

Appendix A

Answers and Hints to Selected Exercises

Introduction

1. Prove first that (for $A, B \in \mathbb{R}^n$) $\text{tr}(AB) = \text{tr}(BA)$.
2. Use the (fundamental) property : $\det(AB) = \det(A) \cdot \det(B)$.
3. The following facts are needed :
 - (1) $\text{rank}(A) = \text{rank}(A^T)$.
 - (2) $\text{rank}(A) + \dim \ker(A) = n$.
 - (3) $\text{rank}(AB) = \text{rank}(B) - \dim \ker(A) \cap \text{im}(B)$.
 - (4) $U \subseteq V \Rightarrow \dim(U) \leq \dim(V)$ (as vector spaces).

We have (3) $\Rightarrow \text{rank}(AB) \leq \text{rank}(B)$; (1), (3) $\Rightarrow \text{rank}(AB) \leq \text{rank}(A)$; (4), (2) $\Rightarrow \dim \ker(A) \cap \text{im}(B) \leq n - \text{rank}(A)$.

4. Apply the matrix (linear transformation) $(A - \lambda_2 I_n)(A - \lambda_3 I_n) \cdots (A - \lambda_r I_n)$ to the equation (trivial linear combination) $\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_r w_r = 0$ and hence obtain $\alpha_1 = 0$, etc.
5. Express the characteristic polynomial of A in two different ways : $\text{char}_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = \lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + \cdots + (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$ but also $\text{char}_A(\lambda) = \det(\lambda I_n - A) = \lambda^n - \text{tr}(A)\lambda^{n-1} + \cdots + (-1)^n \det(A)$.
6. Observe that

$$\sum_{k \geq 0} \left\| \frac{t^k}{k!} A^k \right\| \leq \sum_{k \geq 0} \frac{1}{k!} \|tA\|^k$$

and then use the *comparison test* (for numerical series).

7. Induction : $(S^{-1}AS)^n = S^{-1}A^nS$.

8. $uu^T = u_1 u_2 \cdots u_n \begin{bmatrix} u_1 & \dots & u_n \\ \vdots & & \vdots \\ u_1 & \dots & u_n \end{bmatrix}.$

9. Apply the matrix (linear transformation) A^r to the equation $c_0 b + c_1 A b + \cdots + c_r A^r b = 0$ and deduce that $c_0 = 0$, etc.

10. Straightforward computation : $\text{rank}(A^T A) = \text{rank}(A) = \text{rank}(AA^T) = 2$.

11. (a) $\lambda^2 - 4\lambda - 5$; $\lambda_1 = -1$; $\lambda_2 = 5$.

$$E_1 = E_{\lambda_1} = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad E_2 = E_{\lambda_2} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

(b) $(\lambda + 1)(\lambda - 2)$; $\lambda_1 = -1$; $\lambda_2 = 2$.

$$E_1 = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad E_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

(c) $\lambda^2 - \lambda$; $\lambda_1 = 0$; $\lambda_2 = 1$.

$$E_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad E_2 = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}.$$

(d) $\lambda^2 - 4\lambda + 8$; $\lambda_{1,2} = 2 \pm 2i$.

$$E_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad E_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

(e) $\lambda^2 - (a+b)\lambda + ab$; $\lambda_1 = a$; $\lambda_2 = b$.

$$E_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ -a \end{bmatrix} \right\} \quad \text{and} \quad E_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ -b \end{bmatrix} \right\}.$$

(f) $\lambda(\lambda^2 - 5\lambda + 4)$; $\lambda_1 = 0$; $\lambda_2 = 1$; $\lambda_3 = 4$.

$$E_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}, \quad E_2 = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \right\} \quad \text{and} \quad E_3 = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(g) $(\lambda - 1)^3$; $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

$$E = E_\lambda = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(h) $\lambda^2(\lambda - 3)$; $\lambda_1 = \lambda_2 = 0$; $\lambda_3 = 3$.

$$E_{1,2} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad E_3 = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(i) $(\lambda - 6)(\lambda^2 - 6\lambda - 16); \quad \lambda_1 = -2; \lambda_2 = 6; \lambda_3 = 8.$

$$E_1 = \text{span} \left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}, \quad E_2 = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad E_3 = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \right\}.$$

(j) $\lambda^2 - (2 \cos \theta)\lambda + 1; \quad \lambda_{1,2} = \cos \theta \pm i \sin \theta.$

$$E_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad E_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

12. λ is an eigenvalue of A if and only if $\lambda - \mu$ is an eigenvalue of $A - \mu I_n$, etc.

13. (a) Observe that $(\lambda I_n - A)^T = \lambda I_n - A^T$. (b) From $\lambda I_n - S^{-1}AS = S^{-1}(\lambda I_n - A)S$ derive that $\det(\lambda I_n - S^{-1}AS) = \det(\lambda I_n - A)$.

14. (b) Yes. (c) Yes. (d) Yes.

15. (a) Show that $q(x) = -q(x)$ for all $x \in \mathbb{R}^{n \times 1}$. (b) $A_1 = \frac{1}{2}(A + A^T)$, etc.

17. (a) $\exp(tA) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}; \exp(tB) = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}; \exp(t(A+B)) = \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}.$

18. (a) $\exp(tA) = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$. (b) $\exp(tA) = \begin{bmatrix} e^{at} & bte^{at} \\ 0 & e^{at} \end{bmatrix}$. (c) $\begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$
and $\begin{bmatrix} 1 & 0 & 0 \\ 2t & 1 & 0 \\ 3t + 4t^2 & 4t & 1 \end{bmatrix}.$

19. (a) $\begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$. (b) $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$. (c) $\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. (d) $\begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{-t} \end{bmatrix}$.

(e) $\begin{bmatrix} \frac{1}{2}(1+e^{2t}) & \frac{1}{2}(e^{2t}-1) \\ \frac{1}{2}(e^{2t}-1) & \frac{1}{2}(1+e^{2t}) \end{bmatrix}$. (f) $\begin{bmatrix} 1 & t & t + \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$. (g) $\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{7t} \end{bmatrix}$.

20. FFTTTTFTT.

Linear Dynamical Systems

22. Let $a_k := \frac{\alpha^k}{(k+1)!}$. Show that $0 < a_k \leq a_{k+1}$ (for $k \geq \alpha - 2$) and then use the fact that *every bounded non-increasing sequence of numbers is convergent*.

23. Let $Y(\cdot)$ be the unique solution of the Cauchy problem (in matrices) :

$$\dot{Y} = -YA(t), \quad Y(0) = I_n.$$

Then $Y(t) \cdot X(t) = I_n$ (for all t), etc.

24. Straightforward computation.

- 25.** There is no (linear) transformation $w = Tx$ such that $TAT^{-1} = C$ and $Tb = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

26.

$$(a) \quad \begin{aligned} x_1(t) &= (-x_1(0) + x_2(0))e^{-3t} + (2x_1(0) - x_2(0))e^{-2t} \\ x_2(t) &= 2(-x_1(0) + x_2(0))e^{-3t} + (2x_1(0) - x_2(0))e^{-2t}. \end{aligned}$$

$$(b) \quad \begin{aligned} x_1(t) &= \frac{1}{2}(x_1(0) + x_2(0))e^t + \frac{1}{2}(x_1(0) - x_2(0))e^{3t} \\ x_2(t) &= \frac{1}{2}(x_1(0) + x_2(0))e^t - \frac{1}{2}(x_1(0) - x_2(0))e^{-3t}. \end{aligned}$$

$$(c) \quad \begin{aligned} x_1(t) &= \frac{1}{2}(x_1(0) - x_2(0))e^{-t} + \frac{1}{2}(x_1(0) + x_2(0))e^t \\ x_2(t) &= -\frac{1}{2}(x_1(0) - x_2(0))e^{-t} + \frac{1}{2}(x_1(0) + x_2(0))e^{3t}. \end{aligned}$$

$$(d) \quad \begin{aligned} x_1(t) &= (-x_2(0) + \frac{1}{2}x_3(0))e^t + 2(x_1(0) + x_2(0))e^{2t} - (x_1(0) + x_2(0) + \frac{1}{2}x_3(0))e^{3t} \\ x_2(t) &= -(x_2(0) + \frac{1}{2}x_3(0))e^t - (x_1(0) + x_2(0))e^{2t} + (x_1(0) + x_2(0) + \frac{1}{2}x_3(0))e^{3t} \\ x_3(t) &= -2(x_1(0) + x_2(0))e^{2t} + 2(x_1(0) + x_2(0) + \frac{1}{2}x_3(0))e^{3t}. \end{aligned}$$

$$(e) \quad \begin{aligned} x_1(t) &= -(x_2(0) + x_3(0))e^t + (x_1(0) + x_2(0) + x_3(0))e^{2t} \\ x_2(t) &= \frac{1}{2}(x_2(0) - x_3(0))e^{-t} + \frac{1}{2}(x_2(0) + x_3(0))e^t \\ x_3(t) &= -\frac{1}{2}(x_2(0) - x_3(0))e^{-t} + \frac{1}{2}(x_2(0) + x_3(0))e^t. \end{aligned}$$

- 27.** (a) $Z_1 = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$, $Z_2 = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$, etc. (b) $Z_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $Z_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, etc. (c) $Z_1 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $Z_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, etc. (d) ... (e)

$$\mathbf{28.} \quad A = \begin{bmatrix} 0 & 1 \\ -\omega & 0 \end{bmatrix}; \quad \Phi(t, 0) = \exp(tA) = \begin{bmatrix} \cos(\sqrt{\omega}t) & \sqrt{\omega} \sin(\sqrt{\omega}t) \\ -\sqrt{\omega} \sin(\sqrt{\omega}t) & \cos(\sqrt{\omega}t) \end{bmatrix}.$$

- 29.** $\text{char}(\lambda) = \lambda(\lambda - a_1 + a_4) \Rightarrow \lambda_1 = 0$ and $\lambda_2 = a_1 - a_4$. We assume that $a_1 \neq a_4$.

$$\begin{aligned} x(t) &= \left(Z_1 + e^{(a_1-a_4)t} Z_2 \right) \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{a_1-a_4} (-a_4 x_1(0) + a_2 x_2(0) + (a_1 x_1(0) - a_2 x_2(0)) e^{(a_1-a_4)t}) \\ \frac{1}{a_1-a_4} (-a_3 x_1(0) + a_1 x_2(0) + (a_3 x_1(0) - a_4 x_2(0)) e^{(a_1-a_4)t}) \end{bmatrix}. \end{aligned}$$

30.

$$\begin{aligned} \Phi(t, 0) &= e^{-2t} Z_1 + e^{5t} Z_2 \\ &= \begin{bmatrix} \frac{1}{7}(4e^{-2t} + 3e^{5t}) & \frac{1}{7}(-4e^{-2t} + 4e^{5t}) \\ \frac{1}{7}(-3e^{-2t} + 3e^{5t}) & \frac{1}{7}(3e^{-2t} + 4e^{5t}) \end{bmatrix}. \\ x(t) &= \Phi(t, 0) \left[x_0 + \int_0^t \Phi(0, \tau) \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(\tau) d\tau \right]. \end{aligned}$$

31.

$$\begin{aligned} \Phi(t, 0) &= e^{-2t} Z_1 + e^{-t} Z_2 \\ &= \begin{bmatrix} -4e^{-2t} + 2e^{-t} & -e^{-2t} + e^{-t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix}. \\ z(2) &= \frac{1 - e^2}{4e^4} + \frac{e - 1}{e^2}. \end{aligned}$$

- 32.** Existence : simple verification. Uniqueness : let W be a solution; consider the product $\exp(-tA) W \exp(-tB)$ and differentiate, etc.

33. $\exp(tA) = \begin{bmatrix} e^t & 2te^t \\ 0 & e^t \end{bmatrix}.$

- 34.** Take $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. Then consider $\det(X(t)) = x_1 x_4 - x_2 x_3$ and differentiate, etc.

- 36.** $\frac{d}{dt} \exp(B(t)) = \frac{d}{dt} \left(I_n + B(t) + \frac{B^2(t)}{2!} + \dots \right)$. The solution is *unique*.

38.

$$\begin{aligned} \Phi(t, 0) &= \begin{bmatrix} \frac{1}{2}(3^{-3t} + e^t) & e^{-3t} - e^t \\ \frac{1}{4}(e^{-3t} - e^t) & \frac{1}{2}(e^{-3t} + e^t) \end{bmatrix} \\ x(t) &= \Phi(t, 0) \begin{bmatrix} \frac{e^{5t}}{10} - \frac{e^t}{2} + \frac{7}{5} \\ \frac{e^{10}}{20} + \frac{e^t}{4} + \frac{7}{10} \end{bmatrix}, \text{ etc.} \end{aligned}$$

39. (a) $T = \frac{1}{2} \begin{bmatrix} -3 & 1 \\ 5 & -1 \end{bmatrix}$; $TAT^{-1} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$. (b) $T = \begin{bmatrix} \frac{2}{3} & -\frac{5}{3} & \frac{2}{3} \\ -1 & -2 & 1 \\ -1 & -2 & 2 \end{bmatrix}$;
 $TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$.

40. $P = \begin{bmatrix} \frac{1}{12} & \frac{1}{12} & \frac{7}{12} \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$; $E = P^{-1}AP = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 9 \\ 0 & 1 & 0 \end{bmatrix}$.

Linear Control Systems

41.

$$\int_0^\infty e^{-st} \cdot e^{at} dt = \lim_{R \rightarrow \infty} \frac{1 - e^{-(s-a)R}}{s-a} = \frac{1}{s-a} \quad (\text{for } s > a).$$

The matrix(-valued mapping) $\Phi(t, 0) = \exp(tA)$ is completely determined by the conditions : (1) $\dot{\Phi}(t, 0) = A\Phi(t, 0)$ and (2) $\Phi(0, 0) = I_n$.

42. Show first that $\mathcal{C}(\tilde{A}, \tilde{B}) = P\mathcal{C}(A, B)$ and $\mathcal{O}(\tilde{A}, \tilde{C}) = \mathcal{O}(A, C)P^{-1}$.

43. $\tilde{C} \left(sI_n - \tilde{A} \right)^{-1} \tilde{B} = C (sI_n - A)^{-1} B$.

44. $\text{rank}(AB) \leq n$ and $\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB)$.

45. $\text{rank}[B \ AB] = 2$.

46. $b \in \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \cup \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

47.

$$\begin{aligned} \exp(tA) &= \begin{bmatrix} \frac{1}{3}(e^{-8t} + 2e^{-2t}) & \frac{1}{3}(-e^{-8t} + e^{-2t}) \\ \frac{2}{3}(-e^{-8t} + e^{-2t}) & \frac{1}{3}(2e^{-8t} + e^{2t}) \end{bmatrix}; \\ (1) - 2 &= \frac{1}{3} \int_0^1 (-e^{8\tau} + 4e^{2\tau})(C_1 + C_2e^{-2\tau}) d\tau \\ (2) - 3 &= \frac{1}{3} \int_0^1 (2e^{8\tau} + 4e^{2\tau})(C_1 + C_2e^{-2\tau}) d\tau. \end{aligned}$$

48. (1) $\alpha = 10$ or $\alpha = 12$. (2) $\alpha = 0$ or $\alpha = 1$. For $u_1 \equiv 0$: $\dot{x} = \begin{bmatrix} 2 & \alpha-3 \\ 0 & 2 \end{bmatrix}x + \begin{bmatrix} 1 \\ \alpha^2 - \alpha \end{bmatrix}u_2$. $\alpha = 0$ or $\alpha = 1$.

49. $u^*(t) = -[e^t \ 3e^{2t}] U^{-1}(0, 1) \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, where $U(0, 1) = \begin{bmatrix} \frac{e^2-1}{2} & e^3-1 \\ e^3-1 & \frac{9}{4}(e^4-1) \end{bmatrix}$, etc.

50. Verification.

51. $W(t) = \exp((t-t_1)A) \cdot C \cdot \exp((t-t_1)A^T)$, etc.

52. The system is completely observable.

53. $x(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

54. $\text{rank } \mathcal{O} = 1 < 2$.

$$\begin{aligned} x(t) &= \exp(tA)x(0) \\ &= \begin{bmatrix} -e^{-2t} + 2e^{-t} & e^{-t} \\ 2(e^{-2t} - e^{-t}) & 2e^{-2t} - e^{-t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}. \end{aligned}$$

$$x_1(0) + 2x_2(0) = 0.$$

55. Use the fact that

$$\dot{\Phi}(\tau, t) = \frac{d}{dt}\Phi(\tau, t) = -\Phi(\tau, t) A(t).$$

56.

$$\begin{aligned} K &= \underline{k} T = \frac{1}{2} \begin{bmatrix} -14 & -4 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 11 & -5 \end{bmatrix}. \end{aligned}$$

57.

$$\begin{aligned} K &= \underline{k} T = \begin{bmatrix} -11 & 4 & -9 \end{bmatrix} T \\ &= -\frac{1}{3} \begin{bmatrix} -61 & 129 & 88 \end{bmatrix}. \end{aligned}$$

58. $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 3$. $W = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 4 & 1 & 9 \end{bmatrix}$. $\mu_1 = 1, \lambda_2 = 1, \lambda_3 = 3$ ($p = 1$).

$$\begin{aligned} K &= fg \widetilde{W} \\ &= \begin{bmatrix} -1 \\ -1 \end{bmatrix} 1 \begin{bmatrix} -3 & 4 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -4 & 1 \\ 3 & -4 & 1 \end{bmatrix}. \end{aligned}$$

59. $A + bK = \begin{bmatrix} -1 + \alpha & 0 & 3 + \beta \\ \alpha & -3 & \beta \\ 1 - \alpha & 0 & -(\beta + 3) \end{bmatrix}$. $\alpha = \beta$.

60. Take $X^{-1} = \begin{bmatrix} A^{-1} & X_1 \\ X_2 & C^{-1} \end{bmatrix}$, etc.

61. $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $B = \begin{bmatrix} \frac{10}{3} & 2 \\ -\frac{7}{2} & -2 \\ \frac{7}{6} & 1 \end{bmatrix}$. $C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$.

62. $\text{rank}(K_1) = 1$ and $\text{rank}(K_2) = 2$. The *order* of the minimal realization is $1 + 2 = 3$.

63.

$$C(sI_n - A)^{-1}B = C_1(sI_{n_1} - A_1)^{-1}B_1C_2(sI_{n_2} - A_2)^{-1}B_2.$$

65. $b_1 = 2, b_2 = -1; c_1 = 4, c_2 = 1. P = \frac{1}{2} \begin{bmatrix} 1 & 2 \\ -2 & -6 \end{bmatrix}$.

Stability

66. Sketch the graph of f .

67.

$$\|\exp(tA)\| \leq \|e^{\lambda_1 t} Z_1 + \dots + e^{\lambda_n t} Z_n\| \leq \dots \leq e^{-at} (\|Z_1\| + \dots + \|Z_n\|)$$

where $a = \min\{-\text{Re}(\lambda_1), \dots, -\text{Re}(\lambda_n)\}$.

68. The new state equations are :

$$\begin{aligned} \dot{x}_1 &= -2x_1x_2 - 4x_2 \\ \dot{x}_2 &= \frac{1}{2}x_1 + x_1x_2. \end{aligned}$$

69. (a) Yes. (b) No. (c) No.

70. $k < 2$.

71. $0 < k < 2. u = -\frac{2}{3}x_1 - x_2 - \frac{2}{3}x_3$

72. $\dot{x} = A(t)x \Rightarrow \ddot{x}_2 - 4\dot{x}_2 + 3x_2 = 0$. The linear system $\dot{y} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} y$ has solution

$$\begin{aligned} y(t) &= (e^t Z_1 + e^{3t} Z_2)y(0) \\ &= \begin{bmatrix} \frac{1}{2}(3e^t - e^{3t}) & \frac{1}{2}(e^t + e^{3t}) \\ \frac{3}{2}(e^t + -e^{3t}) & \frac{1}{2}(-e^t + 3e^{3t}) \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}. \end{aligned}$$

$x_2(t) = y_1(t) = ae^t + be^{3t}$ and $|x_2(t)| \rightarrow \infty$ (as $t \rightarrow \infty$).

73. $q = \frac{1}{2}, b = \frac{1}{2}, c = 1$ and $d = 1$. The origin is *asymptotically stable*.

74. $\dot{V} = -2(x_1 - x_2)^2 - 2(x_1 + 2x_2)^2(1 - x_2^2) < 0$ for $|x_2| < 1$. $5x_1^2 + 2x_1x_2 + 2x_2^2 \leq \frac{9}{5}$.

75. The origin is *unstable*.

76. $P = \frac{1}{30} \begin{bmatrix} 13 & -1 \\ -1 & 4 \end{bmatrix}$. The origin is *asymptotically stable*.

77. $-C + \lim_{t \rightarrow \infty} W(t) = A \int_0^\infty W(\tau) d\tau + (\int_0^\infty W(\tau) d\tau) B$, etc.

78. The state equations are $\dot{x}_1 = x_2$ and $\dot{x}_2 = -a_2 x_1 - a_1 x_2$.

$$\begin{aligned} V &= x^T Px = ax_1^2 + 2bx_1x_2 + cx_2^2 \Rightarrow \\ \dot{V} &= -2ba_2x_1^2 + 2bx_2^2 - 2a_1cx_2^2 + 2(a - ba_1 - ca_2)x_1x_2, \text{ etc.} \end{aligned}$$

79. The origin is *asymptotically stable*. $k \in (\frac{1}{4}, 4)$.

80. The linearized system is $\dot{x} = \begin{bmatrix} 7 & 2 \\ 1 & -3 \end{bmatrix} x$. The origin is *unstable*.

81. (a) The linearized system is $\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix} x$. The origin is *asymptotically stable*. (b) The origin is *asymptotically stable*.

83. For $u(t) \equiv 0$, $t > 0$, the solution is

$$\begin{aligned} x(t) &= \Phi(t, 0)x_0 \\ &= \frac{3x_0}{t+3} \rightarrow 0 \quad (\text{as } t \rightarrow \infty). \end{aligned}$$

When $u(\cdot)$ is the unit step function, we have (for $t \geq 1$)

$$\begin{aligned} x(t) &= \Phi(t, 0) \left[x_0 + \int_0^t \Phi(0, \tau) d\tau \right] \\ &= \frac{1}{t+3} (t^2 + 6t + 3x_0) \rightarrow \infty \quad (\text{as } t \rightarrow \infty). \end{aligned}$$

84. $(-2, -2)$. The transformed state equations are

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= 3x_1 - x_2 - x_1^2. \end{aligned}$$

The linearized system has roots λ_1, λ_2 such that $\lambda_1 < 0 < \lambda_2$.

85. $1 < k < 5$.

86. (a) Neutrally stable. (b) Unstable.

87. $2x_1^2 + x_2^2 < 6$.

88. $x_1^2 + 3x_2^2 < \frac{3}{2}$.

89. (b) The origin is asymptotically stable.

90. (a) The system is stabilizable but not detectable. (b) The system is detectable but not stabilizable.

Optimal Control

91. Induction.

93. $u^* = -\frac{1}{2}p^*$ and $p^* = C_1 e^{2t}$, etc.

94. $p_3^* = -C_1 \frac{t^2}{2} + C_2 t + C_3$ (a parabola), etc.

97. $u^* = -\frac{1}{2}p_2^*$. $p_2^*(t) = c_1 e^t \sinh(\sqrt{2}t + c_2)$. (Observe that $\sinh(\alpha + \beta) = \sinh \alpha \cdot \cosh \beta + \sinh \beta \cdot \cosh \alpha$; hence $a \sinh \alpha + b \cosh \alpha = A \sinh(\alpha + C)$).

98. $u^* = \frac{13}{5} + \frac{e^t}{25}$.

99. $u^*(t) = -R^{-1}B^T Px(t) = -10(ax_1 + bx_2)$, where $a \approx 0.171$ and $b \approx 0.316$.

100. (a) $u^* = \frac{3(e^{4t} - e^4)}{e^{4t} + 3e^4}x_1$. (b) $u^* = \frac{3e^{4t} + e^4}{e^{4t} - e^4}x_1$.