNIQUENESS THEOREM *the matrix* $X(t)$ *is nonsingular (for every* $t \geq 0$*).*

PROOF : Define a matrix-valued mapping $Y(\cdot)$ as the solution of

$$\dot{Y} = -YA(t), \quad Y(0) = I_m. \tag{2.18}$$

(Such a mapping exists and is unique by an argument virtually identical to that which is used in the proof of the EXISTENCE AND UNIQUENESS THEOREM.)

Now

$$\frac{d}{dt}(YX) = \dot{Y}X + Y\dot{X} = -YAX + YAX = 0$$

so $Y(t)X(t)$ is equal to a constant matrix, which must be the unit matrix because of condition at $t = 0$.

Exercise 23 Show that (for every $t \geq 0$)

$$\det\,(X(t)) \neq 0.$$

Hence $X(t)$ is nonsingular (and its inverse is in fact $Y(t)$). □

We can also generalize the idea of *state transition matrix* by writing

$$\boxed{\Phi(t, t_0) := X(t)X^{-1}(t_0)} \tag{2.19}$$

which exists for all $t, t_0 \geq 0$. It is easy to verify that

$$x(t) = \Phi(t, t_0)x_0 \tag{2.20}$$

is the solution of (2.15). Also, $\Phi(t, t_0)^{-1} = \Phi(t_0, t)$.

NOTE : The expression (2.20) has the same form as that for the time invariant case. However, it is most interesting that although, in general, it is *not* possible to obtain an analytic expression for the solution of (2.16), and therefore for $\Phi(t, t_0)$ in (2.20), this latter matrix possesses precisely the same properties as those for the constant case.

When $m = 1$, we can see that

$$\Phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(\tau)\, d\tau\right).$$

Note that the formula above is *generally not true.* However, one can show that it does hold if

$$A(t)\int_{t_0}^{t} A(\tau)\, d\tau = \left(\int_{t_0}^{t} A(\tau)\, d\tau\right) A(t) \quad \text{for all } t.$$

Otherwise this is not necessarily true and *the state transition matrix is not necessarily the exponential of the integral of* A.

The following result is interesting. We shall omit the proof.

2.3.3 Proposition. *If* $\Phi(t, t_0)$ *is the state transition matrix for*

$$\dot{x} = A(t)x\,, \quad x(t_0) = x_0$$

then

$$\det\,(\Phi(t, t_0)) = e^{\int_{t_0}^{t} \operatorname{tr}(A(\tau))\, d\tau}.$$

A further correspondence with the time-invariant case is the following result.

2.3.4 Proposition. *The solution of*

$$\dot{x} = A(t)x + B(t)u(t)\,, \quad x(t_0) = x_0 \tag{2.21}$$

is given by

$$\boxed{x(t) = \Phi(t, t_0)\left[x_0 + \int_{t_0}^{t} \Phi(t_0, \tau)B(\tau)u(\tau)\, d\tau\right]} \tag{2.22}$$

where $\Phi(t, t_0)$ *is defined in (2.19).*

Proof : Put

$$x = X(t)w \iff X^{-1}(t)x = w$$

where $X(t)$ is defined in (2.16). Substitution into (2.21) produces

$$\dot{x} = AXw + X\dot{w} = AXw + Bu(t).$$

Hence $X(t)\dot{w} = Bu(t)$ and so $\dot{w} = X^{-1}(t)Bu(t)$, which gives

$$w(t) = w(t_0) + \int_{t_0}^{t} X^{-1}(\tau)B(\tau)u(\tau)\,d\tau.$$

The desired expression then follows using $x_0 = X(t_0)w(t_0)$ and (2.19). Indeed,

$$x_0 = x(t_0) = X(t_0)w(t_0) \;\Rightarrow\; w(t_0) = X^{-1}(t_0)x_0$$

and we have

$$\begin{aligned} x(t) = X(t)w &= X(t)\left[X^{-1}(t_0)x_0 + \int_{t_0}^{t} X^{-1}(\tau)B(\tau)u(\tau)\,d\tau\right] \\ &= X(t)X^{-1}(t_0)\left[x_0 + \int_{t_0}^{t} X(t_0)X^{-1}(\tau)B(\tau)u(\tau)\,d\tau\right] \\ &= \Phi(t,t_0)\left[x_0 + \int_{t_0}^{t} \Phi(t_0,\tau)B(\tau)u(\tau)\,d\tau\right]. \end{aligned}$$

□

2.4 Relationship between State Space and Classical Forms

Classical linear control theory deals with scalar ODEs of the form

$$z^{(m)} + k_1 z^{(m-1)} + \cdots + k_{m-1}z^{(1)} + k_m z = \beta_0 u^{(\ell)} + \beta_1 u^{(\ell-1)} + \cdots + \beta_\ell u$$

where $k_1, k_2, \ldots, k_m$ and $\beta_0, \beta_1, \ldots, \beta_\ell$ are constants; it is assumed that $\ell < m$.

We shall consider a simplified form

$$\boxed{z^{(m)} + k_1 z^{(m-1)} + \cdots + k_m z = u(t)} \tag{2.23}$$

where $u(\cdot)$ is the single control variable.

It is easy to write (2.23) in matrix form by taking as state variables

$$w_1 = z, \quad w_2 = z^{(1)}, \quad \ldots, \quad w_m = z^{(m-1)}. \tag{2.24}$$

Since

$$\dot{w}_i = w_{i+1}\,, \quad i = 1, 2, \ldots, m-1$$

(2.23) and (2.24) lead to the *state space form*

$$\boxed{\dot{w} = Cw + du(t)} \tag{2.25}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -k_m & -k_{m-1} & -k_{m-2} & \ldots & -k_1 \end{bmatrix}, \; w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}, \; d = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{2.26}$$

The matrix C is called the **companion form matrix**.

Exercise 24 Show that the characteristic polynomial of C is

$$\mathrm{char}_C(\lambda) = \lambda^m + k_1\lambda^{m-1} + k_2\lambda^{m-2} + \cdots + k_m \tag{2.27}$$

which has the same coefficients as those in (2.23).

The state space form (2.25) is very special; we call it the **canonical form**.

NOTE : The classical form (2.23) and the canonical form are equivalent.

Having seen that (2.23) can be put into matrix form, a natural question is to ask whether the converse hold : *can any linear system in state space form with a single control variable*

$$\dot{x} = Ax + bu(t)$$

be put into the classical form (2.23) ?

2.4.1 Example. Consider the linear system in state space form with a single control variable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t). \tag{2.28}$$

The (state) equations describing the system are

$$\begin{cases} \dot{x}_1 = -2x_1 + 2x_2 + u(t) \\ \\ \dot{x}_2 = x_1 - x_2. \end{cases}$$

Differentiating the second equation we get

$$\ddot{x}_2 = \dot{x}_1 - \dot{x}_2 = (-2x_1 + 2x_2 + u(t)) - (x_1 - x_2) = -3x_1 + 3x_2 + u(t).$$

Hence

$$\ddot{x}_2 + 3\dot{x}_2 = u(t).$$

This second-order ODE for x_2 has the form (2.23) for $m = 2$. Its associated canonical form is

$$\dot{w} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)\,. \tag{2.29}$$

We expect that there is a transformation on $\mathbb{R}^2$ that transforms our original control system (2.28) to the canonical form (2.29). Since the differential equations are linear, we expect that the transformation is linear, say $w = Tx$. Differentiation than gives

$$\dot{w} = TAT^{-1}w + Tbu(t).$$

If we set

$$w_1 = x_2 \quad \text{and} \quad w_2 = \dot{x}_2$$

then

$$\begin{cases} w_1 = x_2 \\ \\ w_2 = x_1 - x_2 \end{cases}$$

so a transformation transforming (2.28) into (2.29) is given by the matrix

$$T = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

2.4.2 Example. *The control system* in state space form

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + bu(t)$$

(where $b \in \mathbb{R}^{2\times 1}$) *cannot be transformed to the canonical form* (or, equivalently, to the classical form).

Exercise 25 Prove the preceding statement.

We would like to determine *when* and *how* such a procedure can be carried out in general. Thus the natural questions concerning existence, uniqueness, and computation of T arise. The answer to these questions is provided by the following result.

2.4.3 Theorem. *A linear control system in state space form*

$$\dot{x} = Ax + bu(t)$$

(where $A \in \mathbb{R}^{m\times m}$ and $0 \neq b \in \mathbb{R}^{m\times 1}$) can be transformed by a linear transformation (i.e. invertible linear mapping)

$$w = Tx$$

into the canonical form

$$\dot{w} = Cw + du(t)$$

where C and d are given by (2.26), provided

$$\boxed{\operatorname{rank}\begin{bmatrix} b & Ab & A^2b & \dots & A^{m-1}b \end{bmatrix} = m.} \tag{2.30}$$

Conversely, if such a transformation T exists, then (2.30) holds.

PROOF : ($\Leftarrow$) *Sufficiency.* Substitution of $w = Tx$ into

$$\dot{x} = Ax + bu$$

produces

$$\dot{w} = TAT^{-1}w + Tbu.$$

We take

$$T = \begin{bmatrix} \tau \\ \tau A \\ \tau A^2 \\ \vdots \\ \tau A^{m-1} \end{bmatrix}$$

where τ is any row m-vector such that T is nonsingular, *assuming for the present that at least one suitable* τ *exists.*

Denote the columns of T^{-1} by $s_1, s_2, \ldots, s_m$ and consider

$$TAT^{-1} = \begin{bmatrix} \tau A s_1 & \tau A s_2 & \ldots & \tau A s_m \\ \tau A^2 s_1 & \tau A^2 s_2 & \ldots & \tau A^2 s_m \\ \vdots & \vdots & & \vdots \\ \tau A^m s_1 & \tau A^m s_2 & \ldots & \tau A^m s_m \end{bmatrix}.$$

Comparison with the identity $TT^{-1} = I_m$ (that is,

$$\begin{bmatrix} \tau s_1 & \tau s_2 & \ldots & \tau s_m \\ \tau A s_1 & \tau A s_2 & \ldots & \tau A s_m \\ \vdots & \vdots & & \vdots \\ \tau A^{m-1} s_1 & \tau A^{m-1} s_2 & \ldots & \tau A^{m-1} s_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix})$$

establishes that the ith row of TAT^{-1} is the $(i+1)^{th}$ row of I_m $(1 = 1, 2, \ldots, m-1)$, so TAT^{-1} has the same form as C in (2.26), with last row given by

$$k_i = -\tau A^m s_{m-i+1}\,, \quad i = 1, 2, \ldots, m.$$

For

$$\dot{w} = TAT^{-1}w + Tbu$$

to be identical to

$$\dot{w} = Cw + du$$

we must also have

$$Tb = d$$

and substitution of T into this relation gives

$$\tau b = 0, \quad \tau Ab = 0, \quad \ldots, \quad \tau A^{m-2}b = 0, \quad \tau A^{m-1}b = 1$$

or

$$\tau \begin{bmatrix} b & Ab & A^2b & \ldots & A^{m-1}b \end{bmatrix} = d^T$$

which has a unique solution for τ (in view of condition (2.30)).

It remains to prove that (the matrix) T is *nonsingular*; we shall show that its rows are linearly independent. Suppose that

$$\alpha_1\tau + \alpha_2\tau A + \cdots + \alpha_m\tau A^{m-1} = 0$$

for some scalars α_i, $i = 1, 2, \ldots, m$. Multiplying this relation on the right by b gives $\alpha_m = 0$. Similarly, multiplying on the right successively by Ab, $A^2b, \ldots, A^{m-1}b$ gives $\alpha_{m-1} = 0, \ldots, \alpha_1 = 0$. Thus, the rows of T are linearly independent.

($\Rightarrow$) *Necessity.* Conversely, if such a transformation T exists, then

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} b & Ab & \ldots & A^{m-1}b \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} Tb & TAb & TA^2b & \ldots & TA^{m-1}b \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} Tb & (TAT^{-1})Tb & \ldots & (TAT^{-1})^{m-1}Tb \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} d & Cd & C^2d & \ldots & C^{m-1}d \end{bmatrix}. \end{aligned}$$

It is easy to verify that this last matrix has the triangular form

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & \theta_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \dots & \theta_{m-4} & \theta_{m-3} \\ 0 & 1 & \theta_1 & \dots & \theta_{m-3} & \theta_{m-2} \\ 1 & \theta_1 & \theta_2 & \dots & \theta_{m-2} & \theta_{m-1} \end{bmatrix}$$

and therefore has full rank. This completes the proof. □

NOTE : T can be constructed using

$$T = \begin{bmatrix} \tau \\ \tau A \\ \vdots \\ \tau A^{m-1} \end{bmatrix} \quad \text{and} \quad \tau \begin{bmatrix} b & Ab & A^2 b & \dots & A^{m-1} b \end{bmatrix} = d^T.$$

However, we can also give an *explicit expression for the matrix in the transformation* $x = T^{-1} w$. We have seen that

$$T \begin{bmatrix} b & Ab & A^2 b & \dots & A^{m-1} b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & \theta_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \dots & \theta_{m-4} & \theta_{m-3} \\ 0 & 1 & \theta_1 & \dots & \theta_{m-3} & \theta_{m-2} \\ 1 & \theta_1 & \theta_2 & \dots & \theta_{m-2} & \theta_{m-1} \end{bmatrix}.$$

This latter matrix has elements given by

$$\theta_i = -k_1 \theta_{i-1} - k_2 \theta_{i-2} - \cdots - k_i \,, \quad i = 1, 2, \dots, m-1.$$

It is straightforward to verify that

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & \theta_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \dots & \theta_{m-4} & \theta_{m-3} \\ 0 & 1 & \theta_1 & \dots & \theta_{m-3} & \theta_{m-2} \\ 1 & \theta_1 & \theta_2 & \dots & \theta_{m-2} & \theta_{m-1} \end{bmatrix}^{-1} = \begin{bmatrix} k_{m-1} & k_{m-2} & \dots & k_1 & 1 \\ k_{m-2} & k_{m-3} & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ k_1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Finally,

$$\boxed{T^{-1} = \begin{bmatrix} b & Ab & A^2b & \dots & A^{m-1}b \end{bmatrix} \begin{bmatrix} k_{m-1} & k_{m-2} & \dots & k_1 & 1 \\ k_{m-2} & k_{m-3} & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ k_1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.} \tag{2.31}$$

($k_1, k_2, \dots, k_{m-1}$ are the coefficients in the characteristic equation of C.)

2.4.4 Example. Consider a system in the form

$$\dot{x} = Ax + bu(t)$$

where

$$A = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find the matrix T and the transformed system.

Solution : From

$$\tau b = 0\,, \quad \tau A b = 1$$

with $\tau = \begin{bmatrix} \tau_1 & \tau_2 \end{bmatrix}$ we have

$$\begin{cases} \tau_1 + \tau_2 = 0 \\ -2\tau_1 + 6\tau_2 = 1 \end{cases}$$

whence

$$\tau_1 = -\frac{1}{8} \quad \text{and} \quad \tau_2 = \frac{1}{8}\,.$$

Now

$$T = \begin{bmatrix} \tau \\ \tau A \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & 1 \\ 3 & 5 \end{bmatrix}$$

and

$$T^{-1} = \begin{bmatrix} -5 & 1 \\ 3 & 1 \end{bmatrix}.$$

Then

$$TAT^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -5 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -14 & 3 \end{bmatrix}.$$

Thus the transformed system is

$$\dot{w} = TAT^{-1}w + Tbu = \begin{bmatrix} 0 & 1 \\ -14 & 3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} -1 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} u;$$

that is,

$$\begin{cases} \dot{w}_1 = w_2 \\ \dot{w}_2 = -14w_1 + 3w_2 + u \end{cases}$$

or (for $w_1 = z,\ w_2 = \dot{z}$)

$$\ddot{z} - 3\dot{z} + 14z = u\,.$$

A result similar to Theorem 2.4.3 can be obtained for systems having zero input and scalar output, so that the system equations are

$$\begin{cases} \dot{x} = Ax \\ \\ y = cx \end{cases} \tag{2.32}$$

where $A \in \mathbb{R}^{m\times m}$, $c \in \mathbb{R}^{1\times m}$, and $y(\cdot)$ is the output variable.

2.4.5 THEOREM. *Any system described by (2.32) can be transformed by $x = Sv$, with S nonsingular, into the canonical form*

$$\dot{v} = Ev, \quad y = fv \tag{2.33}$$

where

$$E = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -e_m \\ 1 & 0 & 0 & \dots & 0 & -e_{m-1} \\ 0 & 1 & 0 & \dots & 0 & -e_{m-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -e_1 \end{bmatrix} \quad \textit{and} \quad f = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix} \tag{2.34}$$

provided that

$$\boxed{\operatorname{rank} \begin{bmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{m-1} \end{bmatrix} = m.} \tag{2.35}$$

Conversely, if such a transformation S exists, then condition (2.35) holds.

The proof is very similar to that of THEOREM 2.4.3 and will be omitted.

NOTE : E is also a companion matrix because its characteristic polynomial is

$$\operatorname{char}_E(\lambda) = \det\,(\lambda I_m - E) = \lambda^m + e_1\lambda^{m-1} + \cdots + e_m$$

which again is identical to the characteristic polynomial of A.

2.5 Exercises

Exercise 26 Find the general solution, in *spectral form*, of the (initialized) uncontrolled system

$$\dot{x} = Ax\,, \quad x(0) = x_0$$

in each of the following cases :

(a) $A = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Exercise 27 Find the general solution, in *exponential form*, of the (initialized) uncontrolled system

$$\dot{x} = Ax\,, \quad x(0) = x_0$$

in each of the cases given in **Exercise 26**.

Exercise 28 Consider the equation of *simple harmonic motion*

$$\ddot{z} + \omega^2 z = 0.$$

Take as state variables $x_1 = z$ and $x_2 = \dfrac{1}{\omega}\dot{z}$, and find the state transition matrix $\Phi(t, 0)$.

Exercise 29 Use the exponential matrix to solve the *rabbit-fox environment problem*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_1 & -a_2 \\ a_3 & -a_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (a_1, a_2, a_3, a_4 > 0)$$

subject to the condition

$$\frac{a_1}{a_3} = \frac{a_2}{a_4}\,.$$

Show that for arbitrary initial conditions, the populations will attain a steady state as $t \to \infty$ only if $a_1 - a_4 < 0$, and give an expression for the ultimate size of the rabbit population in this case. Finally, deduce that if the environment is to reach a steady state in which both rabbits and foxes are present, then $x_1(0) > \dfrac{a_1}{a_3} x_2(0)$.

Exercise 30 A linear control system is described by the equations

$$\begin{cases} \dot{x}_1 = x_1 + 4x_2 + u(t) \\ \\ \dot{x}_2 = 3x_1 + 2x_2. \end{cases}$$

Determine the *state transition matrix* and write down the general solution.

Exercise 31 A linear control system is described by

$$\ddot{z} + 3\dot{z} + 2z = u\,, \quad z(0) = \dot{z}(0) = 0$$

where

$$u(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ \\ 0 & \text{if } t \ge 1. \end{cases}$$

Calculate the *state transition matrix* and determine $z(2)$.

Exercise 32 Verify that *the* solution of the matrix differential equation

$$\dot{W} = AW + WB, \quad W(0) = C$$

(where $A, B \in \mathbb{R}^{m\times m}$) is

$$W(t) = \exp(tA)C\exp(tB).$$

Exercise 33 Consider the linear control system

$$\dot{x} = Ax + bu(t)$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and take

$$u(t) = \begin{cases} 1 & \text{if } t \ge 0 \\ \\ 0 & \text{if } t < 0. \end{cases}$$

Evaluate $\exp(tA)$ and show that the solution of this problem, subject to

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is

$$x(t) = \begin{bmatrix} 1 + 3t + \frac{5}{2}t^2 + \frac{7}{6}t^3 + \cdots \\ \\ t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots \end{bmatrix}.$$

Exercise 34 Consider the matrix differential equation

$$\dot{X} = A(t)X, \quad X(0) = I_m.$$

Show that, when $m = 2$,

$$\frac{d}{dt}\det(X(t)) = \operatorname{tr}(A(t)) \cdot \det(X(t))$$

and hence deduce that $X(t)$ is nonsingular, $t \geq 0$.

Exercise 35 Verify that the properties of the *state transition matrix*

(a) $\frac{d}{dt}\Phi(t, t_0) = A\Phi(t, t_0)$.

(b) $\Phi(t_0, t_0) = I_m$.

(c) $\Phi(t_0, t) = \Phi^{-1}(t, t_0)$.

(d) $\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$

do carry over to the time varying case.

Exercise 36 Consider the (initialized) *uncontrolled* system

$$\dot{x} = A(t)x, \quad x(0) = x_0.$$

If $B(t) = \int_0^t A(\tau)\, d\tau$, show that *the* solution in this case is

$$x(t) = \exp(B(t))x_0$$

provided $B(t)$ and $A(t)$ commute with each other (for all $t \geq 0$).

Exercise 37 Verify that *the* solution of the matrix differential equation

$$\dot{W} = A(t)W + WA^T(t), \quad W(t_0) = C$$

is

$$W(t) = \Phi(t, t_0)C\Phi^T(t, t_0).$$

Exercise 38 For the linear control system

$$\dot{x} = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

determine $\Phi(t, 0)$. If

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad u(t) = e^{2t}, \ \ t \geq 0$$

use formula

$$x(t) = \Phi(t, 0) \left[x_0 + \int_0^t \Phi(0, \tau) B u(\tau) \, d\tau \right]$$

to obtain the expression for $x(t)$.

Exercise 39 Consider a single-input control system, written in state space form,

$$\dot{x} = Ax + bu(t).$$

Find the matrix T of the linear tansformation $w = Tx$ and the transformed system (the system put into *canonical form*)

$$\dot{w} = Cw + du(t)$$

for each of the following cases :

(a) $A = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$

(b) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$

Exercise 40 Consider a single-output (uncontrolled) system, written in state space form,

$$\dot{x} = Ax, \quad y = cx.$$

Find the matrix P of the linear transformation $x = Pv$ and the transformed system (the system put into the *canonical form*)

$$\dot{v} = Ev, \quad y = fv$$

when

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}.$$

Chapter 3

Linear Control Systems

Topics :

1. Controllability
2. Observability
3. Linear Feedback
4. Realization Theory

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Intuitively, a control system should be designed so that *the input* $u(\cdot)$ *"controls" all the states*; and also so that *all states can be "observed" from the output* $y(\cdot)$. The concepts of (complete) *controllability* and *observability* formalize these ideas.

Another two fundamental concepts of control theory – *feedback* and *realization* – are introduced. Using (linear) feedback it is possible to exert a considerable influence on the behaviour of a (linear) control system.

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3.1 Controllability

An essential first step in dealing with many control problems is to determine whether a desired objective can be achieved by manipulating the chosen control variables. If not, then either the objective will have to be modified or control will have to be applied in some different fashion.

We shall discuss the general property of *being able to transfer (or steer) a control system from any given state to any other by means of a suitable choice of control functions.*

3.1.1 Definition. The *linear control system* Σ defined by

$$\dot{x} = A(t)x + B(t)u(t) \tag{3.1}$$

where $A(t) \in \mathbb{R}^{m\times m}$ and $B(t) \in \mathbb{R}^{m\times \ell}$, is said to be **completely controllable** (c.c.) if for any t_0, any initial state $x(t_0) = x_0$, and any given final state x_f, there exist a finite time $t_1 > t_0$ and a control $u : [t_0, t_1] \to \mathbb{R}^\ell$ such that $x(t_1) = x_f$.

Note : (1) The qualifying term "completely" implies that the definition holds for *all* x_0 and x_f, and several other types of controllability can be defined.

(2) The control $u(\cdot)$ is assumed *piecewise-continuous* in the interval $[t_0, t_1]$.

3.1.2 Example. Consider the control system described by

$$\begin{cases} \dot{x}_1 = a_1x_1 + a_2x_2 + u(t) \\ \\ \dot{x}_2 = x_2. \end{cases}$$

Clearly, by inspection, this is *not* completely controllable (c.c.) since $u(\cdot)$ has no influence on x_2, which is entirely determined by the second equation and $x_2(t_0)$.

We have

$$x_f = \Phi(t_1, t_0)\left[x_0 + \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\, d\tau\right]$$

or

$$0 = \Phi(t_1, t_0)\left[x_0 - \Phi(t_0, t_1)x_f + \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)u(\tau)\, d\tau\right].$$

Since $\Phi(t_1, t_0)$ is nonsingular it follows that if $u(\cdot)$ transfers x_0 to x_f, it also transfers $x_0 - \Phi(t_0, t_1)x_f$ to the origin in the same time interval. Since x_0 and x_f are arbitrary, it therefore follows that – in the controllability definition – *the given final state can be taken to be the zero vector without loss of generality.*

NOTE : For *time-invariant* control systems – in the controllability definition – *the initial time* t_0 *can be set equal to zero.*

The Kalman rank condition

For linear *time-invariant* control systems a general algebraic criterion (for complete controllability) can be derived.

3.1.3 THEOREM. *The linear time-invariant control system*

$$\dot{x} = Ax + Bu(t) \tag{3.2}$$

(or the pair (A, B) *) is c.c. if and only if the* ***(Kalman) controllability matrix***

$$\boxed{\mathcal{C} = \mathcal{C}(A, B) := \begin{bmatrix} B & AB & A^2B & \dots & A^{m-1}B \end{bmatrix} \in \mathbb{R}^{m \times m\ell}}$$

has rank m.

PROOF : $(\Rightarrow)$ We suppose the system is c.c. and wish to prove that $\operatorname{rank}(\mathcal{C}) = m$. This is done by assuming $\operatorname{rank}(\mathcal{C}) < m$, which leads to a contradiction.

Then there exists a constant row m-vector $q \neq 0$ such that

$$qB = 0, \quad qAB = 0, \quad \dots, \quad qA^{m-1}B = 0.$$

In the expression

$$x(t) = \exp(tA)\left[x_0 + \int_0^t \exp(-\tau A)Bu(\tau)\, d\tau\right]$$

for the solution of (3.2) subject to $x(0) = x_0$, set $t = t_1$, $x(t_1) = 0$ to obtain (since $\exp(t_1 A)$ is nonsingular)

$$-x_0 = \int_0^{t_1} \exp(-\tau A) B u(\tau)\, d\tau.$$

Now, $\exp(-\tau A)$ can be expressed as some polynomial $r(A)$ in A having degree at most $m-1$, so we get

$$-x_0 = \int_0^{t_1} (r_0 I_m + r_1 A + \cdots + r_{m-1} A^{m-1}) B u(\tau)\, d\tau.$$

Multiplying this relation on the left by q gives

$$q x_0 = 0.$$

Since the system is c.c., this must hold for *any* vector x_0, which implies $q = 0$, contradiction.

($\Leftarrow$) We asume $\operatorname{rank}(\mathcal{C}) = m$, and wish to show that for any x_0 there is a function $u : [0, t_1] \to \mathbb{R}^\ell$, which when substituted into

$$x(t) = \exp(tA) \left[x_0 + \int_0^t \exp(-\tau A) B u(\tau)\, d\tau \right] \tag{3.3}$$

produces

$$x(t_1) = 0.$$

Consider the *symmetric* matrix

$$W_c := \int_0^{t_1} \exp(-\tau A) B B^T \exp(-\tau A^T)\, d\tau.$$

One can show that W_c is nonsingular. Indeed, consider the *quadratic form* associated to W_c

$$\begin{aligned} \alpha^T W_c \alpha &= \int_0^{t_1} \psi(\tau) \psi^T(\tau)\, d\tau \\ &= \int_0^{t_1} \|\psi(\tau)\|_e^2\, d\tau \geq 0 \end{aligned}$$

where $\alpha \in \mathbb{R}^{m \times 1}$ is an arbitrary column vector and $\psi(\tau) := \alpha^T \exp(-\tau A) B$. It is clear that W_c is *positive semi-definite*, and will be *singular* only if there

exists an $\bar{\alpha} \neq 0$ such that $\bar{\alpha}^T W_c \bar{\alpha} = 0$. However, in this case, it follows (using the properties of the norm) that $\psi(\tau) \equiv 0, \quad 0 \leq \tau \leq t_1$. Hence, we have

$$\bar{\alpha}^T \left(I_m - \tau A + \frac{\tau^2}{2!} A^2 - \frac{\tau^3}{3!} A^3 + \cdots \right) B = 0\,, \quad 0 \leq \tau \leq t_1$$

from which it follows that

$$\bar{\alpha}^T B = 0\,, \quad \bar{\alpha}^T AB = 0\,, \quad \bar{\alpha}^T A^2 B = 0\,, \quad \cdots$$

This implies that $\bar{\alpha}^T \mathcal{C} = 0$. Since by assumption $\mathcal{C}$ has rank m, it follows that such a nonzero vector $\bar{\alpha}$ cannot exist, so W_c is *nonsingular*.

Now, if we choose as the control vector

$$u(t) = -B^T \exp(-tA^T) W_c^{-1} x_0\,, \quad t \in [0, t_1]$$

then substitution into (3.3) gives

$$\begin{aligned} x(t_1) &= \exp(t_1 A) \left[x_0 - \int_0^{t_1} \exp(-\tau A) B B^T \exp(-\tau A^T)\, d\tau \cdot (W_c^{-1} x_0) \right] \\ &= \exp(t_1 A) \left[x_0 - W_c W_c^{-1} x_0 \right] = 0 \end{aligned}$$

as required. □

3.1.4 Corollary. *If* $\operatorname{rank}(B) = r$, *then the condition in* Theorem 3.1.3 *reduces to*

$$\operatorname{rank} \begin{bmatrix} B & AB & \ldots & A^{m-r} B \end{bmatrix} = m.$$

Proof : Define the matrix

$$\mathcal{C}_k := \begin{bmatrix} B & AB & \cdots & A^k B \end{bmatrix}, \quad k = 0, 1, 2 \ldots$$

If $\operatorname{rank}(\mathcal{C}_j) = \operatorname{rank}(\mathcal{C}_{j+1})$ it follows that all the columns of $A^{j+1}B$ must be linearly dependent on those of $\mathcal{C}_j$. This then implies that all the columns of $A^{j+2}B$, $A^{j+3}B$, ... must also be linearly dependent on those of $\mathcal{C}_j$, so that

$$\operatorname{rank}(\mathcal{C}_j) = \operatorname{rank}(\mathcal{C}_{j+1}) = \operatorname{rank}(\mathcal{C}_{j+2}) = \cdots$$

Hence the rank of $\mathcal{C}_k$ increases by at least one when the index k is increased by one, until the maximum value of $\operatorname{rank}(\mathcal{C}_k)$ is attained when $k = j$. Since $\operatorname{rank}(\mathcal{C}_0) = \operatorname{rank}(B) = r$ and $\operatorname{rank}(\mathcal{C}_k) \leq m$ it follows that $r + j \leq m$, giving $j \leq m - r$ as required. □

3.1.5 Example. Consider the linear control system Σ described by

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)\,.$$

The (Kalman) controllability matrix is

$$\mathcal{C} = \mathcal{C}^{\Sigma} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

which has rank 2, so the control system Σ is c.c.

Note : When $\ell = 1$, B reduces to a column vector b and Theorem 2.4.3 can be restated as : *A linear control system in the form*

$$\dot{x} = Ax + bu(t)$$

can be transformed into the canonical form

$$\dot{w} = Cw + du(t)$$

if and only if it is c.c.

Controllability criterion

We now give a general criterion for (complete) controllability of control systems (time-invariant or time-varying) as well as an explicit expression for a control vector which carry out a required alteration of states.

3.1.6 Theorem. *The linear control system Σ defined by*

$$\dot{x} = A(t)x + B(t)u(t)$$

is c.c. if and only if the symmetric matrix, called the ***controllability Gramian****,*

$$\boxed{W_c(t_0,t_1) := \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)\,d\tau \in \mathbb{R}^{m\times m}} \tag{3.4}$$

is nonsingular. In this case the control

$$u^*(t) = -B^T(t)\Phi^T(t_0,t)W_c(t_0,t_1)^{-1}\left[x_0 - \Phi(t_0,t_1)x_f\right]\ , \quad t \in [t_0,t_1]$$

transfers $x(t_0) = x_0$ *to* $x(t_1) = x_f$.

Proof : ($\Leftarrow$) *Sufficiency.* If $W_c(t_0,t_1)$ is assumed nonsingular, then the control defined by

$$u^*(t) = -B^T(t)\Phi^T(t_0,t)W_c(t_0,t_1)^{-1}\left[x_0 - \Phi(t_0,t_1)x_f\right]\ , \quad t \in [t_0,t_1]$$

exists. Now, substitution of the above expression into the solution

$$x(t) = \Phi(t,t_0)\left[x_0 + \int_{t_0}^{t} \Phi(t_0,\tau)B(\tau)u(\tau)\,d\tau\right]$$

of

$$\dot{x} = A(t)x + B(t)u(t)$$

gives

$$\begin{aligned}
x(t_1) &= \Phi(t_1,t_0)\left[x_0 + \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)(-B^T(\tau)\Phi^T(t_0,\tau)W_c(t_0,t_1)^{-1}\cdot\right.\\
&\quad \left.[x_0 - \Phi(t_0,t_1)x_f])\,d\tau\right]\\
&= \Phi(t_1,t_0)\left[x_0 - W_c(t_0,t_1)W_c(t_0,t_1)^{-1}\left[x_0 - \Phi(t_0,t_1)x_f\right]\right]\\
&= \Phi(t_1,t_0)\left[x_0 - x_0 + \Phi(t_0,t_1)x_f\right]\\
&= \Phi(t_1,t_0)\Phi(t_0,t_1)x_f\\
&= x_f.
\end{aligned}$$

($\Rightarrow$) *Necessity.* We need to show that if Σ is c.c., then $W_c(t_0,t_1)$ is *nonsingular.* First, notice that if $\alpha \in \mathbb{R}^{m\times 1}$ is an arbitrary column vector, then from

(3.4) since $W = W_c(t_0, t_1)$ is symmetric we can construct the *quadratic form*

$$\begin{aligned} \alpha^T W \alpha &= \int_{t_0}^{t_1} \theta^T(\tau, t_0)\theta(\tau, t_0)\, d\tau \\ &= \int_{t_0}^{t_1} \|\theta\|_e^2\, d\tau \geq 0 \end{aligned}$$

where $\theta(\tau, t_0) := B^T(\tau)\Phi^T(t_0, \tau)\alpha$, so that $W_c(t_0, t_1)$ is *positive semi-definite.* Suppose that there exists some $\bar{\alpha} \neq 0$ such that $\bar{\alpha}^T W \bar{\alpha} = 0$. Then we get (for $\bar{\theta} = \theta$ when $\alpha = \bar{\alpha}$)

$$\int_{t_0}^{t_1} \|\bar{\theta}\|_e^2\, d\tau = 0$$

which in turn implies (using the properties of the norm) that $\bar{\theta}(\tau, t_0) \equiv 0\,, \quad t_0 \leq \tau \leq t_1$. However, by assumption Σ is c.c. so there exists a control $v(\cdot)$ making $x(t_1) = 0$ if $x(t_0) = \bar{\alpha}$. Hence

$$\bar{\alpha} = -\int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)v(\tau)\, d\tau.$$

Therefore

$$\begin{aligned} \|\bar{\alpha}\|_e^2 &= \bar{\alpha}^T \bar{\alpha} \\ &= -\int_{t_0}^{t_1} v^T(\tau)B^T(\tau)\Phi^T(t_0, \tau)\bar{\alpha}\, d\tau \\ &= -\int_{t_0}^{t_1} v^T(\tau)\bar{\theta}(\tau, t_0)\, d\tau = 0 \end{aligned}$$

which contradicts the assumption that $\bar{\alpha} \neq 0$. Hence $W_c(t_0, t_1)$ is positive definite and is therefore *nonsingular.* □

3.1.7 Example. The control system is

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t).$$

Observe that $\lambda = 0$ is an eigenvalue of A, and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector, so the controllability rank condition does not hold. However, A is

similar to its companion matrix. Using the matrix $T = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ computed before (see EXAMPLE 2.5.1) and $w = Tx$ we have the system

$$\dot{w} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u\,.$$

Differentiation of the w_1 equation and substitution produces a second-order ODE for w_1 :

$$\ddot{w}_1 + 3\dot{w}_1 = 3u + \dot{u}.$$

One integration produces a first-order ODE

$$\dot{w}_1 + 3w_1 = 3\int u(\tau)\,d\tau + u$$

which shows that *the action of arbitrary inputs* $u(\cdot)$ *affects the dynamics in only a one-dimensional space.* The original x equations might lead us to think that $u(\cdot)$ can fully affect x_1 and x_2, but notice that the w_2 equation says that $u(\cdot)$ has no effect on the dynamics of the difference $x_1 - x_2 = w_2$. Only when the initial condition for w involves $w_2(0) = 0$ can $u(\cdot)$ be used to control a trajectory. That is, the inputs completely control only the states that lie in the subspace

$$\operatorname{span}[\,b\ Ab\,] = \operatorname{span}\{b\} = \operatorname{span}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Solutions starting with $x_1(0) = x_2(0)$ satisfy

$$x_1(t) = x_2(t) = \int_0^t u(\tau)\,d\tau + x_1(0).$$

One can steer along the line $x_1 = x_2$ from any initial point to any final point $x_1(t_1) = x_2(t_1)$ at any finite time t_1 by appropriate choice of $u(\cdot)$. On the other hand, if the initial condition lies off the line $x_1 = x_2$, then the difference $w_2 = x_1 - x_2$ decays exponentially so *there is no chance of steering to an arbitrarily given final state in finite time.*

NOTE : The control function $u^*(\cdot)$ which transfers the system from $x_0 = x(t_0)$ to $x_f = x(t_1)$ requires calculation of the state transition matrix $\Phi(\cdot, t_0)$ and the controllability Gramian $W_c(\cdot, \tau_0)$. However, this is not too dificult for linear time-invariant control systems, although rather tedious. Of course, *there will in general be many other suitable control vectors which achieve the same result*.

3.1.8 PROPOSITION. *If $u(\cdot)$ is any other control taking $x_0 = x(t_0)$ to $x_f = x(t_1)$, then*

$$\int_{t_0}^{t_1} \|u(\tau)\|_e^2 \, d\tau > \int_{t_0}^{t_1} \|u^*(\tau)\|_e^2 \, d\tau.$$

PROOF : Since both u^* and u satisfy

$$x_f = \Phi(t_1, t_0) \left[x_0 + \int_{t_0}^{t_1} \Phi(t_0, \tau) B(\tau) u(\tau) \, d\tau \right]$$

we obtain after subtraction

$$0 = \int_{t_0}^{t_1} \Phi(t_0, \tau) B(\tau) \left[u(\tau) - u^*(\tau) \right] \, d\tau.$$

Multiplication of this equation on the left by

$$\left[x_0 - \Phi(t_0, t_1) x_f \right]^T \left[W_c(t_0, t_1)^{-1} \right]^T$$

gives

$$\int_{t_0}^{t_1} (u^*)^T(\tau) \left[u^*(\tau) - u(\tau) \right] \, d\tau = 0$$

and thus

$$\int_{t_0}^{t_1} \|u^*(\tau)\|_e^2 \, d\tau = \int_{t_0}^{t_1} (u^*)^T(\tau) u(\tau) \, d\tau.$$

Therefore

$$\begin{aligned} 0 < \int_{t_0}^{t_1} \|u^*(\tau) - u(\tau)\|_e^2 \, d\tau &= \int_{t_0}^{t_1} [u^*(\tau) - u(\tau)]^T [u^*(\tau) - u(\tau)] \, d\tau \\ &= \int_{t_0}^{t_1} \left(\|u(\tau)\|_e^2 + \|u^*(\tau)\|_e^2 - 2(u^*)^T(\tau) u(\tau) \right) \, d\tau \\ &= \int_{t_0}^{t_1} \left(\|u(\tau)\|_e^2 - \|u^*(\tau)\|_e^2 \right) \, d\tau \end{aligned}$$

and so

$$\int_{t_0}^{t_1} \|u(\tau)\|_e^2 \, d\tau = \int_{t_0}^{t_1} \left(\|u^*(\tau)\|_e^2 + \|u^*(\tau) - u(\tau)\|_e^2 \right) \, d\tau > \int_{t_0}^{t_1} \|u^*(\tau)\|_e^2 \, d\tau$$

as required. □

NOTE : This result can be interpreted as showing that the control

$$u^*(t) = -B^T(t)\Phi^T(t_0, t)W_c(t_0, t_1)^{-1} \left[x_0 - \Phi(t_0, t_1)x_f \right]$$

is "optimal", in the sense that it *minimizes* the integral

$$\int_{t_0}^{t_1} \|u(\tau)\|_e^2 \, d\tau = \int_{t_0}^{t_1} \left(u_1^2(\tau) + u_2^2(\tau) + \cdots + u_\ell^2(\tau) \right) \, d\tau$$

over the set of *all* (admissible) controls which transfer $x_0 = x(t_0)$ to $x_f = x(t_1)$, and this integral can be thought of as a measure of control "energy" involved.

Algebraic equivalence and decomposition of control systems

We now indicate a further aspect of controllability. Let $P(\cdot)$ be a matrix-valued mapping which is continuous and such that $P(t)$ is nonsingular for all $t \geq t_0$. (The continuous maping $P : [t_0, \infty) \to \mathsf{GL}\,(m, \mathbb{R})$ is a *path* in the general linear group $\mathsf{GL}\,(m, \mathbb{R})$.) Then the system $\widetilde{\Sigma}$ obtained from Σ by the transformation

$$\widetilde{x} = P(t)x$$

is said to be **algebraically equivalent** to Σ.

3.1.9 PROPOSITION. *If $\Phi(t, t_0)$ is the state transition matrix for Σ, then*

$$P(t)\Phi(t, t_0)P^{-1}(t_0) = \widetilde{\Phi}(t, t_0)$$

is the state transition matrix for $\widetilde{\Sigma}$.

PROOF : We recall that $\Phi(t, t_0)$ is the unique matrix-valued mapping satisfying

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0), \quad \Phi(t_0, t_0) = I_m$$

and is nonsingular. Clearly,

$$\widetilde{\Phi}(t_0, t_0) = I_m\,;$$

differentiation of

$$\widetilde{x} = P(t)x$$

gives

$$\begin{aligned}
\dot{\widetilde{x}} &= \dot{P}x + P\dot{x} \\
&= (\dot{P} + PA)x + PBu \\
&= (\dot{P} + PA)P^{-1}\widetilde{x} + PBu.
\end{aligned}$$

We need to show that $\widetilde{\Phi}$ is the state transition matrix for

$$\dot{\widetilde{x}} = (\dot{P} + PA)P^{-1}\widetilde{x} + PBu\,.$$

We have

$$\begin{aligned}
\dot{\widetilde{\Phi}}(t, t_0) &= \frac{d}{dt}\left[P(t)\Phi(t, t_0)P^{-1}(t_0)\right] \\
&= \dot{P}(t)\Phi(t, t_0)P^{-1}(t_0) + P(t)\dot{\Phi}(t, t_0)P^{-1}(t_0) \\
&= \left[\left(\dot{P}(t) + P(t)A(t)\right)P^{-1}(t)\right]P(t)\Phi(t, t_0)P^{-1}(t_0) \\
&= \left[\left(\dot{P}(t) + P(t)A(t)\right)P^{-1}(t)\right]\widetilde{\Phi}(t, t_0).
\end{aligned}$$

□

3.1.10 Proposition. *If Σ is c.c., then so is $\widetilde{\Sigma}$.*

Proof : The system matrices for $\widetilde{\Sigma}$ are

$$\widetilde{A} = (\dot{P} + PA)P^{-1} \quad \text{and} \quad \widetilde{B} = PB$$

so the controllability matrix for $\widetilde{\Sigma}$ is

$$\begin{aligned}
\widetilde{W} &= \int_{t_0}^{t_1} \widetilde{\Phi}(t_0, \tau)\widetilde{B}(\tau)\widetilde{B}^T(\tau)\widetilde{\Phi}^T(t_0, \tau)\,d\tau \\
&= \int_{t_0}^{t_1} P(t_0)\Phi(t_0, \tau)P^{-1}(\tau)P(\tau)B(\tau)B^T(\tau)P^T(\tau)\left(P^{-1}(\tau)\right)^T\Phi^T(t_0, \tau)P^T(t_0)\,d\tau \\
&= P(t_0)W_c(t_0, t_1)P^T(t_0).
\end{aligned}$$

Thus the matrix $\widetilde{W} = \widetilde{W}_c(t_0, t_1)$ is nonsingular since the matrices $W_c(t_0, t_1)$ and $P(t_0)$ each have rank m. □

The following important result on system decomposition then holds :

3.1.11 THEOREM. *When the linear control system Σ is time-invariant then if the controllability matrix $\mathcal{C}^\Sigma$ has rank $m_1 < m$ there exists a control system, algebraically equivalent to Σ, having the form*

$$\begin{aligned} \begin{bmatrix} \dot{x}_{(1)} \\ \dot{x}_{(2)} \end{bmatrix} &= \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} x_{(1)} \\ x_{(2)} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) \\ y &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} x \end{aligned}$$

where $x_{(1)}$ and $x_{(2)}$ have orders m_1 and $m - m_1$, respectively, and (A_1, B_1) is c.c.

We shall postpone the proof of this until a later section (see the proof of THEOREM 3.4.5) where an explicit formula for the transformation matrix will also be given.

NOTE : It is clear that the vector $x_{(2)}$ is completely unaffected by the control $u(\cdot)$. Thus the state space has been divided into two parts, one being c.c. and the other *uncontrollable.*

3.2 Observability

Closely linked to the idea of controllability is that of observability, which in general terms means that *it is possible to determine the state of a system by measuring only the output.*

3.2.1 DEFINITION. The *linear control system* (with outputs) Σ described by

$$\begin{cases} \dot{x} = A(t)x + B(t)u(t) \\ \\ y = C(t)x \end{cases} \tag{3.5}$$

is said to be **completely observable** (c.o.) if for any t_0 and any initial state $x(t_0) = x_0$, there exists a finite time $t_1 > t_0$ such that knowledge of $u(\cdot)$ and $y(\cdot)$ for $t \in [t_0, t_1]$ suffices to determine x_0 uniquely.

NOTE : There is in fact no loss of generality in assuming $u(\cdot)$ is identically zero throughout the interval. Indeed, for any input $u : [t_0, t_1] \to \mathbb{R}^\ell$ and initial state x_0, we have

$$y(t) - \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau = C(t)\Phi(t,t_0)x_0.$$

Defining

$$\widehat{y}(t) := y(t) - \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

we get

$$\widehat{y}(t) = C(t)\Phi(t,t_0)x_0.$$

Thus a linear control system is c.o. if and only if *knowledge of the output* $\widehat{y}(\cdot)$ *with zero input on the interval* $[t_0, t_1]$ *allows the initial state* x_0 *to be determined.*

3.2.2 EXAMPLE. Consider the linear control system described by

$$\begin{cases} \dot{x}_1 = a_1 x_1 + b_1 u(t) \\ \\ \dot{x}_2 = a_2 x_2 + b_2 u(t) \\ \\ y = x_1. \end{cases}$$

The first equation shows that $x_1(\cdot)$ $(= y(\cdot))$ is completely determined by $u(\cdot)$ and $x_1(t_0)$. Thus it is impossible to determine $x_2(t_0)$ by measuring the output, so the system is *not* completely observable (c.o.).

3.2.3 THEOREM. *The linear control system* Σ *is c.o. if and only if the symmetric matrix, called the* ***observability Gramian****,*

$$\boxed{W_o(t_0,t_1) := \int_{t_0}^{t_1} \Phi^T(\tau,t_0)C^T(\tau)C(\tau)\Phi(\tau,t_0)\,d\tau \in \mathbb{R}^{m\times m}} \tag{3.6}$$

is nonsingular.

PROOF : ($\Leftarrow$) *Sufficiency.* Assuming $u(t) \equiv 0,\ t \in [t_0, t_1]$, we have

$$y(t) = C(t)\Phi(t, t_0)x_0\,.$$

Multiplying this relation on the left by $\Phi^T(t, t_0)C^T(t)$ and integrating produces

$$\int_{t_0}^{t_1} \Phi^T(\tau, t_0)C^T(\tau)y(\tau)\,d\tau = W_o(t_0, t_1)x_0$$

so that if $W_o(t_0, t_1)$ is nonsingular, the initial state is

$$x_0 = W_o(t_0, t_1)^{-1} \int_{t_0}^{t_1} \Phi^T(\tau, t_0)C^T(\tau)y(\tau)\,d\tau$$

so Σ is c.o.

($\Rightarrow$) *Necessity.* We now assume that Σ is c.o. and prove that $W = W_o(t_0, t_1)$ is *nonsingular.* First, if $\alpha \in \mathbb{R}^{m\times 1}$ is an arbitrary column vector,

$$\alpha^T W \alpha = \int_{t_0}^{t_1} \left(C(\tau)\Phi(\tau, t_0)\alpha\right)^T C(\tau)\Phi(\tau, t_0)\alpha\,d\tau \geq 0$$

so $W_o(t_0, t_1)$ is *positive semi-definite.* Next, suppose there exists an $\bar{\alpha} \neq 0$ such that $\bar{\alpha}^T W \bar{\alpha} = 0$. It then follows that

$$C(\tau)\Phi(\tau, t_0)\bar{\alpha} \equiv 0\,, \quad t_0 \leq \tau \leq t_1.$$

This implies that when $x_0 = \bar{\alpha}$ the output is identically zero throughout the time interval, so that x_0 cannot be determined in this case from the knowledge of $y(\cdot)$. This contradicts the assumption that Σ is c.o., hence $W_o(t_0, t_1)$ is positive definite, and therefore *nonsingular.* □

NOTE : Since the observability of Σ is independent of B, we may refer to the observability of the *pair* (A, C).

Duality

3.2.4 THEOREM. *The linear control system (with outputs) Σ defined by*

$$\begin{cases} \dot{x} = A(t)x + B(t)u(t) \\ \\ y = C(t)x \end{cases}$$

is c.c. if and only if the ***dual system*** Σ° *defined by*

$$\begin{cases} \dot{x} = -A^T(t)x + C^T(t)u(t) \\ \\ y = B^T(t)x \end{cases}$$

is c.o.; and conversely.

PROOF : We can see that *if* $\Phi(t,t_0)$ *is the state transition matrix for the system* Σ*, then* $\Phi^T(t_0,t)$ *is the state transition matrix for the dual system* Σ°. Indeed, differentiate $I_m = \Phi(t,t_0)\Phi(t,t_0)^{-1}$ to get

$$\begin{aligned} 0 = \frac{d}{dt}I_m &= \dot{\Phi}(t,t_0)\Phi(t,t_0)^{-1} + \Phi(t,t_0)\dot{\Phi}(t_0,t) \\ &= A(t)\Phi(t,t_0)\Phi(t,t_0)^{-1} + \Phi(t,t_0)\dot{\Phi}(t_0,t) \\ &= A(t) + \Phi(t,t_0)\dot{\Phi}(t_0,t). \end{aligned}$$

This implies

$$\dot{\Phi}(t_0,t) = -\Phi(t_0,t)A(t)$$

or

$$\dot{\Phi}^T(t_0,t) = -A^T(t)\Phi^T(t_0,t).$$

Furthermore, the controllability matrix

$$W_c^\Sigma(t_0,t_1) = \int_{t_0}^{t_1} \Phi(t_0,\tau)B(\tau)B^T(\tau)\Phi^T(t_0,\tau)\,d\tau$$

(associated with Σ) is identical to the observability matrix $W_o^\Sigma(t_0,t_1)$ (associated with Σ°).

Conversely, the observability matrix

$$W_o^\Sigma(t_0,t_1) = \int_{t_0}^{t_1} \Phi^T(\tau,t_0)C^T(\tau)C(\tau)\Phi(\tau,t_0)\,d\tau$$

(associated with Σ) is identical to the controllability matrix $W_c^{\Sigma^\circ}(t_0,t_1)$ (associated with Σ°). $\square$

NOTE : This duality theorem is extremely useful, since it enables us to deduce immediately from a controllability result the corresponding one on observability (and conversely). For example, to obtain the observability criterion for the time-invariant case, we simply apply THEOREM 3.1.3 to Σ° to obtain the following result.

3.2.5 THEOREM. *The linear time-invariant control system*

$$\begin{cases} \dot{x} = Ax + Bu(t) \\ \\ y = Cx \end{cases} \tag{3.7}$$

(or the pair (A, C)*) is c.o. if and only if the* ***(Kalman) observability matrix***

$$\boxed{\mathcal{O} = \mathcal{O}(A, C) := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{m-1} \end{bmatrix} \in \mathbb{R}^{mn \times m}}$$

has rank m.

3.2.6 EXAMPLE. Consider the linear control system Σ described by

$$\begin{cases} \dot{x} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ \\ y = x_1. \end{cases}$$

The (Kalman) observability matrix is

$$\mathcal{O} = \mathcal{O}^\Sigma = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}$$

which has rank 2. Thus the control system Σ is c.o.

In the single-output case (i.e. $n = 1$), if $u(\cdot) = 0$ and $y(\cdot)$ is known in the form

$$\gamma_1 e^{\lambda_1 t} + \gamma_2 e^{\lambda_2 t} + \cdots + \gamma_m e^{\lambda_m t}$$

assuming that *all the eigenvalues* λ_i *of* A *are distinct*, then x_0 can be obtained more easily than by using

$$x_0 = W_o(t_0, t_1)^{-1} \int_{t_0}^{t_1} \Phi^T(\tau, t_0) C^T(\tau) y(\tau)\, d\tau.$$

For suppose that $t_0 = 0$ and consider the solution of $\dot{x} = Ax$ in the spectral form, namely

$$x(t) = (v_1 x(0))\, e^{\lambda_1 t} w_1 + (v_2 x(0))\, e^{\lambda_2 t} w_2 + \cdots + (v_m x(0))\, e^{\lambda_m t} w_m.$$

We have

$$y(t) = (v_1 x(0))\,(cw_1) e^{\lambda_1 t} + (v_2 x(0))\,(cw_2) e^{\lambda_2 t} + \cdots + (v_m x(0))\,(cw_m) e^{\lambda_m t}$$

and equating coefficients of the exponential terms gives

$$v_i x(0) = \frac{\gamma_i}{cw_i} \qquad (i = 1, 2, \ldots, m).$$

This represents m linear equations for the m unknown components of $x(0)$ in terms of γ_i, v_i and w_i $(i = 1, 2, \ldots, m)$.

Again, in the single-output case, C reduces to a *row matrix* c and THEOREM 3.2.5 can be restated as :

A linear system (with outputs) in the form

$$\begin{cases} \dot{x} = Ax \\ \\ y = cx \end{cases}$$

can be transformed into the canonical form

$$\begin{cases} \dot{v} = Ev \\ \\ y = fv \end{cases}$$

if and only if it is c.o.

Decomposition of control systems

By *duality*, the result corresponding to THEOREM 3.1.11 is :

3.2.7 THEOREM. *When the linear control system Σ is time-invariant then if the observability matrix $\mathcal{O}^{\Sigma}$ has rank $m_1 < m$ there exists a control system, algebraically equivalent to Σ, having the form*

$$\begin{aligned} \begin{bmatrix} \dot{x}_{(1)} \\ \dot{x}_{(2)} \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} \begin{bmatrix} x_{(1)} \\ x_{(2)} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \\ y &= C_1 x_{(1)} \end{aligned}$$

where $x_{(1)}$ and $x_{(2)}$ have orders m_1 and $m - m_1$, respectively and (A_1, C_1) is c.o.

We close this section with a *decomposition result* which effectively combines together THEOREMS 3.1.11 and 3.2.7 to show that *a linear time-invariant control system can split up into four mutually exclusive parts, respectively*

- c.c. but unobservable
- c.c. and c.o.
- uncontrollable and unobservable
- c.o. but uncontrollable.

3.2.8 THEOREM. *When the linear control system Σ is time-invariant it is algebraically equivalent to*

$$\begin{aligned} \begin{bmatrix} \dot{x}_{(1)} \\ \dot{x}_{(2)} \\ \dot{x}_{(3)} \\ \dot{x}_{(4)} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} x_{(1)} \\ x_{(2)} \\ x_{(3)} \\ x_{(4)} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u(t) \\ y &= C_2 x_{(2)} + C_4 x_{(4)} \end{aligned}$$

where the subscripts refer to the stated classification.

3.3 Linear Feedback

Consider a linear control system Σ defined by

$$\dot{x} = Ax + Bu(t) \tag{3.8}$$

where $A \in \mathbb{R}^{m\times m}$ and $B \in \mathbb{R}^{m\times \ell}$. Suppose that we apply a (linear) **feedback**, that is each control variable is a linear combination of the state variables, so that

$$u(t) = Kx(t)$$

where $K \in \mathbb{R}^{\ell\times m}$ is a **feedback matrix**. The resulting *closed loop system* is

$$\dot{x} = (A + BK)x. \tag{3.9}$$

The pole-shifting theorem

We ask the question whether it is possible to exert some influence on the behaviour of the closed loop system and, if so, to what extent. A somewhat surprising result, called the Spectrum Assignment Theorem, says in essence that for *almost* any linear control system Σ it is possible to obtain *arbitrary* eigenvalues for the matrix $A+BK$ (and hence arbitrary asymptotic behaviour) using suitable feedback laws (matrices) K, subject only to the obvious constraint that complex eigenvalues must appear in pairs. "Almost any" means that this will be true for (completely) *controllable* systems.

Note : This theorem is most often referred to as the **Pole-Shifting Theorem**, a terminology that is due to the fact that the eigenvalues of $A+BK$ are also the *poles* of the (complex) function

$$z \mapsto \frac{1}{\det(zI_n - A - BK)}.$$

This function appears often in classical control design.

The Pole-Shifting Theorem is central to linear control systems theory and is itself the starting point for more interesting analysis. Once we know that arbitrary sets of eigenvalues can be assigned, it becomes of interest to compare the performance

of different such sets. Also, one may ask what happens when certain entries of K are restricted to vanish, which corresponds to constraints on what can be implemented.

3.3.1 THEOREM. *Let $\Lambda = \{\theta_1, \theta_2, \ldots, \theta_m\}$ be an arbitrary set of m complex numbers (appearing in conjugate pairs). If the linear control system Σ is c.c., then there exists a matrix $K \in \mathbb{R}^{\ell \times m}$ such that the eigenvalues of $A + BK$ are the set Λ.*

PROOF (when $\ell = 1$) : Since $\dot{x} = Ax + Bu(t)$ is c.c., it follows that there exists a (linear) transformation $w = Tx$ such that the given system is transformed into

$$\dot{w} = Cw + du(t)$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -k_m & -k_{m-1} & -k_{m-2} & \ldots & -k_1 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The feedback control $u = \underline{k}w$, where

$$\underline{k} := \begin{bmatrix} \underline{k}_m & \underline{k}_{m-1} & \ldots & \underline{k}_1 \end{bmatrix}$$

produces the closed loop matrix $C + d\underline{k}$, which has the same companion form as C but with last row $-\begin{bmatrix} \gamma_m & \gamma_{m-1} & \cdots & \gamma_1 \end{bmatrix}$, where

$$\boxed{\underline{k}_i = k_i - \gamma_i\,, \quad i = 1, 2, \ldots, m.} \tag{3.10}$$

Since

$$C + d\underline{k} = T(A + b\underline{k}T)T^{-1}$$

it follows that the desired matrix is

$$K = \underline{k}T$$

the entries $\underline{k}_i$ $(i = 1, 2, \ldots, m)$ being given by (3.10).

In this equation k_i $(i = 1, 2, \ldots, m)$ are the coefficients in the characteristic polynomial of A ; that is,

$$\det(\lambda I_m - A) = \lambda^m + k_1\lambda^{m-1} + \cdots + k_m$$

and γ_i $(i = 1, 2, \ldots, m)$ are obtained by equating coefficients of λ in

$$\lambda^m + \gamma_1\lambda^{m-1} + \cdots + \gamma_m \equiv (\lambda - \theta_1)(\lambda - \theta_2)\cdots(\lambda - \theta_m).$$

□

NOTE : The solution of (the closed loop system)

$$\dot{x} = (A + BK)x$$

depends on the eigenvalues of $A + BK$, so provided the control system Σ is c.c., the theorem tells us that *using linear feedback it is possible to exert a considerable influence on the time behaviour of the closed loop system by suitably choosing the numbers* $\theta_1, \theta_2, \ldots, \theta_m$.

3.3.2 COROLLARY. *If the linear time-invariant control system*

$$\begin{cases} \dot{x} = Ax + Bu(t) \\ \\ y = cx \end{cases}$$

is c.o., then there exists a matrix $L \in \mathbb{R}^{m\times 1}$ *such that the eigenvalues of* $A + Lc$ *are the set* Λ.

This result can be deduced from THEOREM 3.3.1 using the DUALITY THEOREM.

3.3.3 EXAMPLE. Consider the linear control system

$$\dot{x} = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)\,.$$

The characteristic equation of A is

$$\mathrm{char}_A(\lambda) \equiv \lambda^2 - 3\lambda + 14 = 0$$

which has roots $\frac{3 \pm i\sqrt{47}}{2}$.

Suppose we wish the eigenvalues of the closed loop system to be -1 and -2, so that the characteristic polynomial is

$$\lambda^2 + 3\lambda + 2\,.$$

We have

$$\begin{aligned} \underline{k}_1 &= k_1 - \gamma_1 = -3 - 3 = -6 \\ \underline{k}_2 &= k_2 - \gamma_2 = 14 - 2 = 12. \end{aligned}$$

Hence

$$K = \underline{k}T = \frac{1}{8}\begin{bmatrix} 12 & -6 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 5 \end{bmatrix} = -\begin{bmatrix} \frac{15}{4} & \frac{9}{4} \end{bmatrix}.$$

It is easy to verify that

$$A + bK = \frac{1}{4}\begin{bmatrix} -11 & -21 \\ 1 & -1 \end{bmatrix}$$

does have the desired eigenvalues.

3.3.4 Lemma. *If the linear control system Σ defined by*

$$\dot{x} = Ax + Bu(t)$$

is c.c. and $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_\ell \end{bmatrix}$ with $b_i \neq 0$, $i = 1, 2, \ldots, \ell$, then there exist matrices $K_i \in \mathbb{R}^{\ell \times m}$, $i = 1, 2, \ldots, \ell$ such that the systems

$$\dot{x} = (A + BK_i)x + b_i u(t)$$

are c.c.

Proof : For convenience consider the case $i = 1$. Since the matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \ldots & A^{m-1}B \end{bmatrix}$$

has full rank, it is possible to select from its columns at least one set of m vectors which are linearly independent. Define an $m \times m$ matrix M by choosing such a set as follows :

$$M = \begin{bmatrix} b_1 & Ab_1 & \dots & A^{r_1-1}b_1 & b_2 & Ab_2 & \dots & A^{r_2-1}b_2 & \dots \end{bmatrix}$$

where r_i is the smallest integer such that $A^{r_i}b_i$ is linearly dependent on all the preceding vectors, the process continuing until m columns of U are taken.

Define an $\ell \times m$ matrix N having its r_1^{th} column equal to e_2, the second column of I_ℓ, its $(r_1 + r_2)^{th}$ column equal to e_3, its $(r_1 + r_2 + r_3)^{th}$ column equal to e_4 and so on, all its other columns being zero.

It is then not difficult to show that the desired matrix in the statement of the Lemma is

$$K_1 = NM^{-1}.$$

□

Proof of Theorem 3.3.1 when $\ell > 1$:

Let K_1 be the matrix in the proof of Lemma 3.3.4 and define an $\ell \times m$ matrix K' having as its first row some vector k, and all its other rows zero. Then the control

$$u = (K_1 + K')x$$

leads to the closed loop system

$$\dot{x} = (A + BK_1)x + BK'x = (A + BK_1)x + b_1kx$$

where b_1 is the first column of B.

Since the system

$$\dot{x} = (A + BK_1)x + b_1u$$

is c.c., it now follows from the proof of the theorem when $\ell = 1$, that k can be chosen so that the eigenvalues of $A + BK_1 + b_1k$ are the set Λ, so the desired feedback control is indeed $u = (K_1 + K')x$. □

If $y = Cx$ is the output vector, then again by duality we can immediately deduce

3.3.5 Corollary. *If the linear control system*

$$\begin{cases} \dot{x} = Ax + Bu(t) \\ \\ y = Cx \end{cases}$$

is c.o., then there exists a matrix $L \in \mathbb{R}^{m\times n}$ *such that the eigenvalues of* $A + LC$ *are the set* Λ.

Algorithm for constructing a feedback matrix

The following method gives a practical way of constructing the feedback matrix K. Let all the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ of A be distinct and let

$$W = \begin{bmatrix} w_1 & w_2 & \ldots & w_m \end{bmatrix}$$

where w_i is an eigenvector corresponding to the eigenvalue λ_i. With linear feedback $u = -Kx$, suppose that the eigenvalues of A and $A - BK$ are ordered so that those of $A - BK$ are to be

$$\mu_1, \mu_2, \ldots, \mu_r, \lambda_{r+1}, \ldots, \lambda_m \qquad (r \le m).$$

Then provided the linear system Σ is c.c., a suitable matrix is

$$K = fg\widetilde{W}$$

where $\widetilde{W}$ consists of the first r rows of W^{-1}, and

$$\begin{aligned} g &= \begin{bmatrix} \dfrac{\alpha_1}{\beta_1} & \dfrac{\alpha_2}{\beta_2} & \ldots & \dfrac{\alpha_r}{\beta_r} \end{bmatrix} \\ \alpha_i &= \begin{cases} \dfrac{\prod\limits_{j=1}^{r}(\lambda_i - \mu_j)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{r}(\lambda_i - \lambda_j)} & \text{if } r > 1 \\ \lambda_1 - \mu_1 & \text{if } r = 1 \end{cases} \\ \beta &= \begin{bmatrix} \beta_1 & \beta_2 & \ldots & \beta_r \end{bmatrix}^T = \widetilde{W}Bf \end{aligned}$$

f being any column ℓ-vector such that all $\beta_i \neq 0$.

3.3.6 Example. Consider the linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t)\,.$$

We have

$$\lambda_1 = -1, \quad \lambda_2 = -2 \quad \text{and}$$

$$W = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}, \quad W^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}.$$

Suppose that

$$\mu_1 = -3, \quad \mu_2 = -4, \quad \text{so} \quad \widetilde{W} = W^{-1}\,.$$

We have

$$\alpha_1 = 6, \quad \alpha_2 = -2$$

and $\beta = \widetilde{W} B f$ gives

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} f = \begin{bmatrix} 5f_1 \\ -3f_1 \end{bmatrix}.$$

Hence we can take $f_1 = 1$, which results in

$$g = \begin{bmatrix} \frac{6}{5} & \frac{2}{3} \end{bmatrix}.$$

Finally, the desired feedback matrix is

$$K = 1 \cdot \begin{bmatrix} \frac{6}{5} & \frac{2}{3} \end{bmatrix} W^{-1} = \begin{bmatrix} \frac{26}{15} & \frac{8}{15} \end{bmatrix}.$$

3.3.7 Example. Consider now the linear control system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} u(t)\,.$$

We now obtain

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 5f_1 + 2f_2 \\ -3f_1 - f_2 \end{bmatrix}$$

so that $f_1 = 1, \quad f_2 = 0$ gives

$$K = \begin{bmatrix} \frac{26}{15} & \frac{8}{15} \\ \\ 0 & 0 \end{bmatrix}.$$

However $f_1 = 1, \quad f_2 = -1$ gives

$$\beta_1 = 3, \quad \beta_2 = -2 \quad \text{so that} \quad g = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

and from $K = fg\widetilde{W}$ we now have

$$K = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} W^{-1} = \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix}.$$

3.4 Realization Theory

The *realization problem* may be viewed as "guessing the equations of motion (i.e. state equations) of a control system from its input/output behaviour" or, if one prefers, "setting up a physical model which explains the experimental data".

Consider the linear control system (with outputs) Σ described by

$$\begin{cases} \dot{x} = Ax + Bu(t) \\ \\ y = Cx \end{cases} \tag{3.11}$$

where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times \ell}$ and $C \in \mathbb{R}^{n \times m}$.

Taking *Laplace transforms* of (3.11) and assuming zero initial conditions gives

$$s\overline{x}(s) = A\overline{x}(s) + B\overline{u}(s)$$

and after rearrangement

$$\overline{x}(s) = (sI_m - A)^{-1} B\overline{u}(s).$$

The Laplace transform of the output is

$$\overline{y}(s) = C\overline{x}(s)$$

and thus

$$\overline{y}(s) = C\left(sI_m - A\right)^{-1} B\overline{u}(s) = G(s)\overline{u}(s)$$

where the $n \times \ell$ matrix

$$\boxed{G(s) := C\left(sI_m - A\right)^{-1} B} \tag{3.12}$$

is called the **transfer function matrix** since it relates the Laplace transform of the output vector to that of the input vector.

Exercise 41 Evaluate (the *Laplace transform* of the exponential)

$$\mathcal{L}\left[e^{at}\right](s) := \int_0^\infty e^{-st} e^{at}\, dt$$

and then show that (for $A \in \mathbb{R}^{m\times m}$) :

$$\mathcal{L}\left[\exp(tA)\right](s) = \left(sI_m - A\right)^{-1}.$$

Using relation

$$\boxed{\left(sI_m - A\right)^{-1} = \frac{s^{m-1}I_m + s^{m-2}B_1 + s^{m-3}B_2 + \cdots + B_{m-1}}{\mathrm{char}_A(s)}} \tag{3.13}$$

where the k_i and B_i are determined successively by

$$B_1 = A + k_1 I_m, \quad B_i = AB_{i-1} + k_i I_m\,; \quad i = 2, 3, \ldots, m-1$$

$$k_1 = -\mathrm{tr}\,(A), \quad k_i = -\tfrac{1}{i}\mathrm{tr}\,(AB_{i-1})\,; \quad i = 2, 3, \ldots, m$$

the expression (3.12) becomes

$$G(s) = \frac{s^{m-1}G_0 + s^{m-2}G_1 + \cdots + G_{m-1}}{\chi(s)} = \frac{H(s)}{\chi(s)}$$

where $\chi(s) = \mathrm{char}_A(s)$ and $G_k = \left[g_{ij}^{(k)}\right] \in \mathbb{R}^{n\times\ell}$, $\quad k = 0, 1, 2, \ldots, m-1$. The $n \times \ell$ matrix $H(s)$ is called a **polynomial matrix**, since each of its entries is itself a polynomial; that is,

$$h_{ij} = s^{m-1}g_{ij}^{(0)} + s^{m-2}g_{ij}^{(1)} + \cdots + g_{ij}^{(m-1)}.$$

NOTE : The formulas above, used mainly for theoretical rather than computational purposes, constitute **Leverrier**'s **algorithm**.

3.4.1 EXAMPLE. Consider the *electrically-heated oven* described in section **1.3**, and suppose that the values of the constants are such that the state equations are

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t).$$

Suppose that the output is provided by a thermocouple in the jacket measuring the jacket (excess) temperature, i.e.

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

The expression (3.12) gives

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+2 & -2 \\ -1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{s+1}{s^2+3s}$$

using

$$(sI_2 - A)^{-1} = \frac{1}{\mathrm{char}_A(s)} \mathrm{adj}\,(sI_2 - A).$$

Realizations

In practice it often happens that the mathematical description of a (linear time-invariant) control system – in terms of differential equations – is *not* known, but $G(s)$ can be determined from experimental measurements or other considerations. It is then useful *to find a system – in our usual state space form – to which* $G(\cdot)$ *corresponds.*

In formal terms, given an $n \times \ell$ matrix $G(s)$, whose elements are rational functions of s, we wish to find (constant) matrices A, B, C having dimensions $m \times m$, $m \times \ell$ and $n \times m$, respectively, such that

$$G(s) = C(sI_m - A)^{-1}B$$

and the system equations will then be

$$\begin{cases} \dot{x} = Ax + Bu(t) \\ \\ y = Cx. \end{cases}$$

The triple (A, B, C) is termed a **realization** of $G(\cdot)$ **of order** m, and is not, of course, unique. Amongst all such realizations some will include matrices A having *least dimensions* – these are called **minimal realizations**, since the corresponding systems involve the smallest possible number of state variables.

NOTE : Since each element in

$$(sI_m - A)^{-1} = \frac{\operatorname{adj}(sI_m - A)}{\det(sI_m - A)}$$

has the degree of the numerator less than that of the denominator, it follows that

$$\lim_{s \to \infty} C(sI_m - A)^{-1}B = 0$$

and we shall assume that *any given* $G(s)$ *also has this property*, $G(\cdot)$ then being termed **strictly proper**.

3.4.2 EXAMPLE. Consider the scalar transfer function

$$g(s) = \frac{2s + 7}{s^2 - 5s + 6}\,.$$

It is easy to verify that one realization of $g(\cdot)$ is

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 7 & 2 \end{bmatrix}.$$

It is also easy to verify that a quite different triple is

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} -11 & 13 \end{bmatrix}.$$

NOTE : Both these realizations are minimal, and there is in consequence a simple relationship between them, as we shall see later.

Algebraic equivalence and realizations

It is now appropriate to return to the idea of algebraic equivalence of linear control systems (defined in section **3.1**), and discuss its implications for the realization problem. The (linear) transformation

$$\widetilde{x} = Px$$

produces a linear control system with matrices

$$\boxed{\widetilde{A} = PAP^{-1}, \quad \widetilde{B} = PB, \quad \widetilde{C} = CP^{-1}.} \tag{3.14}$$

Exercise 42 Show that if (A, B, C) represents a c.c. (or c.o.) linear control system, then so does $(\widetilde{A}, \widetilde{B}, \widetilde{C})$.

Exercise 43 Show that if two linear control systems are algebraically equivalent, then their transfer function matrices are identical (i.e.

$$C(sI_m - A)^{-1}B = \widetilde{C}(sI_m - \widetilde{A})^{-1}\widetilde{B}).$$

Characterization of minimal realizations

We can now state and prove the central result of this section, which links together the three basic concepts of controllability, observability, and realization.

3.4.3 THEOREM. *A realization (A, B, C) of a given transfer function matrix $G(\cdot)$ is minimal if and only if the pair (A, B) is c.c. and the pair (A, C) is c.o.*

PROOF : ($\Leftarrow$) *Sufficiency.* Let $\mathcal{C}$ and $\mathcal{O}$ be the controllability and observability matrices, respectively; that is,

$$\mathcal{C} = \mathcal{C}(A,B) = \begin{bmatrix} B & AB & A^2B & \dots & A^{m-1}B \end{bmatrix} \quad \text{and} \quad \mathcal{O} = \mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{m-1} \end{bmatrix}.$$

We wish to show that if these both have rank m, then (the realization of) $G(\cdot)$ has least order m. Suppose that there exists a realization $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ of $G(\cdot)$ with $\widetilde{A}$ having order $\widetilde{m}$. Since

$$C(sI_m - A)^{-1}B = \widetilde{C}(sI_m - \widetilde{A})^{-1}\widetilde{B}$$

it follows that

$$C\exp(tA)B = \widetilde{C}\exp(t\widetilde{A})\widetilde{B}$$

which implies that

$$CA^iB = \widetilde{C}\widetilde{A}^i\widetilde{B}\,, \quad i = 0, 1, 2, \dots.$$

Consider the product

$$\begin{aligned}
\mathcal{O}\mathcal{C} &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \\
&= \begin{bmatrix} CB & CAB & \dots & CA^{m-1}B \\ CAB & CA^2B & \dots & CA^mB \\ \vdots & \vdots & & \vdots \\ CA^{m-1}B & CA^mB & \dots & CA^{2m-2}B \end{bmatrix} \\
&= \begin{bmatrix} \widetilde{C} \\ \widetilde{C}\widetilde{A} \\ \vdots \\ \widetilde{C}\widetilde{A}^{m-1} \end{bmatrix} \begin{bmatrix} \widetilde{B} & \widetilde{A}\widetilde{B} & \dots & \widetilde{A}^{m-1}\widetilde{B} \end{bmatrix} \\
&= \widetilde{\mathcal{O}}\widetilde{\mathcal{C}}.
\end{aligned}$$

Exercise 44 Given $A \in \mathbb{R}^{\ell \times m}$ and $B \in \mathbb{R}^{m \times n}$, show that if $\operatorname{rank}(A) = \operatorname{rank}(B) = m$, then $\operatorname{rank}(AB) = m$. [Hint : Use the results of **Exercise 3**.]

The matrix $\mathcal{O}\mathcal{C}$ has rank m, so the matrix $\widetilde{\mathcal{O}}\widetilde{\mathcal{C}}$ also has rank m. However, the rank of $\widetilde{\mathcal{O}}\widetilde{\mathcal{C}}$ cannot be greater than $\widetilde{m}$. That is, $m \leq \widetilde{m}$, so there can be no realization of $G(\cdot)$ having order less than m.

($\Rightarrow$) *Necessity.* We show that if the pair (A, B) is *not* completely controllable, then there exists a realization of $G(\cdot)$ having order less than m. The corresponding part of the proof involving observability follows from duality.

Let the rank of $\mathcal{C}$ be $m_1 < m$ and let $u_1, u_2, \dots, u_{m_1}$ be any set of m_1 linearly independent columns of $\mathcal{C}$. Consider the (linear) transformation

$$\widetilde{x} = Px$$

with the $m \times m$ matrix P defined by

$$P^{-1} = \begin{bmatrix} u_1 & u_2 & \dots & u_{m_1} & u_{m_1+1} & \dots & u_m \end{bmatrix} \tag{3.15}$$

where the columns $u_{m_1+1}, \ldots, u_m$ are any vectors which make the matrix P^{-1} nonsingular. Since $\mathcal{C}$ has rank m_1 it follows that all its columns can be expressed as a linear combination of the basis $u_1, u_2, \ldots, u_{m_1}$. The matrix

$$A\mathcal{C} = \begin{bmatrix} AB & A^2B & \ldots & A^mB \end{bmatrix}$$

contains all but the first ℓ columns of $\mathcal{C}$, so in particular it follows that the vectors $Au_i, \quad i = 1, 2, \ldots, m_1$ can be expressed in terms of the same basis. Multiplying both sides of (3.15) on the left by P shows that Pu_i is equal to the ith column of I_m. Combining these facts together we obtain

$$\begin{aligned} \widetilde{A} &= PAP^{-1} \\ &= P\begin{bmatrix} Au_1 & \ldots & Au_{m_1} & \ldots & Au_m \end{bmatrix} \\ &= \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \end{aligned}$$

where A_1 is $m_1 \times m_1$. Similarly, since $u_1, u_2, \ldots, u_{m_1}$ also forms a basis for the columns of B we have from (3.14) and (3.15)

$$\widetilde{B} = PB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where B_1 is $m_1 \times \ell$. Writing

$$\widetilde{C} = CP^{-1} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

we have (see **Exercise 43** and also **Exercise 60**)

$$\begin{aligned} G(s) &= \widetilde{C}(sI_m - \widetilde{A})^{-1}\widetilde{B} \\ &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} sI_{m_1} - A_1 & -A_2 \\ 0 & sI_{m-m_1} - A_3 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} (sI_{m_1} - A_1)^{-1} & (sI_{m_1} - A_1)^{-1}A_2(sI_{m-m_1} - A_3)^{-1} \\ 0 & (sI_{m-m_1} - A_3)^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \\ &= C_1(sI_{m_1} - A_1)^{-1}B_1 \end{aligned}$$

showing that (A_1, B_1, C_1) is a realization of $G(\cdot)$ having order $m_1 < m$. This contradicts the assumption that (A, B, C) is minimal, hence the pair (A, B) must be c.c. $\square$

3.4.4 Example. We apply the procedure introduced in the second part of the proof of Theorem 3.4.5 to split up the linear control system

$$\dot{x} = \begin{bmatrix} 4 & 3 & 5 \\ 1 & -2 & -3 \\ 2 & 1 & 8 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} u(t) \tag{3.16}$$

into its *controllable* and *uncontrollable* parts, as displayed below :

$$\begin{bmatrix} \dot{x}_{(1)} \\ \dot{x}_{(2)} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} x_{(1)} \\ x_{(2)} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t)$$

where $x_{(1)}, x_{(2)}$ have orders m_1 and $m - m_1$, respectively, and (A_1, B_1) is c.c.

The controllability matrix for (3.16) is

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 2 & 6 & 18 \\ 1 & 3 & 9 \\ -1 & -3 & -9 \end{bmatrix}$$

which clearly has rank $m_1 = 1$. For the transformation $\widetilde{x} = Px$ we follow (3.15) and set

$$P^{-1} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

where the column in (3.16) has been selected, and the remaining columns are simply arbitrary choices to produce a nonsingular matrix. It is then easy to

compute the inverse of the matrix above, and from (3.14)

$$\begin{aligned}\widetilde{A} &= PAP^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ 0 & 8 & 5 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \\ \widetilde{B} &= PB = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.\end{aligned}$$

Notice that the transformation matrix is *not* unique. However, all possible matrices $\widetilde{A}$ will be similar to 3×3 matrix above. In particular, the eigenvalues of the uncontrollable part are those of A_3, namely the roots of

$$0 = \det(\lambda I_2 - A_3) = \begin{vmatrix} \lambda - 8 & -5 \\ -3 & \lambda + 1 \end{vmatrix} = \lambda^2 - 7\lambda - 23$$

and these roots cannot be altered by applying linear feedback to (3.16).

NOTE : For any given transfer function matrix $G(\cdot)$ there are an infinite number of minimal realizations satisfying the conditions of THEOREM 3.4.3. However, one can show that the relationship between any two minimal realizations is just that of algebraic equivalence : *If* $\mathcal{R} = (A, B, C)$ *is a minimal realization of* $G(\cdot)$*, then* $\widetilde{\mathcal{R}} = (\widetilde{A}, \widetilde{B}, \widetilde{C})$ *is also a minimal realization if and only if the following holds :*

$$\widetilde{A} = PAP^{-1}, \quad \widetilde{B}PB, \quad \widetilde{C} = CP^{-1}.$$

Algorithm for constructing a minimal realization

We do not have room to discuss the general problem of efficient construction of minimal realizations. We will give here one simple but nevertheless useful result.

3.4.5 PROPOSITION. *Let the denominators of the elements* $g_{ij}(s)$ *of* $G(s) \in \mathbb{R}^{n\times\ell}$ *have simple roots* $s_1, s_2, \ldots, s_q$*. Define*

$$K_i := \lim_{s\to s_i} (s - s_i)G(s), \quad i = 1, 2, \ldots, q$$

and let

$$r_i := \operatorname{rank}(K_i), \quad i = 1, 2, \ldots, q.$$

If L_i *and* M_i *are* $n \times r_i$ *and* $r_i \times \ell$ *matrices, respectively, each having rank* r_i *such that*

$$K_i = L_i M_i$$

then a minimal realization of $G(\cdot)$ *is*

$$A = \begin{bmatrix} s_1 I_{r_1} & & & O \\ & s_2 I_{r_2} & & \\ & & \ddots & \\ O & & & s_q I_{r_q} \end{bmatrix}, \quad B = \begin{bmatrix} M_1 \\ M_2 \\ \vdots \\ M_q \end{bmatrix}, \quad C = \begin{bmatrix} L_1 & L_2 & \ldots & L_q \end{bmatrix}.$$

(To verify that (A, B, C) is a realization of $G(\cdot)$ is straightforward. Indeed,

$$\begin{aligned} C(sI_m - A)^{-1}B &= \begin{bmatrix} L_1 & \cdots & L_q \end{bmatrix} \begin{bmatrix} \frac{1}{s-s_1} I_{r_1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \frac{1}{s-s_q} I_{r_q} \end{bmatrix} \begin{bmatrix} M_1 \\ \vdots \\ M_q \end{bmatrix} \\ &= \frac{L_1 M_1}{s - s_1} + \cdots + \frac{L_q M_q}{s - s_q} = \frac{K_1}{s - s_1} + \cdots + \frac{K_q}{s - s_q} = G(s). \end{aligned}$$

Since $\operatorname{rank} \mathcal{C}(A, B) = \operatorname{rank} \mathcal{O}(A, C) = m$, this realization is minimal.)

3.4.6 Example. Consider the scalar transfer function

$$g(s) = \frac{2s + 7}{s^2 - 5s + 6}.$$

We have

$$\begin{aligned} K_1 &= \lim_{s \to 2} \frac{(s-2)(2s+7)}{(s-2)(s-3)} = -11, \quad r_1 = 1 \\ K_2 &= \lim \frac{(s-3)(2s+7)}{(s-2)(s-3)} = 13, \quad r_2 = 1. \end{aligned}$$

Taking

$$L_1 = K_1, \quad M_1 = 1, \quad L_2 = K_2, \quad M_2 = 1$$

produces a minimal realization

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} -11 & 13 \end{bmatrix}.$$

However,

$$b = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \quad \text{and} \quad c = \begin{bmatrix} -\frac{11}{m_1} & \frac{13}{m_2} \end{bmatrix}$$

can be used instead, still giving a minimal realization for arbitrary nonzero values of m_1 and m_2.

3.5 Exercises

Exercise 45 Verify that the control system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

is c.c.

Exercise 46 Given the control system described by

$$\dot{x} = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix} x + bu(t)$$

find for what vector b the system is *not* c.c.

Exercise 47 For the (initialized) control system

$$\dot{x} = \begin{bmatrix} -4 & 2 \\ 4 & -6 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

apply a control in the form

$$u(t) = c_1 + c_2 e^{-2t}$$

so as to bring the system to the origin at time $t = 1$. Obtain, but do not solve, the equations which determine the constants c_1 and c_2.

Exercise 48 For each of the following cases, determine for what values of the real parameter α the control system is *not* c.c.

$$(1) \quad \dot{x} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & \alpha \\ 0 & 1 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} u(t)\,;$$

$$(2) \quad \dot{x} = \begin{bmatrix} 2 & \alpha - 3 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & \alpha^2 - \alpha \end{bmatrix} u(t).$$

In part (2), if the first control variable $u_1(\cdot)$ ceases to operate, *for what additional values* (if any) of α the system is not c.c. under the remaining scalar control $u_2(\cdot)$?

Exercise 49 Consider the control system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(t).$$

Such a system could be thought of as a simple representation of a *vehicle suspension system*: in this interpretation, x_1 and x_2 are the displacements of the end points of the platform from equilibrium. Verify that the system is c.c. If the ends of the platform are each given an initial displacement of 10 units, find using

$$u^*(t) = -B^T(t)\Phi^T(t_0, t)W_c^{-1}(t_0, t_1)[x_0 - \Phi(t_0, t_1)x_f]$$

a control function which returns the system to equilibrium at $t = 1$.

Exercise 50 Prove that $U(t_0, t_1)$ defined by

$$U(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau)B(\tau)B^T(\tau)\Phi^T(t_0, \tau)d\tau$$

satisfies the matrix differential equation

$$\dot{U}(t, t_1) = A(t)U(t, t_1) + U(t, t_1)A^T(t) - B(t)B^T(t), \quad U(t_1, t_1) = 0.$$

Exercise 51 In the preceding exercise let A and B be time-invariant, and put

$$W(t, t_1) = U(t, t_1) - U_0$$

where the constant matrix U_0 satisfies

$$AU_0 + U_0A^T = BB^T.$$

Write down the solution of the resulting differential equation for W using the result in **Exercise 32** and hence show that

$$U(t,t_1) = U_0 - \exp\left((t-t_1)A\right) U_0 \exp\left((t-t_1)A^T\right).$$

Exercise 52 Consider again the *rabbit-fox environment problem* described in section **1.3** (see also, **Exercise 29**). If it is possible to count only the *total* number of animals, can the individual numbers of rabbits and foxes be determined ?

Exercise 53 For the system (with outputs)

$$\dot{x} = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix} x\,, \quad y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

find $x(0)$ if $y(t) = -20e^{-3t} + 21e^{-2t}$.

Exercise 54 Show that the control system described by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -2x_1 - 3x_2 + u, \quad y = x_1 + x_2$$

is *not* c.o. Determine initial states $x(0)$ such that if $u(t) = 0$ for $t \geq 0$, then the output $y(t)$ is identically zero for $t \geq 0$.

Exercise 55 Prove that $V(t_0,t_1)$ defined by

$$V(t_0,t_1) = \int_{t_0}^{t_1} \Phi^T(\tau,t_0)C^T(\tau)C(\tau)\Phi(\tau,t_0)d\tau$$

satisfies the matrix differential equation

$$\dot{V}(t,t_1) = -A^T(t)V(t,t_1) - V(t,t_1)A(t) - C^T(t)C(t), \quad V(t_1,t_1) = 0.$$

Exercise 56 Consider the time-invariant control system

$$\dot{x} = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(t).$$

Find a 1×2 matrix K such that the closed loop system has eigenvalues -4 and -5.

Exercise 57 For the time-invariant control system

$$\dot{x} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(t)$$

find a suitable matrix K so as to make the closed loop eigenvalues $-1, -1 \pm 2i$.

Exercise 58 Given the time-invariant control system

$$\dot{x} = Ax + Bu(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

find a suitable matrix K which makes the eigenvalues of $A - BK$ equal to $1, 1, 3$.

Exercise 59 Determine whether the system described by

$$\dot{x} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -3 & 0 \\ 1 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} u(t)$$

is c.c. Show that under (linear) feedback of the form $u = \alpha x_1 + \beta x_3$, the *closed loop system* has two fixed eigenvalues, one of which is equal to -3. Determine the second fixed eigenvalue, and also values of α and β such that the third closed loop eigenvalue is equal to -4.

Exercise 60 Show that if $X = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ is a block matrix with A and C invertible, then X is invertible and

$$X^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}.$$

Exercise 61 Use Proposition 3.4.5 to obtain a *minimal realization* of

$$G(s) = \frac{1}{g(s)} \begin{bmatrix} (s^2+6) & (s^2+s+4) \\ (2s^2-7s-2) & (s^2-5s-2) \end{bmatrix}$$

where $g(s) = s^3 + 2s^2 - s - 2$.

Exercise 62 Show that the order of a minimal realization of

$$G(s) = \frac{1}{s^2+3s+2}\begin{bmatrix} (s+2) & 2(s+2) \\ -1 & (s+1) \end{bmatrix}$$

is three. (Notice the fallacy of assuming that the order is equal to the degree of the common denominator.)

Exercise 63 If (A_1, B_1, C_1) and (A_2, B_2, C_2) are realizations of $G_1(\cdot)$ and $G_2(\cdot)$, respectively, show that

$$A = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$$

is a realization of $G_1(\cdot)G_2(\cdot)$, assuming that this product exists.

Exercise 64 Verify that *algebraic equivalence*

$$\widetilde{A} = PAP^{-1}, \quad \widetilde{B} = PB, \quad \widetilde{C} = CP^{-1}$$

can be written as the transformation

$$\begin{bmatrix} P & 0 \\ 0 & I_n \end{bmatrix}\begin{bmatrix} sI_n - A & B \\ -C & 0 \end{bmatrix}\begin{bmatrix} P^{-1} & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} sI_n - \widetilde{A} & \widetilde{B} \\ -\widetilde{C} & 0 \end{bmatrix}.$$

Exercise 65 Determine values of b_1, b_2, c_1 and c_2 such that

$$\mathcal{R} = \left(\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix}\right)$$

and

$$\widetilde{\mathcal{R}} = \left(\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 \end{bmatrix}\right)$$

are realizations of the transfer function

$$g(s) = \frac{s+4}{s^2+5s+6}\,.$$

Determine a matrix P such that the algebraic equivalence relationship holds between the two realizations.

Chapter 4

Stability

Topics :

1. Basic Concepts
2. Algebraic Criteria for Linear Systems
3. Lyapunov Theory with Applications to Linear Systems
4. Stability and Control

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Stability was probably the first question in classical dynamical systems which was dealt with in a satisfactory way. Stability questions motivated the introduction of new mathematical concepts (tools) in engineeering, particularly in control engineering. Stability theory has been of interest to mathematicians and astronomers for a long time and has had a stimulating impact on these fields. The specific problem of attempting to prove that the solar system is stable accounted for the introduction of many new methods.

Our treatment of stability will apply to (control) systems described by sets of linear *or* nonlinear equations. As is to be expected, however, our most explicit results will be obtained for linear systems.

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4.1 Basic Concepts

Consider the *nonlinear* dynamical system Σ described by

$$\boxed{\dot{x} = F(t, x), \qquad x \in \mathbb{R}^m} \tag{4.1}$$

where $x(\cdot)$ is a *curve* in the state space $\mathbb{R}^m$ and F is a vector-valued mapping having components $F_i, \quad i = 1, 2, \ldots, m$.

NOTE : We shall assume that the components F_i are continuous and satisfy standard conditions, such as having continuous first order partial derivatives so that the solution curve of (4.1) exists and is unique for any given initial condition (state).

From a geometric point of view, the right-hand side (rhs) F can be interpreted as a *time-dependent* vector field on $\mathbb{R}^m$. So *a* (nonlinear) *dynamical system is essentially the same as* (and thus can be identified with) *a vector field on the state space.* This point of view is very fruitfull and extremely useful in investigating the properties of the dynamical system (especially when the state space is a *manifold*).

If the functions F_i do not depend explicitly on t, then system Σ is called *autonomous* (or *time-independent*); otherwise, *nonautonomous* (or *time-dependent*).

Equilibrium states

4.1.1 DEFINITION. If $F(t, c) = 0$ for all t, then $c \in \mathbb{R}^m$ is said to be an **equilibrium** (or **critical**) **state**.

It follows at once from (4.1) that (for an equilibrium state c) if $x(t_0) = c$, then $x(t) = c$ for all $t \geq t_0$. Thus *solution curves starting at c remain there.*

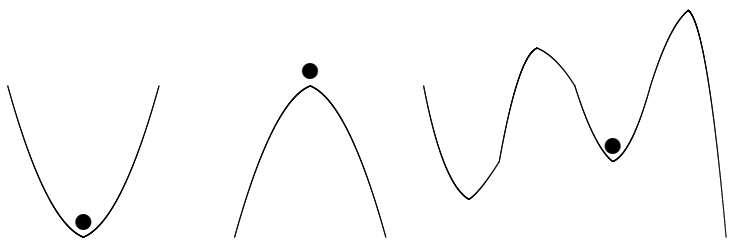
Clearly, by introducing new variables $x_i' = x_i - c_i$ we can arrange for the equilibrium state to be transferred to the origin (of the state space $\mathbb{R}^m$); we shall assume that this has been done for any equilibrium state under consideration (there may well be several for a given system Σ), so that we then have

$$F(t, 0) = 0$$

for all $t \geq t_0$.

We shall also assume that *there is no other constant solution curve in the neighborhood of the origin*, so this is an **isolated equilibrium state**.

4.1.2 Example. The intuitive idea of stability in a dynamical setting is that *for "small" perturbations from the equilibrium state at some time* t_0, *subsequent motions* $t \mapsto x(t)$, $t \geq t_0$ *should not be too "large"*. Consider a ball resting in equilibrium on a sheet of metal bent into various shapes with cross-sections as shown.



(i) (ii) (iii)

If frictional forces can be neglected, then small perturbations lead to :

- oscillatory motion about equilibrium (case (i)) ;
- the ball moving away without returning to equilibrium (case (ii));
- oscillatory motion about equilibrium, unless the initial perturbation is so large that the ball is forced to oscillate about a new equilibrium position (case (iii)).

If friction is taken into account then the oscillatory motions steadily decrease until the equilibrium state is returned to.

Stability

There is no single concept of stability, and many different definitions are possible. We shall consider only the following fundamental statements.

4.1.3 Definition. An equilibrium state $x = 0$ is said to be :

(a) **stable** if for any positive scalar ε there exists a positive scalar δ such that $\|x(t_0)\| < \delta$ implies $\|x(t)\| < \varepsilon$ for all $t \geq t_0$.

(b) **asymptotically stable** if it is stable and if in addition $x(t) \to 0$ as $t \to \infty$.

(c) **unstable** if it is not stable; that is, there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there exists an $x(t_0)$ with $\|x(t_0)\| < \delta$, $\|x(t_1)\| \geq \varepsilon$ for some $t_1 > t_0$.

(d) **completely unstable** if there exists an $\varepsilon > 0$ such that for every $\delta > 0$ and for every $x(t_0)$ with $\|x(t_0)\| < \delta$, $\|x(t_1)\| \geq \varepsilon$ for some $t_1 > t_0$.

Note : The definition (a) is often called "stability in the sense of Lyapunov" (stability i.s.L.) after the Russian mathematician Aleksandr M. Lyapunov (1857-1918), whose important work features prominently in current control theory.

In Example 4.1.2, case (i) represents stability i.s.L. if friction is ignored, and asymptotic stability if friction is taken into account, whereas case (ii) represents instability. If the metal sheet in (i) were thought to extend indefinitely then, if friction is present, the ball would eventually return to equilibrium no matter how large the disturbance. This is an illustration of asymptotic stability *in the large*, which means that *every motion converges to a single equilibrium point (state) as* $t \to \infty$, and clearly does not apply to case (iii). Asymptotic stability in the large implies that all motions are bounded. Generally,

4.1.4 Definition. An equilibrium state $x = 0$ is said to be **bounded** (or **Lagrange stable**) if there exists a constant M, which may depend on t_0 and $x(t_0)$, such that $\|x(t)\| \leq M$ for all $t \geq t_0$.

Some remarks can be made at this stage.

1. Regarded as a function of t in the (n-dimensional) state space, the solution $x(\cdot)$ of (4.1) is called a **trajectory** (or **motion**). In two dimensions we can give the definitions a simple geometric interpretation.

   - If the origin O is stable, then given the outer circle $\mathcal{C}$, radius ε, there exists an inner circle $\mathcal{C}_1$, radius δ_1, such that trajectories starting within $\mathcal{C}_1$ never leave $\mathcal{C}$.
   - If O is asymptotically stable, then there is some circle $\mathcal{C}_2$, radius δ_2, having the same property as $\mathcal{C}_1$ but, in addition, trajectories starting inside $\mathcal{C}_2$ tend to O as $t \to \infty$.

2. We refer to the stability of an equilibrium state of Σ, not the system itself, as different equilibrium states may have different stability properties.

3. A weakness of the definition of stability i.s.L. for practical purposes is that only the *existence* of some positive δ is required, so δ may be very small compared to ε; in other words, only very small disturbances from equilibrium may be allowable.

4. *In engineering applications, asymptotic stability is more desirable than stability* since it ensures eventual return to equilibrium, whereas stability allows continuing deviations "not to far" from the equilibrium state.

Examples

Some further aspects of stability are now illustrated through some examples.

4.1.5 Example. We return again to the environment problem involving rabbits anf foxes, and let the equations have the following numerical form

$$\begin{cases} \dot{x}_1 = 2x_1 - 3x_2 \\ \\ \dot{x}_2 = 2x_1 - x_2. \end{cases}$$

These equations have a single equilibrium point (state) at the origin. With arbitrary initial numbers $x_1(0)$ and $x_2(0)$ of rabbits and foxes, respectively, the solution is

$$x_1(t) = x_1(0)e^{\frac{t}{2}}\left(\cos\frac{\sqrt{15}}{2}t + \frac{3}{\sqrt{15}}\sin\frac{\sqrt{15}}{2}t\right) - \frac{6x_2(0)}{\sqrt{15}}e^{\frac{t}{2}}\sin\frac{\sqrt{15}}{2}t$$

with a similar expression for $x_2(t)$. Clearly, $x_1(t)$ tends to infinity as $t \to \infty$, irrespective of the initial state, so *the origin is unstable.*

4.1.6 Example. Consider the (initialized) dynamical system on $\mathbb{R}$ described by

$$\dot{x} = x^2, \quad x(0) = x_0 \in \mathbb{R}.$$

It is clear that the solution exists and is unique; in fact, by integrating, we easily obtain

$$-\frac{1}{x} = t - \frac{1}{x_0}\cdot$$

Hence

$$x(t) = \frac{1}{\frac{1}{x_0} - t}$$

so that if $x_0 > 0$, $x(t) \to \infty$ as $t \to \frac{1}{x_0}$. The solution is said to "escape" to infinity in a finite time, or to have a *finite escape time.* We shall henceforth exclude this situation and assume that (4.1) *has a finite solution for all finite* $t \geq t_0$, for otherwise (4.1) cannot be a mathematical model of a real-life situation.

4.1.7 Example. We demonstrate that the origin is a stable equilibrium state for the (initialized) system described by

$$\dot{x} = (1 - 2t)x, \quad x(t_0) = x_0 \in \mathbb{R}$$

by determining explicitly the scalar δ in the definition. Integrating the equation gives

$$x(t) = x_0 e^{t-t^2} e^{t_0^2 - t_0}.$$

The condition $|x(t)| < \varepsilon$ leads to

$$|x_0| < \varepsilon\, e^{t^2-t} e^{t_0-t_0^2}.$$

Since $t \mapsto e^{t^2-t}$ has a minimum value of $e^{-\frac{1}{4}}$ when $t = \frac{1}{2}$, it follows that we can take

$$\delta = \varepsilon\, e^{-\left(t_0-\frac{1}{2}\right)^2}.$$

In general, δ will depend upon ε, but in this example it is also a function of the initial time. If δ is independent of t_0, the stability is called **uniform**.

4.1.8 Example. Consider the (initialized) dynamical system on $\mathbb{R}$ described by

$$\dot{x}(t) = \begin{cases} x(t) - 2 & \text{if } x(t) > 2 \\ \\ 0 & \text{if } x(t) \leq 2. \end{cases}, \quad x(t_0) = x_0 \in \mathbb{R}.$$

The solution is easily found to be

$$x(t) = \begin{cases} 2 + (x_0 - 2)e^{t-t_0} & \text{if } x_0 > 2 \\ \\ x_0 & \text{if } x_0 \leq 2. \end{cases}$$

The condition $|x(t)| < \varepsilon$ is implied by $|x_0| < 2$ when $\varepsilon \geq 2$, for then $|x(t)| = |x_0| < 2 < \varepsilon$. When $\varepsilon < 2$, $|x(t)| < \varepsilon$ is implied by $|x_0| < \varepsilon$, for then again $|x(t)| = |x_0| < \varepsilon$. Thus according to the definition, *the origin is a stable equilibrium point (state).* However, if $x_0 > 2$ then $x(t) \to \infty$, so *for initial perturbations* $x_0 > 2$ *from equilibrium motions are certainly unstable in a practical sense.*

4.1.9 Example. Consider the equation

$$\dot{x} = f(t)x, \quad x(0) = x_0 \in \mathbb{R}$$

where

$$f(t) = \begin{cases} \ln 10 & \text{if } 0 \le t \le 10 \\ \\ -1 & \text{if } t > 10. \end{cases}$$

The solution is

$$x(t) = \begin{cases} 10^t\, x_0 & \text{if } 0 \le t \le 10 \\ \\ 10^{10} x_0 e^{10-t} & \text{if } t > 10. \end{cases}$$

Clearly, $x(t) \to 0$ as $t \to \infty$, and *the origin is asymptotically stable.* However, if x_0 changes by a very small amount, say 10^{-5}, then the corresponding change in $x(t)$ is relatively large – for example, when $t = 20$, the change in $x(t)$ is $10^{10} 10^{-5} e^{10-20} \approx 4.5$.

NOTE : EXAMPLES 4.1.8 and 4.1.9 show that *an equilibrium state may be stable according to Lyapunov's definitions and yet the system's behaviour may be unsatisfactory from a practical point of view.* The converse situation is also possible, and this has led to a definition of "practical stability" being coined for systems which are unstable in Lyapunov's sense but have an acceptable performance in practice, namely that *for pre-specified deviations from equilibrium the subsequent motions also lie within specified limits.*

4.2 Algebraic Criteria for Linear Systems

We return to the general *linear* (time-invariant) system given by

$$\boxed{\dot{x} = Ax, \qquad x \in \mathbb{R}^m} \tag{4.2}$$

where $A \in \mathbb{R}^{m \times m}$ and (4.2) may represent the closed *or* open loop system. Provided the matrix A is nonsingular, the only equilibrium state of (4.2) is the origin, so it is meaningful to refer to the stability of the system (4.2). If the system is stable (at the origin) but not asymptotically stable we shall call it **neutrally stable**.

One of the basic results on which the development of linear system stability theory relies is now given. The proof will be omitted.

4.2.1 THEOREM. (STABILITY PROPERTIES OF A LINEAR SYSTEM) *Consider the linear system (4.2), and for each eigenvalue λ of A, suppose that m_λ denotes the algebraic multiplicity of λ and d_λ the geometric multiplicity of λ. Then :*

(a) The system is asymptotically stable if and only if A is a ***stability matrix****; that is, every eigenvalue of A has a negative real part.*

(b) The system is neutrally stable if and only if

- *every eigenvalue of A has a nonpositive real part, <u>and</u>*
- *at least one eigenvalue has a zero real part, and $d_\lambda = m_\lambda$ for every eigenvalue λ with a zero real part.*

(c) The system is unstable if and only if

- *some eigenvalue of A has a positive real part, <u>or</u>*
- *there is an eigenvalue λ with a zero real part and $d_\lambda < m_\lambda$.*

NOTE : (1) Suppose all the eigenvalues of A have nonpositive real parts. One can prove that if all eigenvalues having zero real parts are distinct, then the origin is neutrally stable.

(2) Also, *if every eigenvalue of A has a positive real part, then the system is completely unstable.*

4.2.2 EXAMPLE. In EXAMPLE 4.1.5 the system matrix is of the form

$$A = \begin{bmatrix} a_1 & -a_2 \\ a_3 & -a_4 \end{bmatrix}$$

where $a_1, a_2, a_3, a_4 > 0$. It is easy to show that

$$\det(\lambda I_2 - A) = \lambda(\lambda - d)$$

using the condition

$$\frac{a_1}{a_3} = \frac{a_2}{a_4}\,.$$

Hence A has a single zero eigenvalues, so *the system is neutrally stable provided* $d(= a_1 - a_4)$ *is negative.* (If $a_1 = a_4$ then the system is unstable.)

NOTE : The preceding theorem applies if A is real or complex, so *the stability determination of (4.2) can be carried out by computing the eigenvalues using one of the powerful standard computer programs now available.* However, if m is small (say less than six), or if some of the elements of A are in parametric form, or if access to a digital computer is not possible, then the classical results given below are useful.

Because of its practical importance the linear system stability problem has attracted attention for a considerable time, an early study being by JAMES C. MAXWELL (1831-1879) in connection with the governing of steam engines. The original formulation of the problem was not of course in matrix terms, the system model being

$$z^{(m)} + k_1 z^{(m-1)} + \cdots + k_m z = u(t). \tag{4.3}$$

This is equivalent to working with the *characteristic polynomial* of A, which we shall write in this section as

$$a(\lambda) := \det(\lambda I_m - A) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m. \tag{4.4}$$

The first solutions giving necessary and sufficient conditions for all the roots of $a(\lambda)$ in (4.4) to have negative real parts were given by AUGUSTIN L. CAUCHY (1789-1857), JACQUES C.F. STURM (1803-1855), and CHARLES HERMITE (1822-1901).

We give here a well-known result due to ADOLF HURWITZ (1859-1919) for the case when all the coefficients a_i are real. The proof will be omitted.

4.2.3 THEOREM. (HURWITZ) *The* $m \times m$ ***Hurwitz matrix*** *associated*

with the characteristic polynomial $a(\lambda)$ *of* A *in (4.4) is*

$$H := \begin{bmatrix} a_1 & a_3 & a_5 & \dots & a_{2m-1} \\ 1 & a_2 & a_4 & \dots & a_{2m-2} \\ 0 & a_1 & a_3 & \dots & a_{2m-3} \\ 0 & 1 & a_2 & \dots & a_{2m-4} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_m \end{bmatrix} \tag{4.5}$$

where $a_r = 0,\ r > m$. *Let* H_i *denote the* i^{th} *leading principal minor of* H. *Then all the roots of* $a(\lambda)$ *have negative real parts (*$a(\lambda)$ *is a* ***Hurwitz polynomial*** *) if and only if* $H_i > 0,\ \ i = 1, 2, \dots, m$.

NOTE : A disadvantage of THEOREM 4.2.3 is the need to evaluate determinants of increasing order, and a convenient way of avoiding this is due to EDWARD J. ROUTH (1831-1907). We will give only the test as it applies to polynomials of degree no more than four, although the test can be extended to any polynomial.

4.2.4 PROPOSITION. (THE ROUTH TEST) *All the roots of the polynomial* $a(\lambda)$ *(with real coefficients) have negative real parts precisely when the given conditions are met.*

- $\lambda^2 + a_1\lambda + a_2$: *all the coefficients are positive ;*
- $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$: *all the coefficients are positive and* $a_1a_2 > a_3$ *;*
- $\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4$: *all the coefficients are positive,* $a_1a_2 > a_3$ *and* $a_1a_2a_3 > a_1^2a_4 + a_3^2$.

NOTE : The HURWITZ and ROUTH tests can be useful for determining stability of (4.3) and (4.2) in certain cases. However, it should be noted that a practical disadvantage of application to (4.2) is that it is very difficult to calculate accurately the a_i in (4.4). This is important because small errors in the a_i can lead to large errors in the roots of $a(\lambda)$.

4.2.5 Example. Investigate the stability of the linear system whose characteristic equation is

$$\lambda^4 + 2\lambda^3 + 9\lambda^2 + 4\lambda + 1 = 0.$$

Solution : The polynomial $a(\lambda) = \lambda^4 + 2\lambda^3 + 9\lambda^2 + 4\lambda + 1$ has positive coefficients, and since $a_1 = 2$, $a_2 = 9$, $a_3 = 4$ and $a_4 = 1$, the polynomial meets the Routh conditions :

$$18 = a_1 a_2 > a_3 = 4 \quad \text{and} \quad 72 = a_1 a_2 a_3 > a_1^2 a_4 + a_3^2 = 20\,.$$

So all the roots have negative real parts, and hence the linear system is asymptotically stable.

Since we have assumed that the coefficients a_i are real it is easy to derive a simple *necessary* condition for asymptotic stability :

4.2.6 Proposition. *If the coefficients a_i in (4.4) are real and $a(\lambda)$ corresponds to an asymptotically stable system, then*

$$a_i > 0\,, \quad i = 1, 2, \ldots, m.$$

Proof : Any complex root of $a(\lambda)$ will occur in conjugate pairs $\alpha \pm i\beta$, the corresponding factor of $a(\lambda)$ being

$$(\lambda - \alpha - i\beta)(\lambda - \alpha + i\beta) = \lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2\,.$$

By Theorem 4.2.1, $\alpha < 0$, and similarly any real factor of $a(\lambda)$ can be written $(\lambda + \gamma)$ with $\gamma > 0$. Thus

$$a(\lambda) = \prod(\lambda + \gamma) \prod(\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2)$$

and since all the coefficients above are positive, a_i must also all be positive.□

Note : Of course the condition above is not a sufficient condition, but it provides a useful initial check : *if any a_i are negative or zero, then $a(\lambda)$ cannot be asymptotically stable.*

When we turn to linear *time-varying* systems

$$\boxed{\dot{x} = A(t)x\,, \quad x(t_0) = x_0} \tag{4.6}$$

the situation is much more complicated. In view of THEOREM 4.2.1 it might be thought that if the eigenvalues of $A(t)$ all have negative real parts for all $t \geq t_0$, then the origin of (4.6) would be asymptotically stable. Unfortunately, *this conjecture is not true* (see **Exercise 73**).

4.3 Lyapunov Theory

We shall develop the so-called *"direct" method of Lyapunov* in relation to the (initialized) nonlinear *autonomous* dynamical system Σ given by

$$\boxed{\dot{x} = F(x), \qquad x(0) = x_0 \in \mathbb{R}^m\,; \quad F(0) = 0.} \tag{4.7}$$

NOTE : Modifications needed to deal with the (nonautonomous) case

$$\dot{x} = F(t, x), \qquad x(t_0) = x_0$$

are straightforward.

The aim is to determine the stability nature of the equilibrium state (at the origin) of system Σ *without obtaining the solution* $x(\cdot)$. This of course has been done algebraically for linear time invariant systems in section **4.2**. The essential idea is to generalize the concept of *energy* V for a conservative system in mechanics, where a well-known result states that *an equilibrium point is stable if the energy is minimum.* Thus V is a positive function which has $\dot{V}$ negative in the neighborhood of a stable equilibrium point. More generally,

4.3.1 DEFINITION. We define a **Lyapunov function** $V : \mathbb{R}^m \to \mathbb{R}$ as follows :

- V and all its partial derivatives $\dfrac{\partial V}{\partial x_i}$ are continuous ;

- V is *positive definite*; that is, $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$ in some neighborhood $\{x \mid \|x\| \leq k\}$ of the origin.

Consider now the (directional) derivative of V with respect to (the vector field) F, namely

$$\begin{aligned} \dot{V} &:= \frac{\partial V}{\partial x} F = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_m} \end{bmatrix} \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix} \\ &= \frac{\partial V}{\partial x_1} F_1 + \frac{\partial V}{\partial x_2} F_2 + \cdots + \frac{\partial V}{\partial x_m} F_m. \end{aligned}$$

A Lyapunov function V for the system (4.7) is said to be

- **strong** if the derivative $\dot{V}$ is **negative definite**; that is, $\dot{V}(0) = 0$ and $\dot{V}(x) < 0$ for $x \neq 0$ such that $\|x\| \leq k$.
- **weak** if the derivative $\dot{V}$ is **negative semi-definite**; that is, $\dot{V}(0) = 0$ and $\dot{V}(x) \leq 0$ for all x such that $\|x\| \leq k$.

NOTE : The definitions for positive or negative definiteness or semi-definiteness are generalizations of those for quadratic forms. Here is the definiteness test for planar quadratic forms : *Suppose that* $Q = Q(x, y)$ *is the quadratic form* $ax^2 + 2bxy + cy^2$, *where* $a, b, c \in \mathbb{R}$. *Then* Q *is :*

- *positive definite* $\iff$ $a, c > 0$ *and* $b^2 < ac$;
- *positive semi-definite* $\iff$ $a, c \geq 0$ *and* $b^2 \leq ac$;
- *negative definite* $\iff$ $a, c < 0$ *and* $b^2 < ac$;
- *negative semi-definite* $\iff$ $a, c \leq 0$ *and* $b^2 \leq ac$.

Otherwise, Q *is indefinite.*

The Lyapunov stability theorems

The statements of the two basic theorems of Lyapunov are remarkably simple. The proofs will be omitted.

4.3.2 THEOREM. (LYAPUNOV'S FIRST THEOREM) *Suppose that there is a strong Lyapunov function V for system Σ. Then system Σ is asymptotically stable.*

The conclusion of this theorem is plausible, since the values of the strong Lyapunov function $V(x(t))$ must continually diminish along each orbit $x = x(t)$ as t increases (since $\dot{V}$ is negative definite). This means that the orbit $x = x(t)$ must cut across *level sets* $V(x) = C$ with ever smaller values of C. In fact, $\lim_{t\to\infty} V(x(t)) = 0$, which implies that $x(t) \to 0$ as $t \to \infty$ since $x(t)$ and $V(x)$ are continuous and V has value zero only at the origin (that's where the positive definiteness of V comes into play).

4.3.3 THEOREM. (LYAPUNOV'S SECOND THEOREM) *Suppose that there is a weak Lyapunov function V for system Σ. Then system Σ is stable.*

The result seems reasonable since the negative semi-definiteness of $\dot{V}$ keeps an orbit that starts near the origin close to the origin as t increases. But orbits don't have to cut across level sets of V.

NOTE : If the conditions on V in THEOREM 4.3.2 hold everywhere in state space it does not necessarily follow that the origin is asymptotically stable in the large. For this to be the case V must have the additional property that it is **radially unbounded**, which means that

$$V(x) \to \infty \quad \text{for all } x \text{ such that } \|x\| \to \infty\,.$$

For instance,

$$V = x_1^2 + x_2^2$$

is radially unbounded, but

$$V = \frac{x_1^2}{1 + x_1^2} + x_2^2$$

is not since, for example,

$$V \to 1 \quad \text{as } x_1 \to \infty,\ \ x_2 \to 0.$$

A similar line of reasoning shows that *if Ω is the set of points "outside" a bounded region containing the origin, and if throughout Ω, $V > 0$, $\dot{V} \le 0$ and V is radially unbounded, then the origin is Lagrange stable.*

4.3.4 Example. Consider a unit mass suspended from a fixed support by a spring, z being the displacement from the equilibrium. If first the spring is assumed to obey *Hooke's law*, then the equation of motion is

$$\ddot{z} + kz = 0 \tag{4.8}$$

where k is the spring constant. Taking $x_1 := z$, $x_2 := \dot{z}$, (4.8) becomes

$$\begin{cases} \dot{x}_1 = x_2 \\ \\ \dot{x}_2 = -kx_1. \end{cases}$$

Since the system is *conservative*, the *total energy*

$$E = \frac{1}{2}kx_1^2 + \frac{1}{2}x_2^2$$

is a Lyapunov function and it is easy to see that

$$\dot{E} = kx_1x_2 - kx_2x_1 = 0$$

so by Lyapunov's Second Theorem *the origin is stable.* (Of course, this is trivial since (4.8) represents *simple harmonic motion.*)

Suppose now that the force exerted by the spring, instead of being linear is some function $x_1k(x_1)$ satisfying

$$k(0) = 0\,, \quad k(x_1) > 0 \text{ for } x_1 \neq 0.$$

The total energy is now

$$E = \frac{1}{2}x_2^2 + \int_0^{x_1} \tau k(\tau)\, d\tau$$

and

$$\dot{E} = -kx_1x_2 + kx_1\dot{x}_1 = 0.$$

So again by Lyapunov's Second Theorem *the origin is stable* for any nonlinear spring satisfying the above conditions.

4.3.5 Example. Consider now the system of the previous example but with a *damping force* $d\dot{z}$ added, so that the equation of motion is

$$\ddot{z} + d\dot{z} + kz = 0. \tag{4.9}$$

(Equation (4.9) can also be used to describe an LCR series circuit, motion of a gyroscope, and many other problems.)

Assume first that both d and k are constant, and for simplicity let $d = 1$, $k = 2$. The system equations in state space form are

$$\begin{cases} \dot{x}_1 = x_2 \\ \\ \dot{x}_2 = -2x_1 - x_2 \end{cases}$$

and the total energy is

$$E = x_1^2 + \frac{1}{2}x_2^2$$

so that

$$\dot{E} = 2x_1x_2 + x_2(-2x_1 - x_2) = -x_2^2$$

which is negative semi-definite, so by Lyapunov's Second Theorem *the origin is stable.* However, now consider the function

$$V = 7x_1^2 + 2x_1x_2 + 3x_2^2.$$

Then

$$\dot{V} = -4x_1^2 - 4x_2^2.$$

Clearly, $\dot{V}$ is negative definite and it is easy to verify that the quadratic form V is positive definite, so by Lyapunov's First Theorem *the origin is asymptotically stable* (in fact, in the large).

This example illustrates that *a suitably-chosen Lyapunov function can provide more information than the energy function.* However, when $\dot{V}$ is only negative semi-definite, the following result is often useful.

4.3.6 Proposition. *Suppose that there is a (weak) Lyapunov function V such that $\dot{V}$ does not vanish identically on any nontrivial trajectories of Σ. Then (the origin of) system Σ is asymptotically stable.*

4.3.7 Example. Consider again the damped mass-spring system described by (4.9), but now suppose that both d and k are not constant. Let $k(x_1)$ be as defined in Example 4.3.4 and let $d(x_2)$ have the property

$$d(x_2) > 0 \quad \text{for } x_2 \neq 0; \quad d(0) = 0.$$

The state equations are

$$\begin{cases} \dot{x}_1 = x_2 \\ \\ \dot{x}_2 = -x_1 k(x_1) - x_2 d(x_2). \end{cases}$$

So, if E is

$$E = \frac{1}{2}x_2^2 + \int_0^{x_1} \tau k(\tau)\, d\tau$$

then

$$\dot{E} = x_2(-x_1 k - d x_2) + x_1 k \dot{x}_1 = -x_2^2 d \leq 0.$$

Now E is positive definite and $\dot{E}$ vanishes only when $x_2(t) \equiv 0$, which implies $k(x_1) \equiv 0$, which in turn implies $x_1(t) \equiv 0$. Thus $\dot{E}$ vanishes only on the trivial solution of (4.9), and so by Proposition 4.3.6 *the origin is asymptotically stable.*

4.3.8 Example. The **van der Pol equation**

$$\ddot{z} + \epsilon\,(z^2 - 1)\dot{z} + z = 0 \tag{4.10}$$

where ϵ is a constant, arises in a number of engineering problems. (In a control context it can be thought of as application of *nonlinear feedback*

$$u = -z + \epsilon\,(1 - z^2)\dot{z}$$

to the (linear) system described by

$$\ddot{z} = u.)$$

We shall assume that $\epsilon < 0$. As usual take $x_1 := z$, $x_2 := \dot{z}$ to transform (4.10) into

$$\begin{cases} \dot{x}_1 = x_2 \\ \\ \dot{x}_2 = -x_1 - \epsilon\,(x_1^2 - 1)x_2. \end{cases}$$

(The only equilibrium state of this system is the origin.) Try as a potential Lyapunov function $V = x_1^2 + x_2^2$ which is obviously positive definite. Then

$$\begin{aligned} \dot{V} &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\ &= 2\epsilon\, x_2^2(1 - x_1^2). \end{aligned}$$

Thus $\dot{V} \leq 0$ if $x_1^2 < 1$, and then by PROPOSITION 4.3.6 the origin is asymptotically stable. It follows that *all trajectories starting inside the region* $\Gamma : x_1^2 + x_2^2 < 1$ *converge to the origin as* $t \to \infty$, and Γ is therefore called a **region of asymptotic stability**.

NOTE : You may be tempted to think that the infinite strip $S : x_1^2 < 1$ is a region of asymptotic stability. This is not in fact true, since a trajectory starting outside Γ can move inside the strip whilst continuing in the direction of decreasing V circles, and hence lead to divergence.

In general, if a *closed region*

$$R : \quad V(x) \leq \text{constant}$$

is bounded and has $\dot{V}$ *negative throughout, then region* R *is a region of asymptotic stability.*

Suppose that we now take as state variables

$$x_1 := z, \quad x_3 := \int_0^t z(\tau)\, d\tau.$$

The corresponding state equations are

$$\begin{cases} \dot{x}_1 = -x_3 - \epsilon\left(\frac{1}{3}x_1^3 - x_1\right) \\ \\ \dot{x}_3 = x_1. \end{cases}$$

Indeed,

$$\begin{aligned} \dot{x}_1 &= \dot{z} \\ &= -\int_0^t z\,d\tau - \epsilon\int_0^t (z^2-1)\dot{z}\,d\tau \\ &= -x_3 - \epsilon\left(\frac{1}{3}z^3 - z\right) \\ &= -x_3 - \epsilon\left(\frac{1}{3}x_1^3 - x_1\right). \end{aligned}$$

Hence, using $V = x_1^2 + x_3^2$, we have

$$\begin{aligned} \dot{V} &= 2x_1\left(-x_3 - \frac{1}{3}\epsilon\, x_1^3 + \epsilon\, x_1\right) + 2x_3x_1 \\ &= 2\epsilon\, x_1^2\left(1 - \frac{1}{3}x_1^2\right) \leq 0 \quad \text{if } x_1^2 < 3 \end{aligned}$$

so the region of asymptotic stability obtained by this different set of state variables is $R : x_1^2 + x_3^2 < 3$, larger than before.

NOTE : In general, if the origin is an asymptotically stable equilibrium point (state), then the total set of initial points (states) from which trajectories converge to the origin as $t \to \infty$ is called the **domain of attraction**. *Knowledge of this domain is of great value in practical problems since it enables permissible deviations from equilibrium to be determined.* However, EXAMPLE 4.3.8 illustrates the fact that since a particular Lyapunov function gives only sufficient conditions for stability, the region of asymptotic stability obtained can be expected to be only part of the domain of attraction. Different Lyapunov functions or different sets of state variables may well yield different stability regions.

The general problem of finding "optimum" Lyapunov functions, which give best possible estimates for the domain of attraction, is a difficult one.

It may be a waste of effort trying to determine the stability properties of an equilibrium point, since the point may be unstable. The following result is then useful :

4.3.9 THEOREM. (LYAPUNOV'S THIRD THEOREM) *Let a function* $V : \mathbb{R}^m \to \mathbb{R}$ *with* $V(0) = 0$ *have continuous first order partial derivatives. If there is some neighborhood containing the origin in which* V *takes negative values, and if in addition* $\dot{V}$ *is negative semi-definite, then the origin of (4.7) is not asymptotically stable. If* $\dot{V}$ *is negative definite, the origin is unstable, and if both* V *and* $\dot{V}$ *are negative definite, the origin is completely unstable.*

NOTE : In all three Lyapunov's theorems the terms "positive" and "negative" can be interchanged simply by using $-V$ instead of V. It is only the relative signs of the Lyapunov function and its derivative which matter.

Application to linear systems

We now return to the linear time-invariant system

$$\boxed{\dot{x} = Ax, \qquad x \in \mathbb{R}^m.} \tag{4.11}$$

In section **4.2** we gave *algebraic criteria* for determining asymptotic stability via the characteristic equation of A. We now show how Lyapunov theory can be used to deal directly with (4.11) by taking as a potential Lyapunov function the quadratic form

$$\boxed{V = x^T P x} \tag{4.12}$$

where the matrix $P \in \mathbb{R}^{m \times m}$ is *symmetric*. The (directional) derivative of V with respect to (4.11) (in fact, with respect to the vector field $x \mapsto Ax$) is

$$\begin{aligned} \dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\ &= x^T A^T P x + x^T P A x \\ &= -x^T Q x \end{aligned}$$

where

$$A^T P + PA = -Q \tag{4.13}$$

and it is easy to see that Q is also symmetric. If P and Q are both positive definite, then by LYAPUNOV'S FIRST THEOREM the (origin of) system (4.11) is asymptotically stable. If Q is positive definite and P is negative definite or indefinite, then in both cases V can take negative values in the neighborhood of the origin so by LYAPUNOV'S THIRD THEOREM, (4.11) is unstable. We have therefore proved :

4.3.10 PROPOSITION. *The matrix $A \in \mathbb{R}^{m\times m}$ is a stability matrix if and only if for any given positive definite symmetric matrix Q, there exists a positive definite symmetric matrix P that satisfies the Lyapunov matrix equation (4.13).*

Moreover, if the matrix A is a stability matrix, then P is the unique solution of the *Lyapunov matrix equation* (see **Exercise 77**).

It would be no use choosing P to be positive definite and calculating Q from (4.13). For unless Q turned out to be definite or semi-definite (which is unlikely) nothing could be inferred about asymptotic stability from the Lyapunov theorems.

NOTE : Equations similar in form to (4.13) also arise in other areas of control theory. However, it must be admitted that since a digital computer will be required to solve (4.13) except for small values of n, so far as stability determination of (4.11) is concerned it will be preferable instead to find the eigenvalues of A. The true value and importance of PROPOSITION 4.3.10 lies in its use as a theoretical tool.

Linearization

The usefulness of linear theory can be extended by use of the idea of *linearization*. Suppose the components of (the vector field) F in

$$\dot{x} = F(x), \qquad x(0) = x_0\,; \quad F(0) = 0$$

are such that we can apply *Taylor's theorem* to obtain

$$F(x) = Ax + g(x). \tag{4.14}$$

Here $A = DF(0) := \frac{\partial F}{\partial x}\big|_{x=0} \in \mathbb{R}^{m\times m}$, $g(0) = 0 \in \mathbb{R}^m$, and the components of g have power series expansions in $x_1, x_2, \ldots, x_m$ beginning with terms of at least second degree. The linear system

$$\dot{x} = Ax \tag{4.15}$$

is called the **linearization** (or **first approximation**) of the given (nonlinear) system (at the origin). We then have:

4.3.11 Theorem. (Lyapunov's Linearization Theorem) *If (4.15) is asymptotically stable or unstable, then the origin for $\dot{x} = F(x)$, where F is given by (4.14), has the same stability property.*

Proof : Consider the function

$$V = x^T P x$$

where P satisfies

$$A^T P + PA = -Q$$

Q being an arbitrary positive definite symmetric matrix. If (4.15) is asymptotically stable, then by Proposition 4.3.10 P is positive definite. The derivative of V with respect to (4.14) is

$$\dot{V} = -x^T Q x + 2g^T P x\,.$$

Because of the nature of g, the term $2g^T Px$ has degree three at least, and so for x sufficiently close to the origin, $\dot{V} < 0$.

Exercise 66 Let $a, b > 0$ and consider the function

$$f : \mathbb{R} \to \mathbb{R}, \quad t \mapsto -at^2 + bt^3.$$

Show that, for t sufficiently close to the origin (i.e. for $|t| < \epsilon$), we have $f(t) < 0$.

Hence, by Lyapunov's First Theorem, *the origin of (4.14) is asymptotically stable.*

If (4.15) is unstable, $\dot{V}$ remains negative definite but P is indefinite, so V can take negative values and therefore satisfies the conditions of Lyapunov's Third Theorem for instability. □

Note : If (4.15) is stable but not asymptotically stable, Lyapunov's Linearization Theorem provides no information about the stability of the origin of (4.14), and other methods must be used.

Furthermore, it is clear that *linearization cannot provide any information about regions of asymptotic stability for nonlinear systems*, since if the first approximation is asymptotically stable, then it is so in the large. Thus the extent of asymptotic stability for

$$\dot{x} = F(x), \quad F(0) = 0$$

is determined by the nonlinear terms in (4.14).

4.3.12 Example. Consider the differential equation

$$\ddot{z} + a\dot{z} + bz + g(z, \dot{z}) = 0$$

or

$$\begin{cases} \dot{x}_1 = x_2 \\ \\ \dot{x}_2 = -bx_1 - ax_2 - g(x_1, x_2). \end{cases}$$

The linear part of this system is asymptotically stable if and only if $a > 0$ and $b > 0$, so if g is *any* function of x_1 and x_2 satisfying the conditions of Theorem 4.3.11, the origin of the system is also asymptotically stable.

4.4 Stability and Control

We now consider some stability problems associated explicitly with the control variables.

Input-output stability

Our definitions in section **4.1** referred to stability with respect to perturbations from an equilibrium state. When a system is subject to inputs it is useful to define a new type of stability.

4.4.1 Definition. The control system (with outputs) Σ described by

$$\begin{cases} \dot{x} = F(t, x, u)\,, \quad F(t, 0, 0) = 0 \\ \\ y = h(t, x, u) \end{cases}$$

is said to be **bounded input-bounded output stable** (b.i.b.o. stable) if any bounded input produces a bounded output; that is, given

$$\|u(t)\| < L_1\,, \quad t \geq t_0$$

where L_1 is any positive constant, then there exists a number $L_2 > 0$ such that

$$\|y(t)\| < L_2\,, \quad t \geq t_0$$

regardless of initial state $x(t_0)$. The problem of studying b.i.b.o. stability for nonlinear systems is a difficult one, but we can give some results for the usual linear time-invariant system

$$\begin{cases} \dot{x} = Ax + Bu(t) \\ \\ y = Cx. \end{cases} \tag{4.16}$$

Exercise 67 Let $A \in \mathbb{R}^{m \times m}$ be a matrix having m *distinct* eigenvalues with negative real parts. Show that (for all $t \geq 0$)

$$\|\exp(tA)\| \leq Ke^{-at}$$

for some constants $K, a > 0$. (The result remains valid for *any* stability matrix, but the proof is more difficult.)

4.4.2 Proposition. *If the linear system*

$$\dot{x} = Ax$$

is asymptotically stable, then the system described by (4.16) is b.i.b.o. stable.

Proof : Using

$$x(t) = \exp(tA)\left[x_0 + \int_0^t \exp(-\tau A)Bu(\tau)\,d\tau\right]$$

and properties of norms, we have

$$\begin{aligned} \|y(t)\| &\leq \|C\|\,\|x(t)\| \\ &\leq \|C\|\,\|\exp(tA)x_0\| + \|C\|\int_0^t \|\exp(t-\tau)A\|\|Bu\|\,d\tau. \end{aligned}$$

If A is a stability matrix, then

$$\|\exp(tA)\| \leq K\,e^{-at} \leq K\,, \quad t \geq 0$$

for some positive constants K and a. Thus

$$\begin{aligned} \|y(t)\| &\leq \|C\|\left[K\|x_0\| + L_1K\|B\|\frac{1-e^{-at}}{a}\right] \\ &\leq \|C\|\left[K\|x_0\| + \frac{L_1K\|B\|}{a}\right]\,, \quad t \geq 0 \end{aligned}$$

showing that the output is bounded, since $\|C\|$ and $\|B\|$ are positive numbers.□

The converse of this result holds if (A, B, C) is a minimal realization. In other words,

4.4.3 Proposition. *If the control system described by (4.16) is c.c. and c.o. and b.i.b.o. stable, then the linear (dynamical) system*

$$\dot{x} = Ax$$

is asymptotically stable.

We shall not give a proof.

NOTE : For linear time-varying systems PROPOSITION 4.4.2 is not true, unless for all t the norms of $B(t)$ and $C(t)$ are bounded and the norm of the state transition matrix $\Phi(t, t_0)$ is bounded and tends to zero as $t \to \infty$ independently of t_0.

In the definition of complete controllability no restrictions were applied to $u(\cdot)$, but in practical situations there will clearly always be finite bounds on the magnitudes of the control variables and on the duration of their application. It is then intuitively obvious that this will imply that *not all states are attainable.* As a trivial example, if a finite thrust is applied to a rocket for a finite time, then there will be a limit to the final velocity which can be achieved. We give here one formal result for linear systems.

4.4.4 PROPOSITION. *If $A \in \mathbb{R}^{m \times m}$ is a stability matrix, then the linear control system Σ given by*

$$\dot{x} = Ax + Bu(t)$$

with $u(\cdot)$ bounded is not completely controllable.

PROOF : Let V be a quadratic form Lyapunov function (i.e. $V = x^T P x$, where P is a symmetric matrix) for the (unforced) system $\dot{x} = Ax$. Then (with respect to Σ) we have

$$\begin{aligned} \dot{V} = \dot{V}^T &= (Ax + Bu)^T (\nabla V) \\ &= x^T A^T (\nabla V) + u^T B^T (\nabla V) \\ &= -x^T Q x + u^T B^T (\nabla V) \end{aligned}$$

where P and Q satisfy (the Lyapunov matrix equation)

$$A^T P + PA = -Q$$

and

$$\nabla V := \begin{bmatrix} \dfrac{\partial V}{\partial x_1} & \dfrac{\partial V}{\partial x_2} & \cdots & \dfrac{\partial V}{\partial x_m} \end{bmatrix}^T$$

is the *gradient* of V. The term $u^T B^T(\nabla V)$ is linear in x and since $u(\cdot)$ is bounded, it follows that *for* $\|x\|$ *sufficiently large*, $\dot{V} = \dot{x}^T(\nabla V)$ is negative. This shows that $\dot{x}(\cdot)$ points into the interior of the region $R : V(x) = M$ for some M sufficiently large. Hence *points outside this region cannot be reached*, so by definition the system is not c.c. □

Linear feedback

Consider again the linear system (4.16). If the open loop system is unstable (for instance, by THEOREM 4.2.1, if one or more of the eigenvalues of A has a positive real part), then an essential practical objective would be to apply control so as to *stabilize* the system; that is, to make the closed loop system asymptotically stable.

If (4.16) is c.c., then we saw (THEOREM 3.1.3) that stabilization can always be achieved by linear feedback $u = Kx$, since there are an infinity of matrices K which will make $A + BK$ a stability matrix.

If the pair (A, B) is *not* c.c., then we can define the *weaker* property that (A, B) is **stabilizable** if (and only if) *there exists a constant matrix* K *such that* $A + BK$ *is asymptotically stable.*

4.4.5 EXAMPLE. Return to the linear system Σ described by

$$\dot{x} = \begin{bmatrix} 4 & 3 & 5 \\ 1 & -2 & -3 \\ 2 & 1 & 8 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} u(t)$$

in EXAMPLE 3.4.4. The eigenvalues of the uncontrollable part are the roots of the polynomial

$$p(\lambda) = \lambda^2 - 7\lambda - 23$$

which is *not* asymptotically stable since it has negative coefficients (PROPOSITION 4.2.6). The system Σ is *not* stabilizable.

By duality (see THEOREM 3.2.4) we define the pair (A, C) to be **detectable** if (and only if) *the pair* (A^T, C^T) *is stabilizable.*

4.4.6 Example. Consider the linear control system (with outputs)

$$\begin{cases} \dot{x} = Ax + bu(t) \\ \\ y = cx \end{cases}$$

where

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

This system is both stabilizable and detectable : using the *feedback matrix* K and the *observer matrix* L given by

$$K = \begin{bmatrix} -2 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

the matrix $A + bK$ then has eigenvalues -3, -2 and the matrix $A + Lc$ has eigenvalues -2, -1. (Other choices for K and L are also possible.)

Stabilizability and detectability are not guaranteed.

4.4.7 Example. Consider the linear control system (with outputs)

$$\begin{cases} \dot{x} = Ax + bu(t) \\ \\ y = cx \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

In this case, any 1×2 feedback matrix K produces a closed loop matrix $A + bK$ with zero as an eigenvalue; therefore, the system is not stabilizable.

Also, any 2×1 matrix L yields a matrix $A + Lc$ with zero as an eigenvalue; therefore, the system is not detectable.

The following simple *test for stabilizability* holds :

4.4.8 Proposition. *Suppose* $A \in \mathbb{R}^{m\times m}$ *and* $B \in \mathbb{R}^{m\times \ell}$. *The pair* (A, B) *is stabilizable if and only if*

$$\operatorname{rank}\begin{bmatrix} sI_m - A & B \end{bmatrix} = m$$

for all $s \in \mathbb{C}_{e+} := \{s \,|\, \operatorname{Re}(s) \geq 0\}$.

The proof is not difficult and will be omitted.

4.4.9 Corollary. *The pair* (A, B) *is stabilizable if and only if*

$$\operatorname{rank}\begin{bmatrix} \lambda I_m - A & B \end{bmatrix} = m$$

for all eigenvalues $\lambda \in \mathbb{C}_{e+}$ *of* A.

Note : It can be proved that *the pair* (A, B) *is c.c. if and only if*

$$\operatorname{rank}\begin{bmatrix} \lambda I_m - A & B \end{bmatrix} = m$$

for all eigenvalues λ *of* A. (The eigenvalues at which the rank drops below m are the so-called *uncontrollable modes.*) Clearly, if the pair (A, B) is c.c., then it is stabilizable.

By duality, the following (test for detectability) is now immediate.

4.4.10 Proposition. *Suppose* $A \in \mathbb{R}^{m\times m}$ *and* $C \in \mathbb{R}^{\ell\times m}$. *The pair* (A, C) *is detectable if and only if*

$$\operatorname{rank}\begin{bmatrix} sI_m - A \\ C \end{bmatrix} = m$$

for all $s \in \mathbb{C}_{e+} := \{s \,|\, \operatorname{Re}(s) \geq 0\}$.

4.4.11 Corollary. *The pair* (A, C) *is detectable if and only if*

$$\operatorname{rank}\begin{bmatrix} \lambda I_m - A \\ C \end{bmatrix} = m$$

for all eigenvalues $\lambda \in \mathbb{C}_{e+}$ *of* A.

NOTE : By duality again, *the pair* (A, C) *is c.o. if and only if*

$$\operatorname{rank}\begin{bmatrix}\lambda I_m - A\\ C\end{bmatrix} = m$$

for all eigenvalues λ *of* A. (The eigenvalues at which the rank drops below m are the so-called *unobservable modes.*) Clearly, if the pair (A, C) is c.o., then it is detectable.

Let $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times \ell}$, and $C \in \mathbb{R}^{n\times m}$. Recall that if A is a stability (or Hurwitz) matrix, then the usual (time-invariant) linear control system with outputs

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}$$

is b.i.b.o. stable (see PROPOSITION 4.4.2). As we know, the converse is *not* true, in general. (For example, the usual linear control system with $A = \begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$, $b = \begin{bmatrix}1\\ 1\end{bmatrix}$, and $c = \begin{bmatrix}0 & 1\end{bmatrix}$ is b.i.b.o. stable but *not* asymptotically stable.)

The following interesting result is given without a proof.

4.4.12 PROPOSITION. *Suppose a usual linear control system* Σ *is stabilizable and detectable.* Σ *is asymptotically stable if and only if it is b.i.b.o. stable.*

Application

It is interesting to consider here a *simple application of the Lyapunov methods.* First notice that if the linear system

$$\dot{x} = Ax$$

is asymptotically stable with Lyapunov function $V = x^T P x$, where P satisfies

$$A^T P + PA = -Q$$

then

$$\frac{\dot{V}}{V} = -\frac{x^T Q x}{x^T P x} \leq -\sigma$$

where σ is the minimum value of the ratio $\frac{x^T Q x}{x^T P x}$ (in fact, this is equal to the smallest eigenvalue of QP^{-1}). Integrating with respect to t gives

$$V(x(t)) \leq e^{-\sigma t} V(x(0)).$$

Since $V(x(t)) \to 0$ as $t \to \infty$, this can be regarded as a *measure of the way in which trajectories approach the origin*, so the larger σ the "faster" does $x(t) \to 0$.

Suppose now we apply the control

$$u = (S - Q_1) B^T P x \tag{4.17}$$

to (4.16), where P is the solution of

$$A^T P + PA = -Q$$

and S and Q_1 are arbitrary skew-symmetric and positive definite symmetric matrices, respectively. The *closed loop system* is thus

$$\dot{x} = \left(A + B(S - Q_1) B^T P\right) x \tag{4.18}$$

and it is easy to verify that if $V = x^T P x$, then the (directional) derivative with respect to (4.18) is

$$\dot{V} = -x^T Q x - 2x^T P B Q_1 B^T P x < -x^T Q x$$

since $PBQ_1B^TP = (PB)Q_1(PB)^T$ is positive definite. Hence, by the argument just developed, it follows that (4.18) is "more stable" than the open loop system

$$\dot{x} = Ax$$

in the sense that trajectories will approach the origin more quickly.

NOTE : (4.17) is of rather limited practical value because it requires asymptotic stability of the open loop system, but nevertheless the power of Lyapunov theory is apparent by the ease with which the asymptotic stability of (4.18) can be established. This would be impossible using the classical methods requiring the calculation of the characteristic equation of (4.18). Furthermore, *the Lyapunov approach often enables extensions to nonlinear problems to be made.*

4.5 Exercises

Exercise 68 Determine the equilibrium point (other than the origin) of the system described by

$$\begin{cases} \dot{x}_1 = x_1 - 2x_1x_2 \\ \\ \dot{x}_2 = -2x_2 + x_1x_2. \end{cases}$$

Apply a transformation of coordinates which moves this point to the origin, and find the new system equations. (The equations are an example of a *predator-pray population model* due to VITO VOLTERRA (1860-1940) and used in biology, and are more general than the simple linear rabbit-fox model.)

Exercise 69 Determine whether the following polynomials are asymptotically stable :

(a) $\lambda^3 + 17\lambda^2 + 2\lambda + 1$.

(b) $\lambda^4 + \lambda^3 + 4\lambda^2 + 4\lambda + 3$.

(c) $\lambda^4 + 6\lambda^3 + 2\lambda^2 + \lambda + 3$.

Exercise 70 Determine for what *range of values* of k the polynomial

$$(3-k)\lambda^3 + 2\lambda^2 + (5-2k)\lambda + 2$$

is asymptotically stable.

Exercise 71 Determine for what range of values of $k \in \mathbb{R}$ the linear (dynamical) system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k & -1 & -2 \end{bmatrix} x$$

is asymptotically stable. If $k = -1$, and a control term

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t)$$

is added, find a linear feedback control which makes all the eigenvalues of the closed loop system equal to -1.

Exercise 72 Verify that the eigenvalues of

$$A(t) = \begin{bmatrix} -4 & 3e^{-8t} \\ -e^{8t} & 0 \end{bmatrix}$$

are both constant and negative, but the solution of the linear time-varying system

$$\dot{x} = A(t)x$$

diverges as $t \to \infty$.

Exercise 73 Write the system

$$\ddot{z} + \dot{z} + z^3 = 0$$

in state space form. Let

$$V = ax_1^4 + bx_1^2 + cx_1x_2 + dx_2^2$$

and choose the constants a, b, c, d such that

$$\dot{V} = -x_1^4 - x_2^2.$$

Hence investigate the stability nature of the equilibrium point at the origin.

Exercise 74 Using the function

$$V = 5x_1^2 + 2x_1x_2 + 2x_2^2$$

show that the origin of

$$\begin{cases} \dot{x}_1 = x_2 \\ \\ \dot{x}_2 = -x_1 - x_2 + (x_1 + 2x_2)\left(x_2^2 - 1\right) \end{cases}$$

is asymptotically stable by considering the region $|x_2| < 1$. State the *region of asymptotic stability* thus determined.

Exercise 75 Investigate the stability nature of the origin of the system

$$\begin{cases} \dot{x}_1 = x_1^2 - x_2^2 \\ \\ \dot{x}_2 = -2x_1x_2 \end{cases}$$

using the function

$$V = 3x_1x_2^2 - x_1^3.$$

Exercise 76 Given the matrix

$$A = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix}$$

and taking

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

solve the equation

$$A^T P + PA = -I_2.$$

Hence determine the stability nature of A.

Exercise 77 Integrate both sides of the matrix ODE

$$\dot{W}(t) = AW(t) + W(t)B\,, \quad W(0) = C$$

with respect to t from $t = 0$ to $t = \infty$. Hence deduce that if A is a stability matrix, the solution of the equation

$$A^T P + PA = -Q$$

can be written as

$$P = \int_0^\infty \exp(tA^T)Q\exp(tA)\,dt.$$

Exercise 78 Convert the second order ODE

$$\ddot{z} + a_1\dot{z} + a_2 z = 0$$

into the state space form. Using $V = x^T Px$ with $\dot{V} = -x_2^2$, obtain the necessary and sufficient conditions $a_1 > 0$, $a_2 > 0$ for asymptotic stability.

Exercise 79 By using the quadratic Lyapunov function $V = V(x_1, x_2)$ which has derivative $-2(x_1^2 + x_2^2)$, determine the stability of the system

$$\begin{cases} \dot{x}_1 = -kx_1 - 3x_2 \\ \\ \dot{x}_2 = kx_1 - 2x_2 \end{cases}$$

when $k = 1$. Using the *same* function V, obtain sufficient conditions on k for the system to be asymptotically stable.

Exercise 80 Investigate the stability nature of the equilibrium state at the origin for the system

$$\begin{cases} \dot{x}_1 = 7x_1 + 2\sin x_2 - x_2^4 \\ \\ \dot{x}_2 = e^{x_1} - 3x_2 - 1 + 5x_1^2. \end{cases}$$

Exercise 81 Investigate the stability nature of the origin.

(a)

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \\ \dot{x}_2 = (x_1 + x_2)\sin x_1 - 3x_2. \end{cases}$$

(b)

$$\begin{cases} \dot{x}_1 = -x_1^3 + x_2 \\ \\ \dot{x}_2 = -ax_1 - bx_2; \quad a, b > 0. \end{cases}$$

Exercise 82 Show that the system described by

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ \\ y = x \end{cases}$$

is stable i.s.L. and b.i.b.o. stable, but not asymptotically stable. [It is easy to verify that this system is *not* completely controllable.]

Exercise 83 Consider the system described by the ODE

$$\dot{x} = -\frac{1}{t+3}x + u(t), \quad x(0) = x_0 \in \mathbb{R}.$$

Show that if $u(t) \equiv 0,\ t \geq 0$ then the origin is asymptotically stable, but that if $u(\cdot)$ is the unit step function, then $\lim_{t\to\infty} x(t) = \infty$.

Exercise 84 A system is described by the equations

$$\begin{cases} \dot{x}_1 = x_2 - x_1 \\ \\ \dot{x}_2 = -x_1 - x_2 - x_1^2. \end{cases}$$

Determine the equilibrium state which is not at the origin. Transform the system equations so that this point is transferred to the origin, and hence verify that this equilibrium state is unstable.

Exercise 85 Determine for what range of values of the real parameter k the linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -k & k-6 \end{bmatrix} x$$

is asymptotically stable.

Exercise 86 A particle moves in the xy-plane so that its position in any time t is given by

$$\ddot{x} + \dot{y} + 3x = 0, \quad \ddot{y} + \lambda\dot{x} + 3y = 0.$$

Determine the stability nature of the system if $(a)\ \lambda = -4$ and $(b)\ \lambda = 16$.

Exercise 87 Use the Lyapunov function

$$V = 2x_1^2 + x_2^2$$

to find a *region of asymptotic stability* for the origin of the system

$$\begin{cases} \dot{x}_1 = x_1^3 - 3x_1 + x_2 \\ \\ \dot{x}_2 = -2x_1. \end{cases}$$

Exercise 88 Use Lyapunov's Linearization Theorem to show that the origin for the system

$$\begin{cases} \dot{x}_1 = -3x_2 \\ \\ \dot{x}_2 = x_1 - \alpha\left(2x_2^3 - x_2\right) \end{cases}$$

is asymptotically stable provided the real parameter α is negative. Use the Lyapunov function

$$V = \frac{1}{2}\left(x_1^2 + 3x_2^2\right)$$

to determine a region of asymptotic stability about the origin.

Exercise 89 A system has state equations

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 - x_3 + \alpha\, x_1^3 \\ \\ \dot{x}_2 = -x_1 - x_2 \\ \\ \dot{x}_3 = x_1 - x_2 - \beta\, x_3 + u(t). \end{cases}$$

(a) If $\alpha = 0$ show that the system is b.i.b.o. stable for any output which is linear in the state variables, provided $\beta > -1$.

(b) If $\alpha = \beta = 1$ and $u(\cdot) = 0$, investigate the stability nature of the equilibrium state at the origin by using the Lyapunov function

$$V = x_1^2 + x_2^2 + x_3^2.$$

Exercise 90 Investigate for *stabilizability* and *detectability* the following control systems.

(a)

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x. \end{aligned}$$

(b)

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x. \end{aligned}$$

Chapter 5

Optimal Control

Topics :

1. Performance Indices
2. Elements of Calculus of Variations
3. Pontryagin's Principle
4. Linear Regulators with Quadratic Costs

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This section deals with the problem of compelling a system to behave in some "best possible" way. Of course, the precise control strategy will depend upon the criterion used to decide what is meant by "best", and we first discuss some choices for measures of system performance. This is followed by a description of some mathematical techniques for determining optimal control policies, including the special case of linear systems with quadratic performance index when a complete analytical solution is possible.

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5.1 Performance Indices

Consider a (nonlinear) *control system* Σ described by

$$\dot{x} = F(t, x, u), \quad x(t_0) = x_0 \in \mathbb{R}^m. \tag{5.1}$$

Here $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix}$ is the *state vector*, $u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_\ell(t) \end{bmatrix}$ is the *control vector*, and F is a vector-valued mapping having components

$$F_i : t \mapsto F_i(t, x_1(t), x_2(t), \ldots, x_m(t), u_1(t), \ldots, u_\ell(t)), \quad i = 1, 2, \ldots, m.$$

NOTE : We shall assume that the F_i are continuous and satisfy standard conditions, such as having continuous first order partial derivatives (so that the solution exists and is unique for the given initial condition). We say that F is *continuously differentiable* (or of *class* C^1).

The optimal control problem

The general *optimal control problem* (OCP) concerns the *minimization* of some function (functional) $\mathcal{J} = \mathcal{J}[u]$, the **performance index** (or **cost functional**); or, one may want to maximize instead a "utility" functional $\mathcal{J}$, but this amounts to minimizing the cost $-\mathcal{J}$. The performance index $\mathcal{J}$ provides a *measure* by which the performance of the system is judged. We give several examples of performance indices.

(1) Minimum-time problems.

Here $u(\cdot)$ is to be chosen so as to transfer the system from an initial state x_0 to a specified state in the *shortest possible time*. This is equivalent to *minimizing* the performance index

$$\mathcal{J} := t_1 - t_0 = \int_{t_0}^{t_1} dt \tag{5.2}$$

where t_1 is the first instant of time at which the desired state is reached.

5.1.1 Example. An aircraft pursues a ballistic missile and wishes to intercept it as quickly as possible. For simplicity neglect gravitational and aerodynamic forces and suppose that the trajectories are horizontal. At $t = 0$ the aircraft is at a distance a from the missile, whose motion is known to be described by $x(t) = a + bt^2$, where b is a positive constant. The motion of the aircraft is given by $\ddot{x} = u$, where the thrust $u(\cdot)$ is subject to $|u| \leq 1$, with suitably chosen units. Clearly the optimal strategy for the aircraft is to accelerate with maximum thrust $u(t) = 1$. After a time t the aircraft has then travelled a distance $ct + \frac{1}{2}t^2$, where $\dot{x}(0) = c$, so interception will occur at time T where

$$cT + \frac{1}{2}T^2 = a + bT^2.$$

This equation may not have any real positive solution; in other words, *this minimum-time problem may have no solution for certain initial conditions.*

(2) Terminal control.

In this case the final state $x_f = x(t_1)$ is to be brought as near as possible to some desired state $\overline{x}(t_1)$. A suitable performance measure to be *minimized* is

$$\mathcal{J} := e^T(t_1)Me(t_1) \tag{5.3}$$

where $e(t) := x(t) - \overline{x}(t)$ and M is a positive definite symmetric matrix ($M^T = M > 0$).

A special case is when M is the unit matrix and then

$$\mathcal{J} = \|x_f - \overline{x}(t_1)\|^2 .$$

Note : More generally, if $M = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$, then the entries λ_i are chosen so as to weight the relative importance of the deviations $(x_i(t_1) - \overline{x}_i(t_1))$. If some of the $\overline{x}_i(t_1)$ are not specified, then the corresponding elements of M will be zero and M will be only positive semi-definite ($M^T = M \geq 0$).

(3) Minimum effort.

The desired final state is now to be attained with *minimum total expenditure* of control effort. Suitable performance indices to be *minimized* are

$$\mathcal{J} := \int_{t_0}^{t_1} \sum_{i=1}^{\ell} \beta_i |u_i| \, dt \tag{5.4}$$

or

$$\mathcal{J} := \int_{t_0}^{t_1} u^T R u \, dt \tag{5.5}$$

where $R = \left[r_{ij} \right]$ is a positive definite symmetric matrix ($R^T = R > 0$) and the β_i and r_{ij} are *weighting factors.*

(4) Tracking problems.

The aim here is to follow or "track" as closely as possible some desired state $\overline{x}(\cdot)$ throughout the interval $[t_0, t_1]$. A suitable performance index is

$$\mathcal{J} := \int_{t_0}^{t_1} e^T Q e \, dt \tag{5.6}$$

where Q is a positive semi-definite symmetric matrix ($Q^T = Q \geq 0$).

NOTE : Such systems are called **servomechanisms**; the special case when $\overline{x}(\cdot)$ is constant or zero is called a **regulator**. If the $u_i(\cdot)$ are unbounded, then the minimization problem can lead to a control vector having infinite components. This is unacceptable for real-life problems, so to restrict the total control effort, the following index can be used

$$\mathcal{J} := \int_{t_0}^{t_1} \left(e^T Q e + u^T R u \right) \, dt. \tag{5.7}$$

Expressions (costs) of the form (5.5), (5.6) and (5.7) are termed **quadratic performance indices** (or **quadratic costs**).

5.1.2 EXAMPLE. A landing vehicle separates from a spacecraft at time $t_0 = 0$ at an altitude h from the surface of a planet, with initial (downward) velocity $\vec{v}$. For simplicity, assume that gravitational forces are neglected and that the mass of the vehicle is constant. Consider vertical motion only, with

upwards regarded as the positive direction. Let x_1 denote *altitude*, x_2 *velocity* and $u(\cdot)$ the *thrust* exerted by the rocket motor, subject to $|u(t)| \leq 1$ with suitable scaling. The *equations of motion* are

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

and the *initial conditions* are

$$x_1(0) = h, \quad x_2(0) = -v.$$

For a "soft landing" at some time T we require

$$x_1(T) = 0, \quad x_2(T) = 0.$$

A suitable *performance index* might be

$$\mathcal{J} := \int_0^T (|u| + k)\ dt.$$

This expression represents a sum of the *total fuel consumption* and *time to landing*, k being a factor which weights the relative importance of these two quantities.

Simple application

Before dealing with problems of determining optimal controls, we return to the linear time-invariant system

$$\dot{x} = Ax\,, \quad x(0) = x_0 \tag{5.8}$$

and show *how to evaluate associated quadratic indices (costs)*

$$\mathcal{J}_r := \int_0^\infty t^r x^T Q x\, dt\,, \quad r = 0, 1, 2, \ldots \tag{5.9}$$

where Q is a positive definite symmetric matrix ($Q^T = Q > 0$).

NOTE : If (5.8) represents a *regulator*, with $x(\cdot)$ being the *deviation* from some desired constant state, then minimizing $\mathcal{J}_r$ with respect to system parameters is

equivalent to making the system approach its desired state in an "optimal" way. Increasing the value of r in (5.9) corresponds to penalizing large values of t in this process.

To evaluate $\mathcal{J}_0$ we use the techniques of Lyapunov theory (cf. section **4.3**). It was shown that

$$\frac{d}{dt}\left(x^T P x\right) = -x^T Q x \tag{5.10}$$

where P and Q satisfy the *Lyapunov matrix equation*

$$A^T P + PA = -Q. \tag{5.11}$$

Integrating both sides of (5.10) with respect to t gives

$$\mathcal{J}_0 = \int_0^\infty x^T Q x \, dt = -\left(x^T(t) P x(t)\right)\Big|_0^\infty = x_0^T P x_0$$

provided A is a stability matrix, since in this case $x(t) \to 0$ as $t \to \infty$ (cf. Theorem 4.2.1).

Note : The matrix P is positive definite and so $\mathcal{J}_0 > 0$ for all $x_0 \neq 0$.

A repetition of the argument leads to a similar expression for $\mathcal{J}_r$, $r \geq 1$. For example,

$$\frac{d}{dt}\left(t x^T P x\right) = x^T P x - t x^T Q x$$

and integrating we have

$$\mathcal{J}_1 = \int_0^\infty t x^T Q x \, dt = x_0^T P_1 x_0$$

where

$$A^T P_1 + P_1 A = -P.$$

Exercise 91 Show that

$$\mathcal{J}_r := \int_0^\infty t^r x^T Q x \, dt = r! \, x_0^T P_r x_0 \tag{5.12}$$

where

$$A^T P_{r+1} + P_{r+1} A = -P_r \,, \quad r = 0, 1, 2, \ldots \,; \quad P_0 = P. \tag{5.13}$$

Thus *evaluation of (5.9) involves merely successive solution of the linear matrix equations (5.13)*; there is no need to calculate the solution $x(\cdot)$ of (5.8).

5.1.3 EXAMPLE. A general second-order linear system (the *harmonic oscillator* in one dimension) can be written as

$$\ddot{z} + 2\omega k\dot{z} + \omega^2 z = 0$$

where ω is the natural *frecquency* of the undamped system and k is a *damping coefficient*. With the usual choice of state variables $x_1 := z$, $x_2 := \dot{z}$, and taking $Q = \operatorname{diag}(1, q)$ in (5.11), it is easy to obtain the corresponding solution $P = \left[p_{ij}\right]$ with elements

$$p_{11} = \frac{k}{\omega} + \frac{1 + q\omega^2}{4k\omega}, \quad p_{12} = p_{21} = \frac{1}{2\omega^2}, \quad p_{22} = \frac{1 + q\omega^2}{4k\omega^3}.$$

Exercise 92 Work out the preceding computation.

In particular, if $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\mathcal{J}_0 = p_{11}$. Regarding k as a parameter, optimal damping could be defined as that which minimizes $\mathcal{J}_0$. By setting $\frac{d}{dk}\mathcal{J}_0 = 0$, this gives

$$k^2 = \frac{1 + q\omega^2}{4}.$$

For example, if $q = \frac{1}{\omega^2}$ then the "optimal" value of k is $\frac{1}{\sqrt{2}}$.

NOTE : In fact by determining $x(t)$ it can be deduced that *this value does indeed give the desirable system transient behaviour.* However, there is no *a priori* way of deciding on a suitable value for the factor q, which weights the relative importance of reducing $z(\cdot)$ and $\dot{z}(\cdot)$ to zero. This illustrates a disadvantage of the performance index approach, although in some applications it is possible to use physical arguments to choose values for weighting factors.

5.2 Elements of Calculus of Variations

The *calculus of variations* is the name given to the theory of the optimization of integrals. The name itself dates from the mid-eighteenth century and describes the method used to derive the theory. We have room for only a very brief treatment (in particular, we shall not mention the well-known *Euler-Lagrange equation* approach).

We consider the problem of *minimizing* the functional

$$\boxed{\mathcal{J}[u] := \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} L(t, x, u)\, dt} \tag{5.14}$$

subject to

$$\dot{x} = F(t, x, u), \quad x(t_0) = x_0 \in \mathbb{R}^m.$$

We assume that

- there are *no constraints* on the control functions $u_i(\cdot), \quad i = 1, 2, \ldots, \ell$ (that is, the control set U is $\mathbb{R}^\ell$);
- $\mathcal{J} = \mathcal{J}[u]$ is *differentiable* (that is, if u and $u+\delta u$ are two controls for which $\mathcal{J}$ is defined, then

  $$\Delta\mathcal{J} := \mathcal{J}[u + \delta u] - \mathcal{J}[u] = \delta\mathcal{J}[u, \delta u] + j(u, \delta u) \cdot \|\delta u\|$$

  where $\delta\mathcal{J}$ is *linear* in δu and $j(u, \delta u) \to 0$ as $\|\delta u\| \to 0$).

NOTE : (1) The cost functional $\mathcal{J}$ is in fact a function on the *function space* $\mathcal{U}$ (of all admissible controls) :

$$\mathcal{J} : u \in \mathcal{U} \mapsto \mathcal{J}[u] \in \mathbb{R}.$$

(2) $\delta\mathcal{J}$ is called the (first) **variation** of $\mathcal{J}$ corresponding to the variation δu in u.

The control u^* is an **extremal**, and $\mathcal{J}$ has a (relative) *minimum*, provided there exists an $\varepsilon > 0$ such that for all functions u satisfying $\|u - u^*\| < \varepsilon$,

$$\mathcal{J}[u] - \mathcal{J}[u^*] \geq 0.$$

A fundamental result (given without proof) is the following :

5.2.1 Proposition. *A necessary condition for u^* to be an extremal is that*

$$\delta\mathcal{J}\,[u^*, \delta u] = 0 \quad \textit{for all } \delta u.$$

We now apply Proposition 5.2.1. Introduce a covector function of *Lagrange multipliers* $p(t) = \begin{bmatrix} p_1(t) & p_2(t) \dots & p_m(t) \end{bmatrix} \in \mathbb{R}^{1\times m}$ so as to form an *augmented functional* incorporating the constraints :

$$\mathcal{J}_a := \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} (L(t,x,u) + p(F(t,x,u) - \dot{x}))\, dt.$$

Integrating the last term on the rhs by parts gives

$$\begin{aligned} \mathcal{J}_a &= \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} (L + pF + \dot{p}x)\, dt - px\Big|_{t_0}^{t_1} \\ &= \varphi(x(t_1), t_1) - px\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} (H + \dot{p}x)\, dt \end{aligned}$$

where the (control) **Hamiltonian function** is defined by

$$\boxed{H(t,p,x,u) := L(t,x,u) + pF(t,x,u).} \tag{5.15}$$

Assume that u is differentiable on $[t_0, t_1]$ and that t_0 and t_1 are fixed. The variation in $\mathcal{J}_a$ corresponding to a variation δu in u is

$$\delta\mathcal{J}_a = \left[\left(\frac{\partial\varphi}{\partial x} - p\right)\delta x\right]_{t=t_1} + \int_{t_0}^{t_1} \left(\frac{\partial H}{\partial x}\,\delta x + \frac{\partial H}{\partial u}\,\delta u + \dot{p}\,\delta x\right) dt$$

where δx is the variation in x in the differential equation

$$\dot{x} = F(t,x,u)$$

due to δu. (We have used the notation

$$\frac{\partial H}{\partial x} := \begin{bmatrix} \frac{\partial H}{\partial x_1} & \frac{\partial H}{\partial x_2} & \dots & \frac{\partial H}{\partial x_m} \end{bmatrix}$$

and similarly for $\dfrac{\partial\varphi}{\partial x}$ and $\dfrac{\partial H}{\partial u}$.)

NOTE : Since $x(t_0)$ is specified, $\delta x|_{t=t_0} = 0$.

It is convenient to remove the term (in the expression δJ_a) involving δx by suitably choosing p, i.e. by taking

$$\dot{p} = -\frac{\partial H}{\partial x} \quad \text{and} \quad p(t_1) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1} . \tag{5.16}$$

It follows that

$$\delta \mathcal{J}_a = \int_{t_0}^{t_1} \left(\frac{\partial H}{\partial u} \, \delta u \right) dt \, .$$

Thus *a necessary condition for* u^* *to be an extremal is that*

$$\boxed{\left.\frac{\partial H}{\partial u}\right|_{u=u^*} = 0 \, , \qquad t_0 \le t \le t_1 .} \tag{5.17}$$

We have therefore "established"

5.2.2 THEOREM. *Necessary conditions for* u^* *to be an extremal for*

$$\mathcal{J}\,[u] = \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} L(t, x, u) \, dt$$

subject to

$$\dot{x} = F(t, x, u), \quad x(t_0) = x_0$$

are the following :

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial x} \\ p(t_1) &= \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1} \\ \left.\frac{\partial H}{\partial u}\right|_{u=u^*} &= 0 \, , \qquad t_0 \le t \le t_1 . \end{aligned}$$

NOTE : The (vector) *state equation*

$$\dot{x} = F(t, x, u)$$

and the (vector) *co-state equation* (or *adjoint equation*)

$$\dot{p} = -\frac{\partial H}{\partial x}$$

give a total of $2m$ linear *or* nonlinear ODEs with (mixed) *boundary conditions* $x(t_0)$ and $p(t_1)$. In general, *analytical solution is not possible and numerical techniques have to be used.*

5.2.3 Example. Choose $u(\cdot)$ so as to minimize

$$\mathcal{J} = \int_0^T \left(x^2 + u^2\right)\, dt$$

subject to

$$\dot{x} = -ax + u, \quad x(0) = x_0 \in \mathbb{R}$$

where $a, T > 0$. We have

$$H = L + pF = x^2 + u^2 + p(-ax + u).$$

Also,

$$\dot{p}^* = -\frac{\partial H}{\partial x} = -2x^* + ap^*$$

and

$$\left.\frac{\partial H}{\partial u}\right|_{u=u^*} := 2u^* + p^* = 0$$

where x^* and p^* denote the *state* and *adjoint* variables for an optimal solution.

Substitution produces

$$\dot{x}^* = -ax^* - \frac{1}{2}p^*$$

and since $\varphi \equiv 0$, the *boundary condition* is just

$$p(T) = 0.$$

The linear system

$$\begin{bmatrix} \dot{x}^* \\ \dot{p}^* \end{bmatrix} = \begin{bmatrix} -a & -\frac{1}{2} \\ -2 & a \end{bmatrix} \begin{bmatrix} x^* \\ p^* \end{bmatrix}$$

can be solved using the methods of Chapter 2. (It is easy to verify that x^* and p^* take the form $c_1\, e^{\lambda t} + c_2\, e^{-\lambda t}$, where $\lambda = \sqrt{1 + a^2}$ and the constants c_1 and c_2 are found using the conditions at $t = 0$ and $t = T$.)

It follows that the *optimal control* is

$$u^*(t) = -\frac{1}{2}p^*(t).$$

NOTE : We have only found *necessary conditions* for optimality; further discussion of this point goes far beyond the scope of this course.

If the functions L and F do not *explicitly* depend upon t, then from

$$H(p, x, u) = L(x, u) + pF(x, u)$$

we get

$$\begin{aligned}
\dot{H} = \frac{d\,H}{dt} &= \frac{\partial L}{\partial u}\dot{u} + \frac{\partial L}{\partial x}\dot{x} + p\left(\frac{\partial F}{\partial u}\dot{u} + \frac{\partial F}{\partial x}\dot{x}\right) + \dot{p}F \\
&= \left(\frac{\partial L}{\partial u} + p\frac{\partial F}{\partial u}\right)\dot{u} + \left(\frac{\partial L}{\partial x} + p\frac{\partial F}{\partial x}\right)\dot{x} + \dot{p}F \\
&= \frac{\partial H}{\partial u}\dot{u} + \frac{\partial H}{\partial x}\dot{x} + \dot{p}F \\
&= \frac{\partial H}{\partial u}\dot{u} + \left(\frac{\partial H}{\partial x} + \dot{p}\right)F.
\end{aligned}$$

Since on an optimal trajectory

$$\dot{p} = -\frac{\partial H}{\partial x} \quad \text{and} \quad \left.\frac{\partial H}{\partial u}\right|_{u=u^*} = 0$$

it follows that $\dot{H} = 0$ when $u = u^*$, so that

$$H_{u=u^*} = \text{constant}, \qquad t_0 \le t \le t_1.$$

Discussion

We have so far assumed that t_1 is *fixed* and $x(t_1)$ is *free*. If this is not necessary the case, then we obtain

$$\delta\mathcal{J}_a = \left[\left(\frac{\partial\varphi}{\partial x} - p\right)\delta x + \left(H + \frac{\partial\varphi}{\partial t}\right)\delta t\right]_{\substack{u=u^* \\ t=t_1}} + \int_{t_0}^{t_1}\left(\frac{\partial H}{\partial x}\,\delta x + \frac{\partial H}{\partial u}\,\delta u + \dot{p}\,\delta x\right)dt.$$

The expression outside the integral must be zero (by virtue of PROPOSITION 5.2.1), making the integral zero. The implications of this for some important special cases are now listed. The initial condition $x(t_0) = x_0$ holds throughout.

$\boxed{\text{A}}$ Final time t_1 specified.

(i) $x(t_1)$ free

We have $\delta t|_{t=t_1} = 0$ but $\delta x|_{t=t_1}$ is arbitrary, so the condition

$$p(t_1) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1}$$

must hold (with $H_{u=u^*} =$ constant, $t_0 \leq t \leq t_1$ when appropriate), as before.

(ii) $x(t_1)$ specified

In this case $\delta t|_{t=t_1} = 0$ and $\delta x|_{t=t_1} = 0$ so

$$\left[\left(\frac{\partial \varphi}{\partial x} - p\right)\delta x + \left(H + \frac{\partial \varphi}{\partial t}\right)\delta t\right]_{\substack{u=u^* \\ t=t_1}}$$

is automatically zero. The condition is thus

$$x^*(t_1) = x_f$$

(and this replaces $p(t_1) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1}$).

$\boxed{\text{B}}$ Final time t_1 free.

(iii) $x(t_1)$ free

Both $\delta t|_{t=t_1}$ and $\delta x|_{t=t_1}$ are now arbitrary so for the expression

$$\left[\left(\frac{\partial \varphi}{\partial x} - p\right)\delta x + \left(H + \frac{\partial \varphi}{\partial t}\right)\delta t\right]_{\substack{u=u^* \\ t=t_1}}$$

to vanish, the conditions

$$p(t_1) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1} \quad \text{and} \quad \left.\left(H + \frac{\partial \varphi}{\partial t}\right)\right|_{\substack{u=u^* \\ t=t_1}} = 0$$

must hold.

NOTE : In particular, if φ, L, and F do not explicitly depend upon t, then

$$H_{u=u^*} = 0, \qquad t_0 \le t \le t_1.$$

(iv) $x(t_1)$ specified

Only $\delta t|_{t=t_1}$ is now arbitrary, so the conditions are

$$x^*(t_1) = x_f \quad \text{and} \quad \left.\left(H + \frac{\partial \varphi}{\partial t}\right)\right|_{\substack{u=u^* \\ t=t_1}} = 0.$$

5.2.4 EXAMPLE. A particle of unit mass moves along the x-axis subject to a force $u(\cdot)$. It is required to determine the control which transfers the particle from rest at the origin to rest at $x = 1$ in unit time, so as to minimize the effort involved, measured by

$$\mathcal{J} := \int_0^1 u^2 \, dt.$$

SOLUTION : The equation of motion is

$$\ddot{x} = u$$

and taking $x_1 := x$ and $x_2 := \dot{x}$ we obtain the *state equations*

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u.$$

We have

$$H = L + pF = p_1 x_2 + p_2 u + u^2.$$

From

$$\left.\frac{\partial H}{\partial u}\right|_{u=u^*} = 0$$

the *optimal control* is given by

$$2u^* + p_2^* = 0$$

and the *adjoint equations* are

$$\dot{p}_1^* = 0, \quad \dot{p}_2^* = -p_1^*.$$

Integration gives

$$p_2^* = C_1 t + C_2$$

and thus

$$\dot{x}_2^* = -\frac{1}{2}(C_1 t + C_2)$$

which on integrating, and using the given conditions $x_2(0) = 0 = x_2(1)$, produces

$$x_2^*(t) = \frac{1}{2}C_2\left(t^2 - t\right), \quad C_1 = -2C_2.$$

Finally, integrating the equation $\dot{x}_1 = x_2$ and using $x_1(0) = 0, \quad x_1(1) = 1$ gives

$$x_1^*(t) = \frac{1}{2}t^2(3 - 2t), \quad C_2 = -12.$$

Hence the *optimal control* is

$$u^*(t) = 6(1 - 2t).$$

An interesting case

If the state at final time t_1 (assumed fixed) is to lie on a "surface" S (more precisely, an $(m-k)$-*submanifold* of $\mathbb{R}^m$) defined by

$$\begin{aligned} g_1(x_1, x_2 \ldots, x_m) &= 0 \\ g_2(x_1, x_2, \ldots, x_m) &= 0 \\ &\vdots \\ g_k(x_1, x_2, \ldots, x_m) &= 0 \end{aligned}$$

(i.e. $S = g^{-1}(0) \subset \mathbb{R}^m$, where $g = (g_1, \ldots, g_k) : \mathbb{R}^m \to \mathbb{R}^k$, $m \geq k$ is such that $\operatorname{rank} \frac{\partial g}{\partial x} = k$), then (it can be shown that) in addition to the k conditions

$$g_1\left(x^*(t_1)\right) = 0, \ \ldots, \ g_k(x^*(t_1)) = 0 \tag{5.18}$$

there are a further m conditions which can be written as

$$\frac{\partial \varphi}{\partial x} - p = d_1 \frac{\partial g_1}{\partial x} + d_2 \frac{\partial g_2}{\partial x} + \cdots + d_k \frac{\partial g_k}{\partial x} \tag{5.19}$$

both sides being evaluated at $t = t_1$, $u = u^*$, $x = x^*$, $p = p^*$. The d_i are constants to be determined. Together with the $2m$ constants of integration there are thus $2m + k$ unknowns and $2m + k$ conditions (5.18), (5.19), and $x(t_0) = x_0$. If t_1 is free, then in addition

$$\left(H + \frac{\partial \varphi}{\partial t}\right)\Bigg|_{\substack{u=u^* \\ t=t_1}} = 0$$

holds.

5.2.5 Example. A system is described by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + u$$

is to be transformed (steered) from $x(0) = 0$ to the line $\mathcal{L}$ with equation

$$ax_1 + bx_2 = c$$

at time T so as to minimize

$$\int_0^T u^2 \, dt.$$

The values of $a, b, c,$ and T are given.

From

$$H = u^2 + p_1 x_2 - p_2 x_2 + p_2 u$$

we get

$$u^* = -\frac{1}{2} p_2^* \,. \tag{5.20}$$

The *adjoint equations* are

$$\dot{p}_1^* = 0, \quad \dot{p}_2^* = -p_1^* + p_2^*$$

so that

$$p_1^* = c_1\,, \quad p_2^* = c_2 e^t + c_1 \tag{5.21}$$

where c_1 and c_2 are constants. We obtain

$$x_1^* = c_3 e^{-t} - \frac{1}{4} c_2 e^t - \frac{1}{2} c_1 t + c_4, \quad x_2^* = -c_3 e^{-t} - \frac{1}{4} c_2 e^t - \frac{1}{2} c_1$$

and the conditions

$$x_1^*(0) = 0, \quad x_2^*(0) = 0, \quad a x_1^*(T) + b x_2^*(T) = c \tag{5.22}$$

must hold.

It is easy to verify that (5.19) produces

$$\frac{p_1^*(T)}{p_2^*(T)} = \frac{a}{b} \tag{5.23}$$

and (5.22) and (5.23) give four equations for the four unknown constants c_i. The optimal control $u^*(\cdot)$ is then obtained from (5.20) and (5.21).

NOTE : In some problems the restriction on the total amount of control effort which can be expended to carry out a required task may be expressed in the form

$$\int_{t_0}^{t_1} L_0(t, x, u)\, dt = c \tag{5.24}$$

where c is a given constant, such a constraint being termed **isoperimetric**. A convenient way of dealing with (5.24) is to define a new variable

$$x_{m+1}(t) := \int_{t_0}^{t} L_0(t, x, u)\, d\tau$$

so that

$$\dot{x}_{m+1} = L_0(t, x, u).$$

This ODE is simply added to the original one (5.1) together with the conditions

$$x_{m+1}(t_0) = 0, \quad x_{m+1}(t_1) = c$$

and the previous procedure continues as before, ignoring (5.24).

5.3 Pontryagin's Principle

In real-life problems the control variables are usually subject to constraints on their magnitudes, typically of the form

$$|u_i(t)| \leq K_i, \qquad i = 1, 2, \ldots, \ell.$$

This implies that *the set of final states which can be achieved is restricted.*

Our aim here is to derive the necessary conditions for optimality corresponding to THEOREM 5.2.2 for the unbounded case.

An **admissible control** is one which satisfies the constraints, and we consider variations such that

- $u^* + \delta u$ is admissible
- $\|\delta u\|$ is sufficiently small so that the sign of

$$\Delta\mathcal{J} = \mathcal{J}\left[u^* + \delta u\right] - \mathcal{J}\left[u^*\right]$$

  where

$$\mathcal{J}\left[u\right] = \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} L(t, x, u)\, dt$$

  is determined by $\delta\mathcal{J}$ in

$$\mathcal{J}\left[u + \delta u\right] - \mathcal{J}\left[u\right] = \delta\mathcal{J}\left[u, \delta u\right] + j(u, \delta u) \cdot \|\delta u\|.$$

Because of the restriction on δu, PROPOSITION 5.2.1 no longer applies, and instead a necessary condition for u^* to minimize $\mathcal{J}$ is

$$\delta\mathcal{J}\left[u^*, \delta u\right] \geq 0.$$

The development then proceeds as in the previous section; Lagrange multipliers $p = \begin{bmatrix} p_1 & p_2 & \ldots & p_m \end{bmatrix}$ are introduced to define $\mathcal{J}_a$ and are chosen so as to satisfy

$$\dot{p} = -\frac{\partial H}{\partial x} \quad \text{and} \quad p(t_1) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1}.$$

The only difference is that the expression for $\delta\mathcal{J}_a$ becomes

$$\delta\mathcal{J}_a\,[u,\delta u] = \int_{t_0}^{t_1} \left(H(t,p,x,u+\delta u) - H(t,p,x,u)\right)\,dt.$$

It therefore follows that a necessary condition for $u = u^*$ to be a minimizing control is that

$$\delta\mathcal{J}_a\,[u^*,\delta u] \geq 0$$

for all admissible δu. This in turn implies that

$$\boxed{H(t,p^*,x^*,u^*+\delta u) \geq H(t,p^*,x^*,u^*)} \tag{5.25}$$

for all admissible δu and all t in $[t_0,t_1]$. This states that u^* *minimizes* H, so we have "established"

5.3.1 Theorem. (Pontryagin's Minimum Principle) *Necessary conditions for u^* to minimize*

$$\mathcal{J}\,[u] = \varphi(x(t_1),t_1) + \int_{t_0}^{t_1} L(t,x,u)\,dt$$

are the following :

$$\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial x} \\ p(t_1) &= \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1} \\ H(t,p^*,x^*,u^*+\delta u) &\geq H(t,p^*,x^*,u^*) \ \textit{for all admissible}\ \delta u,\ t_0 \leq t \leq t_1. \end{aligned}$$

Note : (1) With a slighty different definition of H, the principle becomes one of maximizing J, and is then referred to as the **Pontryagin's Maximum Principle**.

(2) $u^*(\cdot)$ is now allowed to be *piecewise continuous.* (A rigorous proof is beyond the scope of this course.)

(3) Our derivation assumed that t_1 was fixed and $x(t_1)$ free; the boundary conditions for other situations are *precisely the same* as those given in the preceding section.

5.3.2 EXAMPLE. Consider again the *"soft landing" problem* (cf. EXAMPLE 5.1.2), where the performance index

$$\mathcal{J} = \int_0^T (|u| + k)\ dt$$

is to be *minimized* subject to

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u.$$

The *Hamiltonian* is

$$H = |u| + k + p_1 x_2 + p_2 u.$$

Since the admissible range of control is $-1 \le u(t) \le 1$, it follows that H *will be minimized* by the following :

$$u^*(t) = \begin{cases} -1 & \text{if } 1 < p_2^*(t) \\ \\ 0 & \text{if } -1 < p_2^*(t) < 1 \\ \\ +1 & \text{if } p_2^* < -1. \end{cases} \tag{5.26}$$

NOTE : (1) Such a control is referred to by the graphic term **bang-zero-bang**, since only maximum thrust is applied in a forward or reverse direction; no intermediate nonzero values are used. If there is no period in which u^* is zero, the control is called **bang-bang**. For example, a racing-car driver approximates to bang-bang operation, since he tends to use either full throttle or maximum braking when attempting to circuit a track as quickly as possible.

(2) In (5.26) $u^*(\cdot)$ *switches* in value according to the value of $p_2^*(\cdot)$, which is therefore termed (in this example) the **switching function**.

The *adjoint equations* are

$$\dot{p}_1^* = 0, \quad \dot{p}_2^* = -p_1^*$$

and integrating these gives

$$p_1^*(t) = c_1\,, \quad p_2^*(t) = -c_1 t + c_2$$

where c_1 and c_2 are constants. Since p_2^* is linear in t, it follows that it can take each of the values $+1$ and -1 at most once in $[0, T]$, so $u^*(\cdot)$ can switch at most twice. *We must however use physical considerations to determine an actual optimal control.*

Since the landing vehicle begins with a downwards velocity at altitude h, logical sequences of control would seem to either

$$u^* = 0\,, \quad \text{followed by} \;\; u^* = +1$$

(upwards is regarded as positive), or

$$u^* = -1\,, \text{then} \;\; u^* = 0\,, \text{ then} \;\; u^* = +1.$$

Consider the first possibility and suppose that u^* switches from 0 to $+1$ in time t_1. By virtue of (5.26) this sequence of control is possible if p_2^* decreases with time. It is easy to verify (exercise !) that the solution of

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

subject to the initial conditions

$$x_1(0) = h\,, \quad x_2(0) = -v$$

is

$$x_1^* = \begin{cases} h - vt & \text{if } \; 0 \le t \le t_1 \\ h - vt + \frac{1}{2}(t - t_1)^2 & \text{if } \; t_1 \le t \le T \end{cases} \tag{5.27}$$

$$x_2^* = \begin{cases} -v & \text{if } \; 0 \le t \le t_1 \\ -v + (t - t_1) & \text{if } \; t_1 \le t \le T. \end{cases} \tag{5.28}$$

Substituting the soft landing requirements

$$x_1(T) = 0, \quad x_2(T) = 0$$

into (27) and (28) gives

$$T = \frac{h}{v} + \frac{1}{2}v, \quad t_1 = \frac{h}{v} - \frac{1}{2}v \cdot$$

Because the final time is not specified and because of the form of H equation $H_{u=u^*} = 0$ holds, so in particular $H_{u=u^*} = 0$ at $t = 0$; that is,

$$k + p_1^*(0)x_2^*(0) = 0$$

or

$$p_1^*(0) = \frac{k}{v} \cdot$$

Hence we have

$$p_1^*(t) = \frac{k}{v}, \quad t \geq 0$$

and

$$p_2^*(t) = -\frac{kt}{v} - 1 + \frac{kt_1}{v}$$

using the assumption that $p_2^*(t_1) = -1$. Thus the assumed optimal control will be valid if $t_1 > 0$ and $p_2^*(0) < 1$ (the latter conditions being necessary since $u^* = 0$), and these conditions imply that

$$h > \frac{1}{2}v^2, \quad k < \frac{2v^2}{h - \frac{1}{2}v^2} \cdot \tag{5.29}$$

NOTE : If these inequalities do not hold, then some different control strategy (such as $u^* = -1$, then $u^* = 0$, then $u^* = +1$), becomes optimal. For example, if k is increased so that the second inequality in (5.29) is violated, then this means that more emphasis is placed on the time to landing in the performance index. It is therefore reasonable to expect this time would be reduced by first accelerating downwards with $u^* = -1$ before coasting with $u^* = 0$.

A general regulator problem

We can now discuss a general *linear regulator problem* in the usual form

$$\dot{x} = Ax + Bu \tag{5.30}$$

where $x(\cdot)$ is the *deviation* from the desired constant state. The aim is to transfer the system from some initial state to the origin in minimum time, subject to

$$|u_i(t)| \leq K_i, \qquad i = 1, 2, \ldots, \ell.$$

The Hamiltonian is

$$\begin{aligned} H &= 1 + p\,(Ax + Bu) \\ &= 1 + pAx + \begin{bmatrix} pb_1 & pb_2 & \ldots & pb_\ell \end{bmatrix} u \\ &= 1 + pAx + \sum_{i=1}^{\ell} (p\,b_i) u_i \end{aligned}$$

where the b_i are the columns of B. Application of (PMP) (cf. THEOREM 5.3.1) gives the necessary conditions for optimality that

$$u_i^*(t) = -K_i \operatorname{sgn}(s_i(t)), \qquad i = 1, 2, \ldots, \ell$$

where

$$s_i(t) := p^*(t) b_i \tag{5.31}$$

is the *switching function* for the ith variable. The *adjoint* equation is

$$\dot{p}^* = -\frac{\partial}{\partial x}(p^* A x)$$

or

$$\dot{p}^* = -p^* A.$$

The solution of this ODE can be written in the form

$$p^*(t) = p(0) \exp(-tA)$$

so the switching function becomes

$$s_i(t) = p(0) \exp(-tA) b_i.$$

If $s_i(t) \equiv 0$ in some time interval, then $u_i^*(t)$ is *indeterminate* in this interval. We now therefore investigate whether the expression in (5.31) can vanish.

Firstly, we can assume that $b_i \neq 0$. Next, since the final time is free, the condition $H_{u=u^*} = 0$ holds, which gives (for all t)

$$1 + p^* (Ax^* + Bu^*) = 0$$

so clearly $p^*(t)$ cannot be zero for any value of t. Finally, if the product $p^* b_i$ is zero, then $s_i = 0$ implies that

$$\dot{s}_i(t) = -p^*(t) A b_i = 0$$

and similarly for higher derivatives of s_i. This leads to

$$p^*(t) \begin{bmatrix} b_i & Ab_i & A^2 b_i & \dots & A^{m-1} b_i \end{bmatrix} = 0. \tag{5.32}$$

If the system (5.30) is c.c. by the ith input acting alone (i.e. $u_j \equiv 0,\ j \neq i$), then by THEOREM 3.1.3 the matrix in (5.32) is nonsingular, and equation (5.32) then has only the trivial solution $p^* = 0$. However, we have already ruled out this possibility, so s_i cannot be zero. Thus *provided the controllability condition holds, there is no time interval in which* u_i^* *is indeterminate.*

The *optimal control* for the ith variable then has the *bang-bang form*

$$u_i^* = \pm K_i.$$

5.4 Linear Regulators with Quadratic Costs

A general closed form solution of the optimal control problem is possible for a linear regulator with quadratic performance index. Specifically, consider the time-varying system

$$\dot{x} = A(t)x + B(t)u \tag{5.33}$$

with a criterion (obtained by combining together (5.3) and (5.7)) :

$$\mathcal{J} := \frac{1}{2} x^T(t_1) M x(t_1) + \frac{1}{2} \int_0^{t_1} \left(x^T Q(t) x + u^T R(t) u \right) dt \tag{5.34}$$

with $R(t)$ positive definite and M and $Q(t)$ positive semi-definite symmetric matrices for $t \geq 0$ (the factors $\frac{1}{2}$ enter only for convenience).

NOTE : The quadratic term in u in (5.34) ensures that the total amount of control effort is restricted, so that *the control variables can be assumed unbounded.*

The Hamiltonian is

$$H = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru + p(Ax + Bu)$$

and the necessary condition (5.17) for optimality gives

$$\frac{\partial}{\partial u}\left(\frac{1}{2}(u^*)^T Ru^* + p^* Bu^*\right) = (Ru^*)^T + p^* B = 0$$

so that

$$u^* = -R^{-1}B^T (p^*)^T \tag{5.35}$$

$R(t)$ being nonsingular (since it is positive definite). The *adjoint equation* is

$$(\dot{p}^*)^T = -Qx^* - A^T (p^*)^T . \tag{5.36}$$

Substituting (5.35) into (5.33) gives

$$\dot{x}^* = Ax^* - BR^{-1}B^T (p^*)^T$$

and combining this equation with (5.36) produces the system of $2m$ linear ODEs

$$\frac{d}{dt}\begin{bmatrix} x^* \\ (p^*)^T \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & -A^T(t) \end{bmatrix}\begin{bmatrix} x^* \\ (p^*)^T \end{bmatrix} . \tag{5.37}$$

Since $x(t_1)$ is not specified, the *boundary condition* is

$$(p^*)^T(t_1) = Mx^*(t_1) . \tag{5.38}$$

It is convenient to express the solution of (5.37) as follows :

$$\begin{aligned} \begin{bmatrix} x^* \\ (p^*)^T \end{bmatrix} &= \Phi(t, t_1)\begin{bmatrix} x^*(t_1) \\ (p^*)^T(t_1) \end{bmatrix} \\ &= \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix}\begin{bmatrix} x^*(t_1) \\ (p^*)^T(t_1) \end{bmatrix} \end{aligned}$$

where Φ is the transition matrix for (5.37). Hence

$$\begin{aligned} x^* &= \phi_1 x^*(t_1) + \phi_2 (p^*)^T(t_1) \\ &= (\phi_1 + \phi_2 M) x^*(t_1)\,. \end{aligned}$$

Also we get

$$\begin{aligned} (p^*)^T &= (\phi_3 + \phi_4 M) x^*(t_1) \\ &= (\phi_3 + \phi_4 M)(\phi_1 + \phi_2 M)^{-1} x^*(t) \\ &= P(t) x^*(t). \end{aligned}$$

(It can be shown that $\phi_1 + \phi_2 M$ is nonsingular for all $t \geq 0$). It now follows that the *optimal control is of linear feedback form*

$$u^*(t) = -R^{-1}(t) B^T(t) P(t) x^*(t). \tag{5.39}$$

To determine the matrix $P(t)$, differentiating $(p^*)^T = P x^*$ gives

$$\dot{P} x^* + P \dot{x}^* - (\dot{p}^*)^T = 0$$

and substituting for $\dot{x}^*$, $(\dot{p}^*)^T$ (from (5.37)) and $(p^*)^T$, produces

$$\left(\dot{P} + PA - PBR^{-1}B^T P + Q + A^T P \right) x^*(t) = 0.$$

Since this must hold throughout $0 \leq t \leq t_1$ it follows that $P(t)$ satisfies

$$\boxed{\dot{P} = PBR^{-1}B^T P - A^T P - PA - Q} \tag{5.40}$$

with boundary condition

$$P(t_1) = M.$$

Equation (5.40) is often referred to as a **matrix Riccati differential equation**.

NOTE : (1) Since the matrix M is symmetric, it follows that $P(t)$ is symetric for all t, so the (vector) ODE (5.40) represents $\frac{m(m+1)}{2}$ scalar first order (quadratic) ODEs, which can be integrated numerically.

(2) Even when the matrices $A, B, Q,$ and R are all time-invariant the solution $P(t)$ of (5.40), and hence the feedback matrix in (5.39), will in general still be time-varying.

However, of particular interest is the case when in addition the *final time* t_1 tends to infinity. Then there is no need to include the terminal expression in the performance index since the aim is to make $x(t_1) \to 0$ as $t_1 \to \infty$, so we set $M = 0$. Let Q_1 be a matrix having the same rank as Q and such that $Q = Q_1^T Q_1$. It can be shown that *the solution $P(t)$ of (5.40) does become a constant matrix P, and we have :*

5.4.1 PROPOSITION. *If the linear time-invariant control system*

$$\dot{x} = Ax + Bu(t)$$

is c.c. and the pair (A, Q_1) is c.o., then the control which minimizes

$$\int_0^\infty \left(x^T Q x + u^T R u\right) dt \tag{5.41}$$

is given by

$$u^*(t) = -R^{-1} B^T P x(t) \tag{5.42}$$

where P is the unique positive definite symmetric matrix which satisfies the so-called ***algebraic Riccati equation***

$$\boxed{PBR^{-1}B^T P - A^T P - PA - Q = 0.} \tag{5.43}$$

NOTE : Equation (5.43) represents $\frac{m(m+1)}{2}$ quadratic algebraic equations for the unknown elements (entries) of P, so the solution will not in general be unique. However, it can be shown that *if a positive definite solution of (5.43) exists, then there is only one such solution.*

Interpretation

The matrix Q_1 can be interpreted by defining an *output vector* $y = Q_1 x$ and replacing the quadratic term involving the state in (5.41) by

$$y^T y \left(= x^T Q_1^T Q_1 x\right).$$

The *closed loop system* obtained by substituting (5.42) into (5.33) is

$$\dot{x} = \mathbf{A}x \tag{5.44}$$

where $\mathbf{A} := A - BR^{-1}B^TP$. It is easy to verify that

$$\mathbf{A}^TP + P\mathbf{A} = A^TP + PA - 2PBR^{-1}B^TP = -PBR^{-1}B^TP - Q \tag{5.45}$$

using the fact that it is a solution of (5.43). Since R^{-1} is positive definite and Q is positive semi-definite, the matrix on the RHS in (5.45) is negative semi-definite, so PROPOSITION 4.3.10 is *not* directly applicable, unless Q is actually positive definite.

It can be shown that *if the triplet* (A, B, Q_1) *is neither c.c. nor c.o. but is stabilizable and detectable, then the algebraic Riccati equation (5.43) has a unique solution, and the closed loop system (5.44) is asymptotically stable.*

NOTE : Thus a solution of the algebraic Riccati equation leads to a *stabilizing linear feedback control* (5.42) irrespective of whether or not the open loop system is stable. (This provides an alternative to the methods of section **3.3**.)

If $x^*(\cdot)$ is the solution of the closed loop system (5.44), then (as in (5.10)) equation (5.45) implies

$$\begin{aligned} \frac{d}{dt}\left((x^*)^TPx^*\right) &= -(x^*)^T\left(PBR^{-1}B^TP + Q\right)x^* \\ &= -(u^*)^TRu^* - (x^*)^TQx^*. \end{aligned}$$

Since $\mathbf{A}$ is a stability matrix, we can integrate both sides of this equality with respect to t (from 0 to ∞) to obtain the *minimum value* of (5.41) :

$$\int_0^\infty \left((x^*)^TQx^* + (u^*)^TRu^*\right)\,dt = x_0^TPx_0. \tag{5.46}$$

NOTE : When $B \equiv 0$, (5.43) and (5.46) reduce simply to

$$A^TP + PA = -Q$$

and

$$\mathcal{J}_0 = \int_0^\infty x^TQx\,dt = x_0^TPx_0$$

respectively.

5.5 Exercises

Exercise 93 A system is described by

$$\dot{x} = -2x + u$$

and the control $u(\cdot)$ is to be chosen so as to *minimize* the performance index

$$\mathcal{J} = \int_0^1 u^2 \, dt.$$

Show that the optimal control which transfers the system from $x(0) = 1$ to $x(1) = 0$ is

$$u^*(t) = -\frac{4e^{2t}}{e^4 - 1} \cdot$$

Exercise 94 A system is described by

$$\dddot{z} = u(t)$$

where $z(\cdot)$ denotes *displacement*. Starting from some given initial position with given velocity and acceleration it is required to choose $u(\cdot)$ which is constrained by $|u(t)| \leq k$, so as to make displacement, velocity, and acceleration equal to zero in the least possible time. Show using (PMP) that the optimal control consists of

$$u^* = \pm k$$

with zero, one, or two switchings.

Exercise 95 A linear system is described by

$$\ddot{z} + a\dot{z} + bz = u$$

where $a > 0$ and $a^2 < 4b$. The control variable is subject to $|u(t)| \leq k$ and is to be chosen so that the system reaches the state $z(T) = 0$, $\dot{z}(T) = 0$ in *minimum possible time*. Show that the optimal control is

$$u^*(t) = k \operatorname{sgn} p(t)$$

where $p(\cdot)$ is a periodic function.

Exercise 96 A system is described by

$$\dot{x} = -2x + 2u, \qquad x \in \mathbb{R}.$$

The unconstrained control variable $u(\cdot)$ is to be chosen so as to *minimize* the performance index

$$\mathcal{J} = \int_0^1 \left(3x^2 + u^2\right)\, dt$$

whilst transferring the system from $x(0) = 0$ to $x(1) = 1$. Show that the optimal control is

$$u^*(t) = \frac{3e^{4t} + e^{-4t}}{e^4 - e^{-4}}\,.$$

Exercise 97 A system is described by the equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - 2x_2 + u$$

and is to be transferred to the origin from some given initial state.

(a) If the control $u(\cdot)$ is unbounded, and is to be chosen so that

$$\mathcal{J} = \int_0^T u^2\, dt$$

is *minimized*, where T is fixed, show that the optimal control has the form

$$u^*(t) = c_1 e^t \sinh(t\sqrt{2} + c_2)$$

where c_1 and c_2 are certain constants. (DO NOT try to determine their values.)

(b) If $u(\cdot)$ is such that $|u(t)| \leq k$, where k is a constant, and the system is to be brought to the origin in the *shortest possible time*, show that the optimal control is *bang-bang*, with at most one switch.

Exercise 98 For the system described by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + u$$

determine the control which transfers it from $x(0) = 0$ to the line $\mathcal{L}$ with equation

$$x_1 + 5x_2 = 15$$

and *minimizes* the performance index

$$\mathcal{J} = \frac{1}{2}\left(x_1(2) - 5\right)^2 + \frac{1}{2}\left(x_2(2) - 2\right)^2 + \frac{1}{2}\int_0^2 u^2\, dt.$$

Exercise 99 Use PROPOSITION 5.4.1 to find the feedback control which *minimizes*

$$\int_0^\infty \left(x_2^2 + \frac{1}{10} u^2 \right) \, dt$$

subject to

$$\dot{x}_1 = -x_1 + u, \quad \dot{x}_2 = x_1.$$

Exercise 100

(a) Use the *Riccati equation* formulation to determine the feedback control for the system

$$\dot{x} = -x + u, \qquad x \in \mathbb{R}$$

which *minimizes*

$$\mathcal{J} = \frac{1}{2} \int_0^1 \left(3x^2 + u^2 \right) \, dt.$$

[Hint : In the Riccati equation for the problem put $P(t) = -\dfrac{\dot{w}(t)}{w(t)}$.]

(b) If the system is to be transferred to the origin from an arbitrary initial state with the same performance index, use the calculus of variations to determine the optimal control.

Appendix A

Answers and Hints to Selected Exercises

Introduction

1. Prove first that (for $A, B \in \mathbb{R}^n$) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

2. Use the (fundamental) property : $\det(AB) = \det(A) \cdot \det(B)$.

3. The following facts are needed :

(1) $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

(2) $\operatorname{rank}(A) + \dim\ker(A) = n$.

(3) $\operatorname{rank}(AB) = \operatorname{rank}(B) - \dim\ker(A) \cap \operatorname{im}(B)$.

(4) $U \subseteq V \Rightarrow \dim(U) \leq \dim(V)$ (as vector spaces).

We have (3) $\Rightarrow$ $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$; (1), (3) $\Rightarrow$ $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$; (4), (2) $\Rightarrow$ $\dim\ker(A) \cap \operatorname{im}(B) \leq n - \operatorname{rank}(A)$.

4. Apply the matrix (linear transformation) $(A - \lambda_2 I_n)(A - \lambda_3 I_n) \cdots (A - \lambda_r I_n)$ to the equation (trivial linear combination) $\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_r w_r = 0$ and hence obtain $\alpha_1 = 0$, etc.

5. Express the characteristic polynomial of A in two different ways : $\operatorname{char}_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = \lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + \cdots + (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$ but also $\operatorname{char}_A(\lambda) = \det(\lambda I_n - A) = \lambda^n - \operatorname{tr}(A)\lambda^{n-1} + \cdots + (-1)^n \det(A)$.

6. Observe that

$$\sum_{k \geq 0} \left\| \frac{t^k}{k!} A^k \right\| \leq \sum_{k \geq 0} \frac{1}{k!} \|tA\|^k$$

and then use the *comparison test* (for numerical series).

7. Induction : $\left(S^{-1}AS\right)^n = S^{-1}A^nS$.

8. $uu^T = u_1u_2\cdots u_n \begin{bmatrix} u_1 & \dots & u_n \\ \vdots & & \vdots \\ u_1 & \dots & u_n \end{bmatrix}.$

9. Apply the matrix (linear transformation) A^r to the equation $c_0b+c_1Ab+\cdots+c_rA^rb=0$ and deduce that $c_0=0$, etc.

10. Straightforward computation : $\operatorname{rank}(A^TA)=\operatorname{rank}(A)=\operatorname{rank}(AA^T)=2$.

11. (a) $\lambda^2-4\lambda-5;\quad \lambda_1=-1;\ \lambda_2=5.$

$$E_1=E_{\lambda_1}=\operatorname{span}\left\{\begin{bmatrix}2\\-1\end{bmatrix}\right\}\quad\text{and}\quad E_2=E_{\lambda_2}=\operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

(b) $(\lambda+1)(\lambda-2);\quad \lambda_1=-1;\ \lambda_2=2.$

$$E_1=\operatorname{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}\quad\text{and}\quad E_2=\operatorname{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}.$$

(c) $\lambda^2-\lambda;\quad \lambda_1=0;\ \lambda_2=1.$

$$E_1=\operatorname{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}\quad\text{and}\quad E_2=\operatorname{span}\left\{\begin{bmatrix}2\\-1\end{bmatrix}\right\}.$$

(d) $\lambda^2-4\lambda+8;\quad \lambda_{1,2}=2\pm 2i.$

$$E_1=\operatorname{span}\left\{\begin{bmatrix}1\\i\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}+i\begin{bmatrix}0\\1\end{bmatrix}\right\}\quad\text{and}\quad E_2=\operatorname{span}\left\{\begin{bmatrix}1\\-i\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}-i\begin{bmatrix}0\\1\end{bmatrix}\right\}.$$

(e) $\lambda^2-(a+b)\lambda+ab;\quad \lambda_1=a;\ \lambda_2=b.$

$$E_1=\operatorname{span}\left\{\begin{bmatrix}1\\-a\end{bmatrix}\right\}\quad\text{and}\quad E_2=\operatorname{span}\left\{\begin{bmatrix}1\\-b\end{bmatrix}\right\}.$$

(f) $\lambda(\lambda^2-5\lambda+4);\quad \lambda_1=0;\ \lambda_2=1;\ \lambda_3=4.$

$$E_1=\operatorname{span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\},\quad E_2=\operatorname{span}\left\{\begin{bmatrix}2\\-1\\-1\end{bmatrix}\right\}\quad\text{and}\quad E_3=\operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$$

(g) $(\lambda-1)^3;\quad \lambda_1=\lambda_2=\lambda_3=1.$

$$E=E_\lambda=\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}.$$

(h) $\lambda^2(\lambda-3);\quad \lambda_1=\lambda_2=0;\ \lambda_3=3.$

$$E_{1,2}=\operatorname{span}\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}\quad\text{and}\quad E_3=\operatorname{span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}.$$

(i) $(\lambda-6)(\lambda^2-6\lambda-16)$; $\lambda_1=-2$; $\lambda_2=6$; $\lambda_3=8$.

$$E_1=\operatorname{span}\left\{\begin{bmatrix}3\\-1\\0\end{bmatrix}\right\},\quad E_2=\operatorname{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\quad\text{and}\quad E_3=\operatorname{span}\left\{\begin{bmatrix}1\\3\\0\end{bmatrix}\right\}.$$

(j) $\lambda^2-(2\cos\theta)\lambda+1$; $\lambda_{1,2}=\cos\theta\pm i\sin\theta$.

$$E_1=\operatorname{span}\left\{\begin{bmatrix}1\\-i\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}-i\begin{bmatrix}0\\1\end{bmatrix}\right\}\quad\text{and}\quad E_2=\operatorname{span}\left\{\begin{bmatrix}1\\i\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}+i\begin{bmatrix}0\\1\end{bmatrix}\right\}.$$

12. λ is an eigenvalue of A if and only if $\lambda-\mu$ is an eigenvalue of $A-\mu I_n$, etc.

13. (a) Observe that $(\lambda I_n-A)^T=\lambda I_n-A^T$. (b) From $\lambda I_n-S^{-1}AS=S^{-1}(\lambda I_n-A)S$ derive that $\det(\lambda I_n-S^{-1}AS)=\det(\lambda I_n-A)$.

14. (b) Yes. (b) Yes. (c) Yes. (d) Yes.

15. (a) Show that $q(x)=-q(x)$ for all $x\in\mathbb{R}^{n\times 1}$. (b) $A_1=\frac{1}{2}\left(A+A^T\right)$, etc.

17. (a) $\exp(tA)=\begin{bmatrix}1&t\\0&1\end{bmatrix}$; $\exp(tB)=\begin{bmatrix}e^t&0\\0&1\end{bmatrix}$; $\exp(t(A+B))=\begin{bmatrix}e^t&e^t-1\\0&1\end{bmatrix}$.

18. (a) $\exp(tA)=\begin{bmatrix}e^{at}&0\\0&e^{bt}\end{bmatrix}$. (b) $\exp(tA)=\begin{bmatrix}e^{at}&bte^{at}\\0&e^{at}\end{bmatrix}$. (c) $\begin{bmatrix}1&t&\frac{t^2}{2}\\0&1&t\\0&0&1\end{bmatrix}$ and $\begin{bmatrix}1&0&0\\2t&1&0\\3t+4t^2&4t&1\end{bmatrix}$.

19. (a) $\begin{bmatrix}e^t&0\\0&e^{2t}\end{bmatrix}$. (b) $\begin{bmatrix}1&t\\0&1\end{bmatrix}$. (c) $\begin{bmatrix}\cos t&\sin t\\-\sin t&\cos t\end{bmatrix}$. (d) $\begin{bmatrix}e^{-t}&te^{-t}\\0&e^{-t}\end{bmatrix}$.
(e) $\begin{bmatrix}\frac{1}{2}(1+e^{2t})&\frac{1}{2}(e^{2t}-1)\\\frac{1}{2}(e^{2t}-1)&\frac{1}{2}(1+e^{2t})\end{bmatrix}$. (f) $\begin{bmatrix}1&t&t+\frac{t^2}{2}\\0&1&t\\0&0&1\end{bmatrix}$. (g) $\begin{bmatrix}e^{2t}&0&0\\0&e^{-3t}&0\\0&0&e^{7t}\end{bmatrix}$.

20. FFTTTTFT.

Linear Dynamical Systems

22. Let $a_k:=\frac{\alpha^k}{(k+1)!}\cdot$ Show that $0<a_k\le a_{k+1}$ (for $k\ge\alpha-2$) and then use the fact that *every bounded non-increasing sequence of numbers is convergent.*

23. Let $Y(\cdot)$ be the unique solution of the Cauchy problem (in matrices) :

$$\dot{Y}=-YA(t),\quad Y(0)=I_n.$$

Then $Y(t)\cdot X(t)=I_n$ (for all t), etc.

24. Straightforward computation.

25. There is no (linear) transformation $w = Tx$ such that $TAT^{-1} = C$ and $Tb = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

26.

$$(a) \quad \begin{aligned} x_1(t) &= (-x_1(0) + x_2(0))e^{-3t} + (2x_1(0) - x_2(0))e^{-2t} \\ x_2(t) &= 2(-x_1(0) + x_2(0))e^{-3t} + (2x_1(0) - x_2(0))e^{-2t}. \end{aligned}$$

$$(b) \quad \begin{aligned} x_1(t) &= \frac{1}{2}(x_1(0) + x_2(0))e^{t} + \frac{1}{2}(x_1(0) - x_2(0))e^{3t} \\ x_2(t) &= \frac{1}{2}(x_1(0) + x_2(0))e^{t} - \frac{1}{2}(x_1(0) - x_2(0))e^{-3t}. \end{aligned}$$

$$(c) \quad \begin{aligned} x_1(t) &= \frac{1}{2}(x_1(0) - x_2(0))e^{-t} + \frac{1}{2}(x_1(0) + x_2(0))e^{t} \\ x_2(t) &= -\frac{1}{2}(x_1(0) - x_2(0))e^{-t} + \frac{1}{2}(x_1(0) + x_2(0))e^{3t}. \end{aligned}$$

$$(d) \quad \begin{aligned} x_1(t) &= (-x_2(0) + \frac{1}{2}x_3(0))e^{t} + 2(x_1(0) + x_2(0))e^{2t} - (x_1(0) + x_2(0) + \frac{1}{2}x_3(0))e^{3t} \\ x_2(t) &= -(x_2(0) + \frac{1}{2}x_3(0))e^{t} - (x_1(0) + x_2(0))e^{2t} + (x_1(0) + x_2(0) + \frac{1}{2}x_3(0))e^{3t} \\ x_3(t) &= -2(x_1(0) + x_2(0))e^{2t} + 2(x_1(0) + x_2(0) + \frac{1}{2}x_3(0))e^{3t}. \end{aligned}$$

$$(e) \quad \begin{aligned} x_1(t) &= -(x_2(0) + x_3(0))e^{t} + (x_1(0) + x_2(0) + x_3(0))e^{2t} \\ x_2(t) &= \frac{1}{2}(x_2(0) - x_3(0))e^{-t} + \frac{1}{2}(x_2(0) + x_3(0))e^{t} \\ x_3(t) &= -\frac{1}{2}(x_2(0) - x_3(0))e^{-t} + \frac{1}{2}(x_2(0) + x_3(0))e^{t}. \end{aligned}$$

27. (a) $Z_1 = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$, $Z_2 = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$, etc. (b) $Z_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $Z_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, etc. (c) $Z_1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $Z_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, etc. (d) ... (e) ...

28. $A = \begin{bmatrix} 0 & 1 \\ -\omega & 0 \end{bmatrix}$; $\Phi(t,0) = \exp(tA) = \begin{bmatrix} \cos(\sqrt{\omega}\,t) & \sqrt{\omega}\sin(\sqrt{\omega}\,t) \\ -\sqrt{\omega}\sin(\sqrt{\omega}\,t) & \cos(\sqrt{\omega}\,t) \end{bmatrix}$.

29. $\mathrm{char}(\lambda) = \lambda(\lambda - a_1 + a_4) \Rightarrow \lambda_1 = 0$ and $\lambda_2 = a_1 - a_4$. We assume that $a_1 \neq a_4$.

$$\begin{aligned} x(t) &= \left(Z_1 + e^{(a_1-a_4)t} Z_2\right) \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{a_1-a_4}\left(-a_4 x_1(0) + a_2 x_2(0) + (a_1 x_1(0) - a_2 x_2(0)) e^{(a_1-a_4)t}\right) \\ \frac{1}{a_1-a_4}\left(-a_3 x_1(0) + a_1 x_2(0) + (a_3 x_1(0) - a_4 x_2(0)) e^{(a_1-a_4)t}\right) \end{bmatrix}. \end{aligned}$$

30.

$$\begin{aligned} \Phi(t,0) &= e^{-2t} Z_1 + e^{5t} Z_2 \\ &= \begin{bmatrix} \frac{1}{7}(4e^{-2t} + 3e^{5t}) & \frac{1}{7}(-4e^{-2t} + 4e^{5t}) \\ \frac{1}{7}(-3e^{-2t} + 3e^{5t}) & \frac{1}{7}(3e^{-2t} + 4e^{5t}) \end{bmatrix}. \\ x(t) &= \Phi(t,0) \left[x_0 + \int_0^t \Phi(0,\tau) \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(\tau)\, d\tau \right]. \end{aligned}$$

31.

$$\begin{aligned} \Phi(t,0) &= e^{-2t} Z_1 + e^{-t} Z_2 \\ &= \begin{bmatrix} -4e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t}) \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t}) \end{bmatrix}. \\ z(2) &= \frac{1 - e^2}{4e^4} + \frac{e-1}{e^2}. \end{aligned}$$

32. Existence : simple verification. Uniqueness : let W be a solution; consider the product $\exp(-tA)\, W \exp(-tB)$ and differentiate, etc.

33. $\exp(tA) = \begin{bmatrix} e^t & 2te^t \\ 0 & e^t \end{bmatrix}$.

34. Take $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. Then consider $\det(X(t)) = x_1 x_4 - x_2 x_3$ and differentiate, etc.

36. $\frac{d}{dt} \exp(B(t)) = \frac{d}{dt}\left(I_n + B(t) + \frac{B^2(t)}{2!} + \cdots\right)$. The solution is *unique*.

38.

$$\begin{aligned} \Phi(t,0) &= \begin{bmatrix} \frac{1}{2}(3^{-3t} + e^t) & e^{-3t} - e^t \\ \frac{1}{4}(e^{-3t} - e^t) & \frac{1}{2}(e^{-3t} + e^t) \end{bmatrix} \\ x(t) &= \Phi(t,0) \begin{bmatrix} \frac{e^{5t}}{10} - \frac{e^t}{2} + \frac{7}{5} \\ \frac{e^{5t}}{20} + \frac{e^t}{4} + \frac{7}{10} \end{bmatrix}, \text{ etc.} \end{aligned}$$

39. (a) $T = \frac{1}{2}\begin{bmatrix} -3 & 1 \\ 5 & -1 \end{bmatrix}$; $TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$. (b) $T = \begin{bmatrix} \frac{2}{3} & -\frac{5}{3} & \frac{2}{3} \\ -1 & -2 & 1 \\ -1 & -2 & 2 \end{bmatrix}$; $TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$.

40. $P = \begin{bmatrix} \frac{1}{12} & \frac{1}{12} & \frac{7}{12} \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$; $E = P^{-1}AP = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 9 \\ 0 & 1 & 0 \end{bmatrix}$.

Linear Control Systems

41.
$$\int_0^\infty e^{-st} \cdot e^{at}\, dt = \lim_{R\to\infty} \frac{1 - e^{-(s-a)R}}{s-a} = \frac{1}{s-a} \quad (\text{for } s > a).$$

The matrix(-valued mapping) $\Phi(t,0) = \exp(tA)$ is completely determined by the conditions : (1) $\dot{\Phi}(t,0) = A\,\Phi(t,0)$ and (2) $\Phi(0,0) = I_n$.

42. Show first that $\mathcal{C}(\widetilde{A}, \widetilde{B}) = P\,\mathcal{C}(A,B)$ and $\mathcal{O}(\widetilde{A}, \widetilde{C}) = \mathcal{O}(A,C)\,P^{-1}$.

43. $\widetilde{C}\left(sI_n - \widetilde{A}\right)^{-1} \widetilde{B} = C\,(sI_n - A)^{-1}\,B$.

44. $\operatorname{rank}(AB) \le n$ and $\operatorname{rank}(A) + \operatorname{rank}(B) - n \le \operatorname{rank}(AB)$.

45. $\operatorname{rank}\begin{bmatrix} B & AB \end{bmatrix} = 2$.

46. $b \in \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\} \bigcup \operatorname{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

47.
$$\begin{aligned} \exp(tA) &= \begin{bmatrix} \frac{1}{3}(e^{-8t} + 2e^{-2t}) & \frac{1}{3}(-e^{-8t} + e^{-2t}) \\ \frac{2}{3}(-e^{-8t} + e^{-2t}) & \frac{1}{3}(2e^{-8t} + e^{2t}) \end{bmatrix}; \\ (1) \quad -2 &= \frac{1}{3}\int_0^1 (-e^{8\tau} + 4e^{2\tau})(C_1 + C_2 e^{-2\tau})\, d\tau \\ (2) \quad -3 &= \frac{1}{3}\int_0^1 (2e^{8\tau} + 4e^{2\tau})(C_1 + C_2 e^{-2\tau})\, d\tau. \end{aligned}$$

48. (1) $\alpha = 10$ or $\alpha = 12$. (2) $\alpha = 0$ or $\alpha = 1$. For $u_1 \equiv 0$: $\dot{x} = \begin{bmatrix} 2 & \alpha - 3 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ \alpha^2 - \alpha \end{bmatrix} u_2$. $\alpha = 0$ or $\alpha = 1$.

49. $u^*(t) = -\begin{bmatrix} e^t & 3e^{2t} \end{bmatrix} U^{-1}(0,1) \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, where $U(0,1) = \begin{bmatrix} \frac{e^2-1}{2} & e^3 - 1 \\ e^3 - 1 & \frac{9}{4}(e^4 - 1) \end{bmatrix}$, etc.

50. Verification.

51. $W(t) = \exp((t-t_1)A) \cdot C \cdot \exp((t-t_1)A^T)$, etc.

52. The system is completely observable.

53. $x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

54. $\operatorname{rank} \mathcal{O} = 1 < 2$.

$$\begin{aligned} x(t) &= \exp(tA)x(0) \\ &= \begin{bmatrix} -e^{-2t} + 2e^{-t} & e^{-t} \\ 2(e^{-2t} - e^{-t}) & 2e^{-2t} - e^{-t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_(0) \end{bmatrix}. \end{aligned}$$

$x_1(0) + 2x_2(0) = 0$.

55. Use the fact that

$$\dot{\Phi}(\tau, t) = \frac{d}{dt}\Phi(\tau, t) = -\Phi(\tau, t)\, A(t).$$

56.

$$\begin{aligned} K &= \underline{k}\, T = \frac{1}{2}\begin{bmatrix} -14 & -4 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 11 & -5 \end{bmatrix}. \end{aligned}$$

57.

$$\begin{aligned} K &= \underline{k}\, T = \begin{bmatrix} -11 & 4 & -9 \end{bmatrix} T \\ &= -\frac{1}{3}\begin{bmatrix} -61 & 129 & 88 \end{bmatrix}. \end{aligned}$$

58. $\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 3.\ W = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 4 & 1 & 9 \end{bmatrix}.\ \mu_1 = 1,\ \lambda_2 = 1,\ \lambda_3 = 3 \quad (p = 1).$

$$\begin{aligned} K &= fg\,\widetilde{W} \\ &= \begin{bmatrix} -1 \\ -1 \end{bmatrix} 1 \begin{bmatrix} -3 & 4 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -4 & 1 \\ 3 & -4 & 1 \end{bmatrix}. \end{aligned}$$

59. $A + bK = \begin{bmatrix} -1+\alpha & 0 & 3+\beta \\ \alpha & -3 & \beta \\ 1-\alpha & 0 & -(\beta+3) \end{bmatrix}.\ \alpha = \beta.$

60. Take $X^{-1} = \begin{bmatrix} A^{-1} & X_1 \\ X_2 & C^{-1} \end{bmatrix}$, etc.

61. $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $B = \begin{bmatrix} \frac{10}{3} & 2 \\ -\frac{7}{2} & -2 \\ \frac{7}{6} & 1 \end{bmatrix}$. $C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$.

62. $\operatorname{rank}(K_1) = 1$ and $\operatorname{rank}(K_2) = 2$. The *order* of the minimal realization is $1 + 2 = 3$.

63.
$$C\,(sI_n - A)^{-1}\,B = C_1\,(sI_{n_1} - A_1)^{-1}\,B_1 C_2\,(sI_{n_2} - A_2)^{-1}\,B_2.$$

65. $b_1 = 2$, $b_2 = -1$; $c_1 = 4$, $c_2 = 1$. $P = \frac{1}{2}\begin{bmatrix} 1 & 2 \\ -2 & -6 \end{bmatrix}$.

Stability

66. Sketch the graph of f.

67.
$$\|\exp(tA)\| \le \left\|e^{\lambda_1 t}Z_1 + \cdots + e^{\lambda_n t}Z_n\right\| \le \cdots \le e^{-at}\left(\|Z_1\| + \cdots + \|Z_n\|\right)$$
where $a = \min\{-\mathrm{Re}(\lambda_1), \ldots, -\mathrm{Re}(\lambda_n)\}$.

68. The new state equations are :
$$\begin{aligned} \dot{x}_1 &= -2x_1x_2 - 4x_2 \\ \dot{x}_2 &= \frac{1}{2}x_1 + x_1x_2. \end{aligned}$$

69. (a) Yes. (b) No. (c) No.

70. $k < 2$.

71. $0 < k < 2$. $u = -\frac{2}{3}x_1 - x_2 - \frac{2}{3}x_3$

72. $\dot{x} = A(t)x \Rightarrow \ddot{x}_2 - 4\dot{x}_2 + 3x_2 = 0$. The linear system $\dot{y} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$ has solution
$$\begin{aligned} y(t) &= (e^tZ_1 + e^{3t}Z_2)y(0) \\ &= \begin{bmatrix} \frac{1}{2}(3e^t - e^{3t}) & \frac{1}{2}(e^t + e^{3t}) \\ \frac{3}{2}(e^t + -e^{3t}) & \frac{1}{2}(-e^t + 3e^{3t}) \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}. \end{aligned}$$
$x_2(t) = y_1(t) = ae^t + be^{3t}$ and $|x_2(t)| \to \infty$ (as $t \to \infty$).

73. $q = \frac{1}{2}$, $b = \frac{1}{2}$, $c = 1$ and $d = 1$. The origon is *asymptotically stable.*

74. $\dot{V} = -2(x_1 - x_2)^2 - 2(x_1 + 2x_2)^2(1 - x_2^2) < 0$ for $|x_2| < 1$. $5x_1^2 + 2x_1x_2 + 2x_2^2 \le \frac{9}{5}$.

75. The origin is *unstable.*

76. $P = \frac{1}{30}\begin{bmatrix} 13 & -1 \\ -1 & 4 \end{bmatrix}$. The origin is *asymptotically stable.*

77. $-C + \lim_{t\to\infty} W(t) = A\int_0^\infty W(\tau)\,d\tau + \left(\int_0^\infty W(\tau)\,d\tau\right) B$, etc.

78. The state equations are $\dot{x}_1 = x_2$ and $\dot{x}_2 = -a_2x_1 - a_1x_2$.

$$\begin{aligned} V &= x^TPx = ax_1^2 + 2bx_1x_2 + cx_2^2 \Rightarrow \\ \dot{V} &= -2ba_2x_1^2 + 2bx_2^2 - 2a_1cx_2^2 + 2(a - ba_1 - ca_2)x_1x_2, \text{ etc.} \end{aligned}$$

79. The origin is *asymptotically stable.* $k \in \left(\frac{1}{4}, 4\right)$.

80. The linearized system is $\dot{x} = \begin{bmatrix} 7 & 2 \\ 1 & -3 \end{bmatrix} x$. The origin is *unstable.*

81. (a) The linearized system is $\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} x$. The origin is *asymptotically stable.* (b) The origin is *asympttotically stable.*

83. For $u(t) \equiv 0,\ t > 0$, the solution is

$$\begin{aligned} x(t) &= \Phi(t,0)x_0 \\ &= \frac{3x_0}{t+3} \to 0 \quad (\text{as } t \to \infty). \end{aligned}$$

When $u(\cdot)$ is the unit step function, we have (for $t \geq 1$)

$$\begin{aligned} x(t) &= \Phi(t,0)\left[x_0 + \int_0^t \Phi(0,\tau)\,d\tau\right] \\ &= \frac{1}{t+3}\left(t^2 + 6t + 3x_0\right) \to \infty \quad (\text{as } t \to \infty). \end{aligned}$$

84. $(-2, -2)$. The transformed state equations are

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= 3x_1 - x_2 - x_1^2. \end{aligned}$$

The linearized system has roots λ_1, λ_2 such that $\lambda_1 < 0 < \lambda_2$.

85. $1 < k < 5$.

86. (a) Neutrally stable. (b) Unstable.

87. $2x_1^2 + x_2^2 < 6$.

88. $x_1^2 + 3x_2^2 < \frac{3}{2}\cdot$

89. (b) The origin is asymptotically stable.

90. (a) The system is stabilizable but not detectable. (b) The system is detectable but not stabilizable.

Optimal Control

91. Induction.

93. $u^* = -\frac{1}{2}p^*$ and $p^* = C_1 e^{2t}$, etc.

94. $p_3^* = -C_1 \frac{t^2}{2} + C_2 t + C_3$ (a parabola), etc.

97. $u^* = -\frac{1}{2}p_2^*$. $p_2^*(t) = c_1 e^t \sinh(\sqrt{2}\,t + c_2)$. (Observe that $\sinh(\alpha+\beta) = \sinh\alpha \cdot \cosh\beta + \sinh\beta \cdot \cosh\alpha$; hence $a \sinh\alpha + b \cosh\alpha = A \sinh(\alpha + C)$).

98. $u^* = \frac{13}{5} + \frac{e^t}{25}$.

99. $u^*(t) = -R^{-1}B^T P x(t) = -10(ax_1 + bx_2)$, where $a \approx 0.171$ and $b \approx 0.316$.

100. (a) $u^* = \dfrac{3(e^{4t} - e^4)}{e^{4t} + 3e^4} x_1$. (b) $u^* = \dfrac{3e^{4t} + e^4}{e^{4t} - e^4} x_1$.

Appendix B

Revision Problems

1. Find the solution of the following uncontrolled linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Class test, August 1998

2. Given the linear system described by

$$\dot{x} = \begin{bmatrix} 2 & \alpha - 1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u$$

determine for what values of the real parameter α the system is *not* completely controllable.

Class test, August 1998

3. Show that the linear system described by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -2x_1 - 3x_2 + u, \quad y = x_1 + x_2$$

is *not* completely observable. Determine initial states $x(0)$ such that if $u(t) = 0$ for $t \geq 0$, then the output $y(t)$ is identically zero for $t \geq 0$.

Class test, October 1998

4. Consider a time invariant linear system of the form

$$\dot{x} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

and let $\theta_1, \theta_2 \in \mathbb{R}$. Prove that if the system is completely controllable, then there exists a matrix K such that the eigenvalues of the matrix $A + Kb$ are θ_1 and θ_2.

Application :

$$A = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \theta_1 = -4,\ \theta_2 = -5.$$

Class test, October 1998

5. A linear system is known to be described by

$$\dot{x} = Ax.$$

It is possible to measure the state vector, but because of difficulties in setting up the equipment, this measurement can be started only after an unknown amount T $(T > 2)$ of time has elapsed. It is then found that

$$x(T) = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}, \quad x(T+1) = \begin{bmatrix} 1.5 \\ 1.6 \end{bmatrix}, \quad x(T+2) = \begin{bmatrix} 1.8 \\ 2.1 \end{bmatrix}.$$

Compute $x(T-2)$.

Exam, November 1998

6. Write the equation of motion

$$\ddot{z} = u(t)$$

in state space form, and then solve it.

Exam, November 1998

7. Given the liner control system described by

$$\begin{aligned} \dot{x}_1 &= \alpha\, x_1 + 2x_2 + u \\ \dot{x}_2 &= x_1 - x_2 \\ y &= x_1 + \alpha\, x_2 \end{aligned}$$

determine for what values of the real parameter α the system is (a) completely controllable and completely observable; (b) completely controllable but not completely observable; (c) completely observable but not completely controllable.

Exam, November 1998

8. Given the system described by

$$\dot{x} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -3 & 0 \\ 1 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} u$$

show that under linear feedback of the form $u = \alpha\, x_1 + \beta\, x_3$, the closed loop system has two fixed eigenvalues, one of which is equal to -3. Determine the second fixed eigenvalue, and also values of α and β such that the third closed loop eigenvalue is equal to -4.

Exam, November 1998

9. A control system is described by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

and the control $u(t)$ is to be chosen as to minimize the performance index

$$\mathcal{J} = \int_0^1 u^2\, dt.$$

Find the optimal control which transfers the system from $x_1(0) = 0$, $x_2(0) = 0$ to $x_1(1) = 1$, $x_2(1) = 0$.

Exam, November 1998

10. Find the characteristic polynomial, the eigenvalues and the corresponding eigenvectors for the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Class test, March 1999

11. Let A be an $n \times n$ matrix. How are the eigenvalues of A^3 related to those of A ? If A is invertible, can 0 be an eigenvalues for A^3 ? Explain.

Class test, March 1999

12. Let A and S be $n \times n$ matrices, and assume that S is invertible. Show that the characteristic polynomials of A and $S^{-1}AS$ are the same.

Class test, March 1999

13. Determine the state transition matrix and write down the general solution of the linear control system described by the equations

$$\begin{aligned} \dot{x}_1 &= x_1 + 4x_2 + u \\ \dot{x}_2 &= 3x_1 + 2x_2. \end{aligned}$$

Class test, March 1999

14. Write the equation of motion

$$\ddot{z} + \dot{z} = 0$$

in state space form, and then solve it (by computing the state transition matrix).

Exam, June 1999

15. Split up the linear control system

$$\dot{x} = \begin{bmatrix} 4 & 3 & 5 \\ 1 & -2 & -3 \\ 2 & 1 & 8 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} u(t)$$

into its controllable and uncontrollable parts, as displayed below :

$$\begin{bmatrix} \dot{x}^{(1)} \\ \dot{x}^{(2)} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t).$$

Exam, June 1999

16. Investigate the stability nature of the equilibrium state at the origin of the system described by the scalar equation

$$\ddot{z} + \alpha\, \dot{z} + \beta\, z = z \cdot \dot{z}.$$

Exam, June 1999

17. Determine whether the system described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 4 & 1 & 2 \\ 3 & -1 & 2 \\ 5 & -3 & 8 \end{bmatrix} x + \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} u(t) \\ y &= \begin{bmatrix} 2 & 1 & -1 \end{bmatrix} x \end{aligned}$$

is detectable.

Exam, June 1999

18. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

(a) Find the characteristic polynomial, the eigenvalues and the corresponding eigenvectors for the matrix A.

(b) Determine the matrix exponential $\exp(tA)$.

Class test, March 2000

19. Let $A, B,$ and S be $n \times n$ matrices, and assume that S is nonsingular.

(a) Show that the characteristic polynomials of A and $S^{-1}AS$ are the same.

(b) If $AB = BA$, show that

$$\exp(t(A+B)) = \exp(tA) \cdot \exp(tB).$$

Class test, March 2000

20. Write down the general solution of the linear system described by the equations

$$\begin{aligned} \dot{x}_1 &= x_1 + 4x_2 \\ \dot{x}_2 &= 3x_1 + 2x_2. \end{aligned}$$

Class test, March 2000

21. Consider a (time-invariant) linear control system of the form

$$\dot{x} = Ax + bu(t)$$

where A is a 2×2 matrix and b is a non-zero 2×1 matrix. Assume that

$$\operatorname{rank} \begin{bmatrix} b & Ab \end{bmatrix} = 2$$

and prove that the given system can be transformed by a (nonsingular) transformation $w(t) = Tx(t)$ into the canonical form

$$\dot{w} = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

<u>Application</u> :

$$A = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Class test, May 2000

22. Find the solution of

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

when

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Exam, June 2000

23. Consider the matrix differential equation

$$\dot{X} = A(t)X, \qquad X(0) = I_n.$$

Show that, when $n = 2$,

$$\frac{d}{dt}(\det X) = \operatorname{tr} A(t) \cdot \det X$$

and hence deduce that $X(t)$ is nonsingular, $t \geq 0$.

Exam, June 2000

24. Write the linear system Σ described by

$$\ddot{z} + \dot{z} + 2z = 0$$

in the state space form, and then use Lyapunov functions to investigate the stability nature of the system. Is the origin asymptotically stable ? Justify your answer.

Exam, June 2000

25. Determine whether the system described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned}$$

is detectable.

Exam, June 2000

26. Consider the matrix
$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

(a) Find the characteristic polynomial, the eigenvalues and the corresponding eigenvectors for the matrix A.

(b) Determine the matrix exponential $\exp(tA)$.

Class test, August 2001

27. Consider the linear control system described by the equations
$$\begin{aligned} \dot{x}_1 &= x_1 - 4x_2 + u(t) \\ \dot{x}_2 &= -x_1 - x_2 + u(t). \end{aligned}$$

If $x(0) = (1, 0)$ and $u(t) = 1$ for $t \geq 0$, find the expressions for $x_1(t)$ and $x_2(t)$.

Class test, August 2001

28. Investigate the stability nature of the origin for the linear system
$$\dot{x}_1 = -2x_1 - 3x_2, \quad \dot{x}_2 = 2x_1 - 2x_2.$$

Class test, November 2001

29. Consider the linear system
$$\dot{x}_1 = k\,x_1 - 3x_2, \quad \dot{x}_2 = -k\,x_1 - 2x_2.$$

Use the quadratic form (Lyapunov function)
$$V = \frac{2}{3}x_1^2 + x_2^2 - \frac{2}{3}x_1x_2$$
to obtain *sufficient* conditions on $k \in \mathbb{R}$ for the system to be asympotically stable (at the origin).

Class test, November 2001

30. Investigate the stability nature of the origin for the nonlinear system

$$\begin{aligned}\dot{x}_1 &= 7x_1 + 2\sin x_2 - x_2^4 \\ \dot{x}_2 &= e^{x_1} - 3x_2 - 1 + 5x_1^2.\end{aligned}$$

Class test, November 2001

31. Find the solution of

$$\dot{x} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t)$$

when

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad u(t) = e^{3t},\ t \geq 0.$$

Exam, November 2001

32. Find a minimal realization of

$$G(s) = \frac{s+4}{s^2+5s+6}.$$

Is $\mathcal{R} = \left(\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 1 \end{bmatrix} \right)$ a minimal realization of $G(\cdot)$? Justify your answer.

Exam, November 2001

33. For the control system described by

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2 + u(t) \\ \dot{x}_2 &= 2x_1 - 4x_2 + 3u(t)\end{aligned}$$

find a suitable feedback matrix K such that the closed loop system has eigenvalues -4 and -5.

Exam, November 2001

34. Use the Lyapunov function

$$V = 5x_1^2 + 2x_1x_2 + 2x_2^2$$

to show that the nonlinear system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 + (x_1 + x_2)(x_2^2 - 1)$$

is asymptotically stable at the origin (by considering the region $|x_2| < 1$). Determine a region of asymptotic stability (about the origin).

Exam, November 2001

35. Determine the matrix exponential

$$\exp\left(t\begin{bmatrix}\alpha & 1 & 0\\ 0 & \alpha & 1\\ 0 & 0 & \alpha\end{bmatrix}\right).$$

Class test, August 2002

36. A linear time-invariant control system is described by the equation

$$\ddot{z} + z = u(t).$$

(a) Write the system in state space form.

(b) Compute the state transition matrix $\Phi(t, 0)$ in TWO DIFFERENT ways.

(c) If $z(0) = 0, \dot{z} = 1$, and $u(t) = 1$ for $t \geq 0$, determine $z(t)$ (for $t \geq 0$).

Class test, August 2002

37. Investigate the stability nature of the linear system

$$\dot{x}_1 = \alpha\, x_1 - x_2, \quad \dot{x}_2 = x_1 + \alpha\, x_2$$

where $\alpha < 0$. What happens when $\alpha = 0$?

Class test, October 2002

38. Consider the linear system

$$\dot{x}_1 = \beta\, x_1 - 3x_2, \quad \dot{x}_2 = -\beta\, x_1 - 2x_2.$$

Use the quadratic form (Lyapunov function)

$$V = \frac{2}{3}x_1^2 - \frac{2}{3}x_1x_2 + x_2^2$$

to obtain *sufficient* conditions (on $\beta \in \mathbb{R}$) for the given system to be asymptotically stable.

Class test, October 2002

39. Given the control system (described by the state equations)

$$\begin{aligned} \dot{x}_1 &= -2x_1 + 2x_2 + u(t) \\ \dot{x}_2 &= x_1 - x_2 \end{aligned}$$

compute the state transition matrix $\Phi(t, 0)$ in TWO DIFFERENT ways, and then determine $x(t)$ (for $t \geq 0$) when

$$x_1(0) = 0, \quad x_2(0) = 0, \quad \text{and} \quad u(t) = 2 \ \text{ for } \ t \geq 0.$$

Exam, November 2002

40. Given the control system with outputs (described by the state and observation equations)

$$\dot{x}_1 = 2x_1 + \alpha\, x_2, \quad \dot{x}_2 = 2x_2 + u(t), \quad y = \beta\, x_1 + x_2$$

determine for what values of $\alpha, \beta \in \mathbb{R}$ the system is

(a) completely controllable and completely observable.

(b) completely controllable but *not* completely observable.

(c) neither completely controllable nor completely observable.

(d) completely observable but *not* completely controllable.

Exam, November 2002

41. Find a minimal realization of

$$G(s) = \frac{s+4}{s^2+5s+6}.$$

Exam, November 2002

42. For the control system (described by the state equations)

$$\begin{aligned} \dot{x}_1 &= x_1 - x_3 + u(t) \\ \dot{x}_2 &= x_1 + 2x_2 + x_3 \\ \dot{x}_3 &= 2x_1 + 2x_2 + 3x_3 + u(t) \end{aligned}$$

find a suitable feedback matrix K such that the closed loop system has eigenvalues -1 and $-1 \pm 2i$.

Exam, November 2002

43. Write the (dynamical) system described by the (second-order) differential equation

$$\ddot{z} + \dot{z} + z^3 = 0$$

in state space form, and then use a quadratic Lyapunov function of the form

$$V = a\,x_1^4 + b\,x_1^2 + c\,x_1x_2 + d\,x_2^2$$

in order to investigate for the stability nature of (the equilibrium state at) the origin. [Hint : Choose the coefficients $a, b, c \in \mathbb{R}$ such that $\dot{V} = -x_1^4 - x_2^4$.]

Exam, November 2002

44. Show that, for $t \in \mathbb{R}$,

$$\exp\left(t\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\right) = e^t \begin{bmatrix} 1 & 2t & 2t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 1 \end{bmatrix}.$$

Class test, April 2003

45. For $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$, find the solution curve (in spectral form or otherwise). What happens when $x_1(0) = 2x_2(0)$?

Class test, April 2003

46. Consider the linear control system (described by the state equations)

$$\begin{aligned} \dot{x}_1 &= 2x_1 - x_2 \\ \dot{x}_2 &= -x_1 + 2x_2 + u(t). \end{aligned}$$

(a) Write the system in the form $\dot{x} = Ax + bu(t)$ with $A \in \mathbb{R}^{2\times 2}$ and $b \in \mathbb{R}^{2\times 1}$.

(b) Compute the state transition matrix $\Phi(t, 0) = \exp(tA)$ in TWO DIFFERENT ways.

(c) If $x_1(0) = x_2(0) = 0$ and $u(t) = 1$ for $t \geq 0$, determine $x_2(2)$.

Class test, April 2003

47. Determine the range of values of parameter $k \in \mathbb{R}$ such that the linear dynamical system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -5x_1 - k\,x_2 + (k-6)x_3 \end{aligned}$$

is asymptotically stable.

Class test, May 2003

48. An autonomous dynamical system is described by

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1 \\ \dot{x}_2 &= -x_1 - x_2 - x_1^2. \end{aligned}$$

(a) Determine the equilibrium state which is *not* at the origin.

(b) Transform the state equations (of the system) so that this point is transfered to the origin.

(c) Hence verify that this equilibrium state is *unstable*

Class test, May 2003

49. Let $A \in \mathbb{R}^{n\times n}$.

(a) Define the *transpose* matrix A^T and then show that the linear mapping

$$\mathcal{T} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}, \quad A \mapsto \mathcal{T}(A) := A^T$$

enjoys the property $\mathcal{T}(AB) = \mathcal{T}(B)\mathcal{T}(A)$. Hence deduce that if the matrix A is invertible, then so is its transpose A^T, and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T .$$

(b) Let $\mathcal{Q}$ denote the associated *quadratic form* (i.e. $x \in \mathbb{R}^n \mapsto x^T Ax \in \mathbb{R}$). Explain what is meant by saying that $\mathcal{Q}$ is *positive definite* and *negative semi-definite*. State clearly necessary and sufficient conditions for positive definiteness and negative semi-definiteness, respectively.

(c) Assume that the matrix A is skew-symmetric (i.e. $A^T + A = 0$) and then show that $\mathcal{Q}(x) = 0$ for all $x \in \mathbb{R}^n$. Furthermore, show A can be written as $A_1 + A_2$, where A_1 is symmetric and A_2 is skew-symmetric. Hence deduce that

$$\mathcal{Q}(x) = x^T A_1 x \quad \text{for all } x \in \mathbb{R}^n.$$

Exam, June 2003

50. Let $A \in \mathbb{R}^{n\times n}$. Determine (by direct computation) the characteristic polynomial of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_4 & -k_3 & -k_2 & -k_1 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

Exam, June 2003

51. Find the (invertible) matrix $T \in \mathbb{R}^{n\times n}$ such that the linear transformation $x \mapsto w := Tx$ transforms the linear control system given by

$$\dot{x} = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(t)$$

into the canonical form. Write down the system in canonical form.

Exam, June 2003

52. Show that the control system (with outputs) described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 + u(t) \\ y &= x_1 + x_2 \end{aligned}$$

is completely controllable but *not* completely observable. Determine initial states $x(0)$ such that if $u(t) = 0$ for $t \geq 0$, then the output $y(\cdot)$ is identically zero for $t \geq 0$.

Exam, June 2003

53. For the control system described by

$$\begin{aligned} \dot{x}_1 &= x_2 + u_1(t) \\ \dot{x}_2 &= x_3 + u_2(t) \\ \dot{x}_3 &= 6x_1 - 11x_2 + 6x_3 + u_1(t) + u_2(t) \end{aligned}$$

find a suitable feedback matrix K such that the closed loop system has eigenvalues $1, 1,$ and 3.

Exam, June 2003

54. Investigate the stability nature of the equilibrium state at the origin for the (nonlinear) system given by

$$\begin{aligned} \dot{x}_1 &= -3x_2 \\ \dot{x}_2 &= x_1 - x_2 + 2x_2^3. \end{aligned}$$

Exam, June 2003

55. Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A and B.

(b) Compute

$$\exp(tA), \quad \exp(tB), \quad \text{and} \quad \exp(t(A+B)).$$

(c) Compare $\exp(tA) \cdot \exp(tB)$ and $\exp(t(A+B))$.

Class test, March 2004

56. Find the solution curve of the linear control system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$

when

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u_1(t) - u_2(t) = 1 \text{ for } t \geq 0.$$

Class test, March 2004

57. Consider the matrix differential equation

$$\dot{W} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} W + W \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \quad W \in \mathbb{R}^{2\times 2}.$$

(a) Verify that $W(t) = \exp\left(t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \cdot \exp\left(t \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right)$ is a solution curve through the identity (i.e. such that $W(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$).

(b) Can you find another solution curve through the identity ? Make a clear statement and then prove it.

Class test, March 2004

58. Consider the linear system

$$\dot{x} = \begin{bmatrix} \alpha & -4 \\ 3 & -\beta \end{bmatrix} x$$

where $\alpha, \beta > 0$ and $\alpha\beta = 12$. Investigate the stability nature of the system at the origin. For what values (if any) of the parameters α and β is the system asymptotically stable ?

Class test, May 2004

59. Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 - 3x_1 + x_2 \\ \dot{x}_2 &= -2x_1. \end{aligned}$$

Investigate the stability nature of (the origin of) the system

(a) by using a suitable Lyapunov function.

(Hint : Try for a Lyapunov function of the form $V = ax_1^2 + bx_2^2$.)

(b) by linearizing the system.

Class test, May 2004

60. Let $A \in \mathbb{R}^{n\times n}$.

(a) Define the terms *eigenvalue* and *eigenvector* (of A), and then investigate the relationship between the eigenvalues of (the power) A^k and those of A. Make a clear statement and then prove it.

(b) Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be $r\,(\leq n)$ distinct eigenvalues of A with corresponding eigenvectors $w_1, w_2, \ldots, w_r$. Prove that the vectors $w_1, w_2, \ldots, w_r$ are *linearly independent.*

(c) The matrix A is said to be *nilpotent* if some power A^k is the zero matrix. Show that A is nilpotent if and only if all its eigenvalues are zero.

(d) Define the *exponential* of A and then calculate $\exp(A)$ for

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Exam, June 2004

61. Find the solution of

$$\dot{x} = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

when $u(t) = e^{2t}$, $t \geq 0$.

Exam, June 2004

62. Consider the control system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} x + bu \\ y &= cx. \end{aligned}$$

Find $b \in \mathbb{R}^{2\times 1}$ and $c \in \mathbb{R}^{1\times 2}$ such that the system is

(a) *not* completely controllable;

(b) completely observable.

When $c = \begin{bmatrix} 1 & 1 \end{bmatrix}$, determine (the initial state) $x(0)$ if $y(t) = e^t - 2e^{3t}$.

Exam, June 2004

63. For the control system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + u \\ \dot{x}_3 &= x_1 - x_2 - 2x_3 \end{aligned}$$

find a linear *feedback control* which makes all the eigenvalues (of the system) equal to -1.

Exam, June 2004

64. Define the term *Lyapunov function* (for a general *nonlinear* system) and then use a quadratic Lyapunov function $V(x) = x^T P x$ to investigate the stability of the system described by the equation

$$\ddot{z} + \epsilon\,(z^2 - 1)\dot{z} + z = 0, \quad \epsilon < 0.$$

Exam, June 2004

65. Find the (feedback) control u^* which minimizes the (quadratic) cost functional

$$\mathcal{J} = \frac{1}{2} x^T(t_1) M x(t_1) + \frac{1}{2} \int_0^{t_1} \left(x^T Q(t) x + u^T R(t) u \right) dt$$

subject to

$$\dot{x} = A(t)x + B(t)u(t) \quad \text{and} \quad x(0) = x_0.$$

(It is assumed that $R(t)$ is a positive definite, and M and $Q(t)$ are positive semi-definite symmetric matrices for $t \geq 0$.)

Exam, June 2004

66.

(a) Find the *eigenvalues* and *eigenvectors* of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Compute the characteristic polynomial of the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c_3 & -c_2 & -c_1 \end{bmatrix}.$$

Class test, March 2005

67. Find *the* solution curve of the initialized linear dynamical system described by

$$\dot{x} = Ax, \quad x(0) = x_0$$

when

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

What happens when $x_1(0) = x_2(0)$?

Class test, March 2005

68. Consider a *linear* control system (with outputs) Σ given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

(where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times \ell}$ and $C \in \mathbb{R}^{n\times m}$). Assume that

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} \alpha \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & \beta \end{bmatrix}.$$

For what values of α and β is Σ completely controllable but *not* completely observable ?

Class test, May 2005

69. Consider a *linear* control system (with outputs) Σ given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

(where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times \ell}$ and $C \in \mathbb{R}^{n\times m}$).

(a) Derive the *transfer function matrix* $G(\cdot)$ associated with Σ.

(b) Assume that

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Compute the scalar transfer function $g(\cdot)$.

(c) Find a *minimal realization* of the scalar transfer function

$$g(s) = \frac{2s+7}{s^2 - 5s + 6}.$$

Class test, May 2005

70. Let $M \in \mathbb{R}^{n\times n}$.

(a) Define the *trace* $\operatorname{tr}(M)$ of M, and then show that the function

$$\operatorname{tr} : \mathbb{R}^{n\times n} \to \mathbb{R}, \quad A \mapsto \operatorname{tr}(A)$$

is *linear*.

(b) Prove that (for any $A, B \in \mathbb{R}^{n\times n}$)

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

(c) Hence deduce that (for any invertible $n \times n$ matrix S)

$$\operatorname{tr}(SMS^{-1}) = \operatorname{tr}(M).$$

(d) Define the terms *eigenvalue* and *eigenvector* (of M), and then prove that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (complex) eigenvalues of M (listed with their algebraic multiplicities), then

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \operatorname{tr}(M).$$

Exam, June 2005

71. Find the solution of

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

when $u(t) = e^t, \ t \geq 0$.

Exam, June 2005

72. Consider a single-input linear control system Σ given by

$$\dot{x} = Ax + bu$$

(where $A \in \mathbb{R}^{m \times m}$ and $b \in \mathbb{R}^{m \times 1}$).

(a) Explain what is meant by the *canonical form* of Σ.

(b) Prove that if

$$\operatorname{rank} \begin{bmatrix} b & Ab & A^2 b & \cdots & A^{m-1} b \end{bmatrix} = m$$

then Σ can be transformed by a linear transformation $w = Tx$ into the canonical form.

(c) Reduce the single-input linear control system

$$\dot{x} = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

to the canonical form and determine the linear mapping $x = T^{-1} w$.

Exam, June 2005

73. Let $d, k > 0$. Consider the control system (with outputs) Σ described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kx_1 - dx_2 + u \\ y &= x_1 - x_2. \end{aligned}$$

Write down the equations describing the dual control system Σ°, and then investigate for (complete) controllability and (complete) observability both control systems.

Exam, June 2005

74. Let $A \in \mathbb{R}^{m\times m}$ and $b \in \mathbb{R}^{m\times 1}$. Consider a single-input linear control system Σ given by

$$\dot{x} = Ax + bu.$$

(a) Given an arbitrary set $\Lambda = \{\theta_1, \ldots, \theta_m\}$ of complex numbers (appearing in conjugate pairs), prove that if Σ is completely controllable, then there exists a *feedback matrix* K such that the eigenvalues of $A + bK$ are the set Λ.

(b) <u>Application</u> : Find a linear feedback control $u = Kx$ when

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \{-1, -1-2i, -1+2i\}.$$

Exam, June 2005

75. Consider the dynamical system

$$\begin{aligned} \dot{x}_1 &= -kx_1 - 3x_2 \\ \dot{x}_2 &= kx_1 - 2x_2, \quad k \in \mathbb{R}. \end{aligned}$$

(a) When $k = 1$, use a quadratic Lyapunov function

$$\begin{aligned} V(x) &= x^T P x \\ &= a x_1^2 + 2b x_1 x_2 + c x_2^2 \end{aligned}$$

with derivative $\dot{V} = -2\left(x_1^2 + x_2^2\right)$ to determine the stability of the system (at the origin).

(b) Using the *same* Lyapunov function, find sufficient conditions on k for the system to be asymptotically stable (at the origin).

Exam, June 2005

76. Find u^* so as to *minimize*

$$\mathcal{J} = \int_0^T dt$$

subject to

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad |u_i| \le K_i, \quad i = 1, \ldots, \ell \\ x(0) &= x_0 \\ x(T) &= 0. \end{aligned}$$

Exam, June 2005

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