Appendix B

Revision Problems

1. Find the solution of the following uncontrolled linear system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Class test, August 1998

2. Given the linear system described by

$$\dot{x} = \begin{bmatrix} 2 & \alpha - 1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ \alpha \end{bmatrix} u$$

determine for what values of the real parameter α the system is *not* completely controllable.

Class test, August 1998

3. Show that the linear system described by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -2x_1 - 3x_2 + u, \quad y = x_1 + x_2$$

is *not* completely observable. Determine initial states $x(0)$ such that if $u(t) = 0$ for $t \geq 0$, then the output $y(t)$ is identically zero for $t \geq 0$.

Class test, October 1998

4. Consider a time invariant linear system of the form

$$\dot{x} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

and let $\theta_1, \theta_2 \in \mathbb{R}$. Prove that if the system is completely controllable, then there exists a matrix K such that the eigenvalues of the matrix $A + Kb$ are θ_1 and θ_2.

<u>Application</u> :

$$A = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \theta_1 = -4,\ \theta_2 = -5.$$

Class test, October 1998

5. A linear system is known to be described by

$$\dot{x} = Ax.$$

It is possible to measure the state vector, but because of difficulties in setting up the equipment, this measurement can be started only after an unknown amount T $(T > 2)$ of time has elapsed. It is then found that

$$x(T) = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}, \quad x(T+1) = \begin{bmatrix} 1.5 \\ 1.6 \end{bmatrix}, \quad x(T+2) = \begin{bmatrix} 1.8 \\ 2.1 \end{bmatrix}.$$

Compute $x(T-2)$.

Exam, November 1998

6. Write the equation of motion

$$\ddot{z} = u(t)$$

in state space form, and then solve it.

Exam, November 1998

7. Given the liner control system described by

$$
\begin{aligned}
\dot{x}_1 &= \alpha\, x_1 + 2x_2 + u \\
\dot{x}_2 &= x_1 - x_2 \\
y &= x_1 + \alpha\, x_2
\end{aligned}
$$

determine for what values of the real parameter α the system is (a) completely controllable and completely observable; (b) completely controllable but not completely observable; (c) completely observable but not completely controllable.

Exam, November 1998

8. Given the system described by

$$
\dot{x} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & -3 & 0 \\ 1 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} u
$$

show that under linear feedback of the form $u = \alpha\, x_1 + \beta\, x_3$, the closed loop system has two fixed eigenvalues, one of which is equal to -3. Determine the second fixed eigenvalue, and also values of α and β such that the third closed loop eigenvalue is equal to -4.

Exam, November 1998

9. A control system is described by

$$
\dot{x}_1 = x_2, \quad \dot{x}_2 = u
$$

and the control $u(t)$ is to be chosen as to minimize the performance index

$$
\mathcal{J} = \int_0^1 u^2\, dt.
$$

Find the optimal control which transfers the system from $x_1(0) = 0$, $x_2(0) = 0$ to $x_1(1) = 1$, $x_2(1) = 0$.

Exam, November 1998

10. Find the characteristic polynomial, the eigenvalues and the corresponding eigenvectors for the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Class test, March 1999

11. Let A be an $n \times n$ matrix. How are the eigenvalues of A^3 related to those of A ? If A is invertible, can 0 be an eigenvalues for A^3 ? Explain.

Class test, March 1999

12. Let A and S be $n \times n$ matrices, and assume that S is invertible. Show that the characteristic polynomials of A and $S^{-1}AS$ are the same.

Class test, March 1999

13. Determine the state transition matrix and write down the general solution of the linear control system described by the equations

$$\begin{aligned} \dot{x}_1 &= x_1 + 4x_2 + u \\ \dot{x}_2 &= 3x_1 + 2x_2. \end{aligned}$$

Class test, March 1999

14. Write the equation of motion

$$\ddot{z} + \dot{z} = 0$$

in state space form, and then solve it (by computing the state transition matrix).

Exam, June 1999

15. Split up the linear control system

$$\dot{x} = \begin{bmatrix} 4 & 3 & 5 \\ 1 & -2 & -3 \\ 2 & 1 & 8 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} u(t)$$

into its controllable and uncontrollable parts, as displayed below :

$$\begin{bmatrix} \dot{x}^{(1)} \\ \dot{x}^{(2)} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t).$$

Exam, June 1999

16. Investigate the stability nature of the equilibrium state at the origin of the system described by the scalar equation

$$\ddot{z} + \alpha\,\dot{z} + \beta\,z = z \cdot \dot{z}.$$

Exam, June 1999

17. Determine whether the system described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 4 & 1 & 2 \\ 3 & -1 & 2 \\ 5 & -3 & 8 \end{bmatrix} x + \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} u(t) \\ y &= \begin{bmatrix} 2 & 1 & -1 \end{bmatrix} x \end{aligned}$$

is detectable.

Exam, June 1999

18. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

(a) Find the characteristic polynomial, the eigenvalues and the corresponding eigenvectors for the matrix A.

(b) Determine the matrix exponential $\exp(tA)$.

Class test, March 2000

19. Let A, B, and S be $n \times n$ matrices, and assume that S is nonsingular.

(a) Show that the characteristic polynomials of A and $S^{-1}AS$ are the same.

(b) If $AB = BA$, show that

$$\exp(t(A+B)) = \exp(tA) \cdot \exp(tB).$$

Class test, March 2000

20. Write down the general solution of the linear system described by the equations

$$\begin{aligned} \dot{x}_1 &= x_1 + 4x_2 \\ \dot{x}_2 &= 3x_1 + 2x_2. \end{aligned}$$

Class test, March 2000

21. Consider a (time-invariant) linear control system of the form

$$\dot{x} = Ax + bu(t)$$

where A is a 2×2 matrix and b is a non-zero 2×1 matrix. Assume that

$$\operatorname{rank} \begin{bmatrix} b & Ab \end{bmatrix} = 2$$

and prove that the given system can be transformed by a (nonsingular) transformation $w(t) = Tx(t)$ into the canonical form

$$\dot{w} = \begin{bmatrix} 0 & 1 \\ -k_2 & -k_1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

<u>Application</u> :

$$A = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Class test, May 2000

22. Find the solution of

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

when

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Exam, June 2000

23. Consider the matrix differential equation

$$\dot{X} = A(t)X, \qquad X(0) = I_n.$$

Show that, when $n = 2$,

$$\frac{d}{dt}(\det X) = \operatorname{tr} A(t) \cdot \det X$$

and hence deduce that $X(t)$ is nonsingular, $t \geq 0$.

Exam, June 2000

24. Write the linear system Σ described by

$$\ddot{z} + \dot{z} + 2z = 0$$

in the state space form, and then use Lyapunov functions to investigate the stability nature of the system. Is the origin asymptotically stable ? Justify your answer.

Exam, June 2000

25. Determine whether the system described by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned}$$

is detectable.

Exam, June 2000

26. Consider the matrix

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

(a) Find the characteristic polynomial, the eigenvalues and the corresponding eigenvectors for the matrix A.

(b) Determine the matrix exponential $\exp(tA)$.

Class test, August 2001

27. Consider the linear control system described by the equations

$$\begin{aligned} \dot{x}_1 &= x_1 - 4x_2 + u(t) \\ \dot{x}_2 &= -x_1 - x_2 + u(t). \end{aligned}$$

If $x(0) = (1, 0)$ and $u(t) = 1$ for $t \geq 0$, find the expressions for $x_1(t)$ and $x_2(t)$.

Class test, August 2001

28. Investigate the stability nature of the origin for the linear system

$$\dot{x}_1 = -2x_1 - 3x_2, \quad \dot{x}_2 = 2x_1 - 2x_2.$$

Class test, November 2001

29. Consider the linear system

$$\dot{x}_1 = k\,x_1 - 3x_2, \quad \dot{x}_2 = -k\,x_1 - 2x_2.$$

Use the quadratic form (Lyapunov function)

$$V = \frac{2}{3}x_1^2 + x_2^2 - \frac{2}{3}x_1x_2$$

to obtain *sufficient* conditions on $k \in \mathbb{R}$ for the system to be asympotically stable (at the origin).

Class test, November 2001

30. Investigate the stability nature of the origin for the nonlinear system

$$\begin{aligned} \dot{x}_1 &= 7x_1 + 2\sin x_2 - x_2^4 \\ \dot{x}_2 &= e^{x_1} - 3x_2 - 1 + 5x_1^2. \end{aligned}$$

Class test, November 2001

31. Find the solution of

$$\dot{x} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t)$$

when

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad u(t) = e^{3t},\ t \geq 0.$$

Exam, November 2001

32. Find a minimal realization of

$$G(s) = \frac{s+4}{s^2+5s+6}.$$

Is $\mathcal{R} = \left(\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 1 \end{bmatrix} \right)$ a minimal realization of $G(\cdot)$? Justify your answer.

Exam, November 2001

33. For the control system described by

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_2 + u(t) \\ \dot{x}_2 &= 2x_1 - 4x_2 + 3u(t) \end{aligned}$$

find a suitable feedback matrix K such that the closed loop system has eigenvalues -4 and -5.

Exam, November 2001

34. Use the Lyapunov function

$$V = 5x_1^2 + 2x_1x_2 + 2x_2^2$$

to show that the nonlinear system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 + (x_1 + x_2)(x_2^2 - 1)$$

is asymptotically stable at the origin (by considering the region $|x_2| < 1$). Determine a region of asymptotic stability (about the origin).

Exam, November 2001

35. Determine the matrix exponential

$$\exp\left(t\begin{bmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{bmatrix}\right).$$

Class test, August 2002

36. A linear time-invariant control system is described by the equation

$$\ddot{z} + z = u(t).$$

(a) Write the system in state space form.

(b) Compute the state transition matrix $\Phi(t, 0)$ in TWO DIFFERENT ways.

(c) If $z(0) = 0, \dot{z} = 1$, and $u(t) = 1$ for $t \geq 0$, determine $z(t)$ (for $t \geq 0$).

Class test, August 2002

37. Investigate the stability nature of the linear system

$$\dot{x}_1 = \alpha\, x_1 - x_2, \quad \dot{x}_2 = x_1 + \alpha\, x_2$$

where $\alpha < 0$. What happens when $\alpha = 0$?

Class test, October 2002

38. Consider the linear system

$$\dot{x}_1 = \beta\, x_1 - 3x_2, \quad \dot{x}_2 = -\beta\, x_1 - 2x_2.$$

Use the quadratic form (Lyapunov function)

$$V = \frac{2}{3}x_1^2 - \frac{2}{3}x_1 x_2 + x_2^2$$

to obtain *sufficient* conditions (on $\beta \in \mathbb{R}$) for the given system to be asymptotically stable.

Class test, October 2002

39. Given the control system (described by the state equations)

$$\begin{aligned} \dot{x}_1 &= -2x_1 + 2x_2 + u(t) \\ \dot{x}_2 &= x_1 - x_2 \end{aligned}$$

compute the state transition matrix $\Phi(t,0)$ in TWO DIFFERENT ways, and then determine $x(t)$ (for $t \geq 0$) when

$$x_1(0) = 0, \quad x_2(0) = 0, \quad \text{and} \quad u(t) = 2 \ \text{ for } \ t \geq 0.$$

Exam, November 2002

40. Given the control system with outputs (described by the state and observation equations)

$$\dot{x}_1 = 2x_1 + \alpha\, x_2, \quad \dot{x}_2 = 2x_2 + u(t), \quad y = \beta\, x_1 + x_2$$

determine for what values of $\alpha, \beta \in \mathbb{R}$ the system is

(a) completely controllable and completely observable.

(b) completely controllable but *not* completely observable.

(c) neither completely controllable nor completely observable.

(d) completely observable but *not* completely controllable.

Exam, November 2002

41. Find a minimal realization of

$$G(s) = \frac{s+4}{s^2+5s+6}\cdot$$

Exam, November 2002

42. For the control system (described by the state equations)

$$\begin{aligned}\dot{x}_1 &= x_1 - x_3 + u(t)\\ \dot{x}_2 &= x_1 + 2x_2 + x_3\\ \dot{x}_3 &= 2x_1 + 2x_2 + 3x_3 + u(t)\end{aligned}$$

find a suitable feedback matrix K such that the closed loop system has eigenvalues -1 and $-1 \pm 2i$.

Exam, November 2002

43. Write the (dynamical) system described by the (second-order) differential equation

$$\ddot{z} + \dot{z} + z^3 = 0$$

in state space form, and then use a quadratic Lyapunov function of the form

$$V = a\,x_1^4 + b\,x_1^2 + c\,x_1x_2 + d\,x_2^2$$

in order to investigate for the stability nature of (the equilibrium state at) the origin. [Hint : Choose the coefficients $a, b, c \in \mathbb{R}$ such that $\dot{V} = -x_1^4 - x_2^4$.]

Exam, November 2002

44. Show that, for $t \in \mathbb{R}$,

$$\exp\left(t\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}\right) = e^t\begin{bmatrix}1 & 2t & 2t^2\\ 0 & 1 & 2t\\ 0 & 0 & 1\end{bmatrix}.$$

Class test, April 2003

45. For $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$, find the solution curve (in spectral form or otherwise). What happens when $x_1(0) = 2x_2(0)$?

Class test, April 2003

46. Consider the linear control system (described by the state equations)

$$\begin{aligned} \dot{x}_1 &= 2x_1 - x_2 \\ \dot{x}_2 &= -x_1 + 2x_2 + u(t). \end{aligned}$$

(a) Write the system in the form $\dot{x} = Ax + bu(t)$ with $A \in \mathbb{R}^{2\times 2}$ and $b \in \mathbb{R}^{2\times 1}$.

(b) Compute the state transition matrix $\Phi(t, 0) = \exp(tA)$ in TWO DIFFERENT ways.

(c) If $x_1(0) = x_2(0) = 0$ and $u(t) = 1$ for $t \geq 0$, determine $x_2(2)$.

Class test, April 2003

47. Determine the range of values of parameter $k \in \mathbb{R}$ such that the linear dynamical system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -5x_1 - k\,x_2 + (k-6)x_3 \end{aligned}$$

is asymptotically stable.

Class test, May 2003

48. An autonomous dynamical system is described by

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1 \\ \dot{x}_2 &= -x_1 - x_2 - x_1^2. \end{aligned}$$

(a) Determine the equilibrium state which is *not* at the origin.

(b) Transform the state equations (of the system) so that this point is transfered to the origin.

(c) Hence verify that this equilibrium state is *unstable*

Class test, May 2003

49. Let $A \in \mathbb{R}^{n \times n}$.

(a) Define the *transpose* matrix A^T and then show that the linear mapping

$$\mathcal{T} : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}, \quad A \mapsto \mathcal{T}(A) := A^T$$

enjoys the property $\mathcal{T}(AB) = \mathcal{T}(B)\mathcal{T}(A)$. Hence deduce that if the matrix A is invertible, then so is its transpose A^T, and

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T.$$

(b) Let $\mathcal{Q}$ denote the associated *quadratic form* (i.e. $x \in \mathbb{R}^n \mapsto x^T A x \in \mathbb{R}$). Explain what is meant by saying that $\mathcal{Q}$ is *positive definite* and *negative semi-definite*. State clearly necessary and sufficient conditions for positive definiteness and negative semi-definiteness, respectively.

(c) Assume that the matrix A is skew-symmetric (i.e. $A^T + A = 0$) and then show that $\mathcal{Q}(x) = 0$ for all $x \in \mathbb{R}^n$. Furthermore, show A can be written as $A_1 + A_2$, where A_1 is symmetric and A_2 is skew-symmetric. Hence deduce that

$$\mathcal{Q}(x) = x^T A_1 x \quad \text{for all } x \in \mathbb{R}^n.$$

Exam, June 2003

50. Let $A \in \mathbb{R}^{n\times n}$. Determine (by direct computation) the characteristic polynomial of the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_4 & -k_3 & -k_2 & -k_1 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

Exam, June 2003

51. Find the (invertible) matrix $T \in \mathbb{R}^{n\times n}$ such that the linear transformation $x \mapsto w := Tx$ transforms the linear control system given by

$$\dot{x} = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 3 \end{bmatrix} u(t)$$

into the canonical form. Write down the system in canonical form.

Exam, June 2003

52. Show that the control system (with outputs) described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - 3x_2 + u(t) \\ y &= x_1 + x_2 \end{aligned}$$

is completely controllable but *not* completely observable. Determine initial states $x(0)$ such that if $u(t) = 0$ for $t \geq 0$, then the output $y(\cdot)$ is identically zero for $t \geq 0$.

Exam, June 2003

53. For the control system described by

$$\begin{aligned} \dot{x}_1 &= x_2 + u_1(t) \\ \dot{x}_2 &= x_3 + u_2(t) \\ \dot{x}_3 &= 6x_1 - 11x_2 + 6x_3 + u_1(t) + u_2(t) \end{aligned}$$

find a suitable feedback matrix K such that the closed loop system has eigenvalues $1, 1,$ and 3.

Exam, June 2003

54. Investigate the stability nature of the equilibrium state at the origin for the (nonlinear) system given by

$$\begin{aligned}\dot{x}_1 &= -3x_2\\ \dot{x}_2 &= x_1 - x_2 + 2x_2^3.\end{aligned}$$

Exam, June 2003

55. Consider the matrices

$$A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}.$$

   (a) Find the eigenvalues and eigenvectors of A and B.

   (b) Compute

$$\exp(tA), \quad \exp(tB), \quad \text{and} \quad \exp(t(A+B)).$$

   (c) Compare $\exp(tA) \cdot \exp(tB)$ and $\exp(t(A+B))$.

Class test, March 2004

56. Find the solution curve of the linear control system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0$$

when

$$A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \quad B = \begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}, \quad x(0) = \begin{bmatrix}0\\ 0\end{bmatrix}, \quad u_1(t) - u_2(t) = 1 \ \text{ for } t \geq 0.$$

Class test, March 2004

57. Consider the matrix differential equation

$$\dot{W} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} W + W \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}, \quad W \in \mathbb{R}^{2\times 2}.$$

(a) Verify that $W(t) = \exp\left(t \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \cdot \exp\left(t \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right)$ is a solution curve through the identity (i.e. such that $W(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$).

(b) Can you find another solution curve through the identity ? Make a clear statement and then prove it.

Class test, March 2004

58. Consider the linear system

$$\dot{x} = \begin{bmatrix} \alpha & -4 \\ 3 & -\beta \end{bmatrix} x$$

where $\alpha, \beta > 0$ and $\alpha\beta = 12$. Investigate the stability nature of the system at the origin. For what values (if any) of the parameters α and β is the system asymptotically stable ?

Class test, May 2004

59. Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1^3 - 3x_1 + x_2 \\ \dot{x}_2 &= -2x_1. \end{aligned}$$

Investigate the stability nature of (the origin of) the system

(a) by using a suitable Lyapunov function.

(Hint : Try for a Lyapunov function of the form $V = ax_1^2 + bx_2^2$.)

(b) by linearizing the system.

Class test, May 2004

60. Let $A \in \mathbb{R}^{n \times n}$.

(a) Define the terms *eigenvalue* and *eigenvector* (of A), and then investigate the relationship between the eigenvalues of (the power) A^k and those of A. Make a clear statement and then prove it.

(b) Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be $r\,(\leq n)$ distinct eigenvalues of A with corresponding eigenvectors $w_1, w_2, \ldots, w_r$. Prove that the vectors $w_1, w_2, \ldots, w_r$ are *linearly independent.*

(c) The matrix A is said to be *nilpotent* if some power A^k is the zero matrix. Show that A is nilpotent if and only if all its eigenvalues are zero.

(d) Define the *exponential* of A and then calculate $\exp(A)$ for

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Exam, June 2004

61. Find the solution of

$$\dot{x} = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

when $u(t) = e^{2t}$, $t \geq 0$.

Exam, June 2004

62. Consider the control system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} x + bu \\ y &= cx. \end{aligned}$$

Find $b \in \mathbb{R}^{2 \times 1}$ and $c \in \mathbb{R}^{1 \times 2}$ such that the system is

(a) *not* completely controllable;

(b) completely observable.

When $c = \begin{bmatrix} 1 & 1 \end{bmatrix}$, determine (the initial state) $x(0)$ if $y(t) = e^t - 2e^{3t}$.

Exam, June 2004

63. For the control system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 + u \\
\dot{x}_3 &= x_1 - x_2 - 2x_3
\end{aligned}
$$

find a linear *feedback control* which makes all the eigenvalues (of the system) equal to -1.

Exam, June 2004

64. Define the term *Lyapunov function* (for a general *nonlinear* system) and then use a quadratic Lyapunov function $V(x) = x^T P x$ to investigate the stability of the system described by the equation

$$\ddot{z} + \epsilon\,(z^2 - 1)\dot{z} + z = 0, \quad \epsilon < 0.$$

Exam, June 2004

65. Find the (feedback) control u^* which minimizes the (quadratic) cost functional

$$\mathcal{J} = \frac{1}{2} x^T(t_1) M x(t_1) + \frac{1}{2} \int_0^{t_1} \left(x^T Q(t) x + u^T R(t) u \right) dt$$

subject to

$$\dot{x} = A(t)x + B(t)u(t) \quad \text{and} \quad x(0) = x_0.$$

(It is assumed that $R(t)$ is a positive definite, and M and $Q(t)$ are positive semi-definite symmetric matrices for $t \geq 0$.)

Exam, June 2004

66.

(a) Find the *eigenvalues* and *eigenvectors* of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) Compute the characteristic polynomial of the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c_3 & -c_2 & -c_1 \end{bmatrix}.$$

Class test, March 2005

67. Find *the* solution curve of the initialized linear dynamical system described by

$$\dot{x} = Ax, \quad x(0) = x_0$$

when

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

What happens when $x_1(0) = x_2(0)$?

Class test, March 2005

68. Consider a *linear* control system (with outputs) Σ given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

(where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times \ell}$ and $C \in \mathbb{R}^{n \times m}$). Assume that

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} \alpha \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & \beta \end{bmatrix}.$$

For what values of α and β is Σ completely controllable but *not* completely observable ?

Class test, May 2005

69. Consider a *linear* control system (with outputs) Σ given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

(where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times \ell}$ and $C \in \mathbb{R}^{n\times m}$).

(a) Derive the *transfer function matrix* $G(\cdot)$ associated with Σ.

(b) Assume that

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Compute the scalar transfer function $g(\cdot)$.

(c) Find a *minimal realization* of the scalar transfer function

$$g(s) = \frac{2s+7}{s^2-5s+6}.$$

Class test, May 2005

70. Let $M \in \mathbb{R}^{n\times n}$.

(a) Define the *trace* $\operatorname{tr}(M)$ of M, and then show that the function

$$\operatorname{tr} : \mathbb{R}^{n\times n} \to \mathbb{R}, \quad A \mapsto \operatorname{tr}(A)$$

is *linear*.

(b) Prove that (for any $A, B \in \mathbb{R}^{n\times n}$)

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

(c) Hence deduce that (for any invertible $n \times n$ matrix S)

$$\operatorname{tr}(SMS^{-1}) = \operatorname{tr}(M).$$

(d) Define the terms *eigenvalue* and *eigenvector* (of M), and then prove that if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (complex) eigenvalues of M (listed with their algebraic multiplicities), then

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \mathrm{tr}\,(M).$$

Exam, June 2005

71. Find the solution of

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

when $u(t) = e^t, \;\; t \geq 0$.

Exam, June 2005

72. Consider a single-input linear control system Σ given by

$$\dot{x} = Ax + bu$$

(where $A \in \mathbb{R}^{m\times m}$ and $b \in \mathbb{R}^{m\times 1}$).

(a) Explain what is meant by the *canonical form* of Σ.

(b) Prove that if

$$\mathrm{rank} \begin{bmatrix} b & Ab & A^2b & \cdots & A^{m-1}b \end{bmatrix} = m$$

then Σ can be transformed by a linear transformation $w = Tx$ into the canonical form.

(c) Reduce the single-input linear control system

$$\dot{x} = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

to the canonical form and determine the linear mapping $x = T^{-1}w$.

Exam, June 2005

73. Let $d, k > 0$. Consider the control system (with outputs) Σ described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -kx_1 - dx_2 + u \\ y &= x_1 - x_2. \end{aligned}$$

Write down the equations describing the dual control system Σ°, and then investigate for (complete) controllability and (complete) observability both control systems.

Exam, June 2005

74. Let $A \in \mathbb{R}^{m\times m}$ and $b \in \mathbb{R}^{m\times 1}$. Consider a single-input linear control system Σ given by

$$\dot{x} = Ax + bu.$$

(a) Given an arbitrary set $\Lambda = \{\theta_1, \dots, \theta_m\}$ of complex numbers (appearing in conjugate pairs), prove that if Σ is completely controllable, then there exists a *feedback matrix* K such that the eigenvalues of $A + bK$ are the set Λ.

(b) Application : Find a linear feedback control $u = Kx$ when

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \{-1, -1-2i, -1+2i\}.$$

Exam, June 2005

75. Consider the dynamical system

$$\begin{aligned} \dot{x}_1 &= -kx_1 - 3x_2 \\ \dot{x}_2 &= kx_1 - 2x_2, \quad k \in \mathbb{R}. \end{aligned}$$

(a) When $k = 1$, use a quadratic Lyapunov function

$$\begin{aligned} V(x) &= x^T P x \\ &= a x_1^2 + 2b x_1 x_2 + c x_2^2 \end{aligned}$$

with derivative $\dot{V} = -2\left(x_1^2 + x_2^2\right)$ to determine the stability of the system (at the origin).

(b) Using the *same* Lyapunov function, find sufficient conditions on k for the system to be asymptotically stable (at the origin).

Exam, June 2005

76. Find u^* so as to *minimize*

$$\mathcal{J} = \int_0^T dt$$

subject to

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad |u_i| \leq K_i, \quad i = 1, \dots, \ell \\ x(0) &= x_0 \\ x(T) &= 0. \end{aligned}$$

Exam, June 2005

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