Chapter 2

Linear Dynamical Systems

Topics :

1. Solution of Uncontrolled System
2. Solution of Controlled System
3. Time-Varying Systems
4. Relationship between State Space and Classical Forms

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A *control system* (with outputs) $\Sigma = (\mathbb{R}^m,\, \mathbb{R}^\ell,\, \mathcal{U},\, F,\, \mathbb{R}^n,\, h)$ is *linear* if the dynamics F is linear in (x,u), and the measurement function h is linear, for each fixed $t \in \mathbb{R}$. Such a control system is described by (state equation and observation equation)

$$\dot{x} = A(t)x + B(t)u(t) \quad \text{and} \quad y = C(t)x$$

where $A(t) \in \mathbb{R}^{m\times m}$, $B(t) \in \mathbb{R}^{m\times \ell}$, and $C(t) \in \mathbb{R}^{n\times m}$, each of whose entries is a (continuous) function of time. The system is called *time-invariant* if the structure is independent of time. A system that is not necessarily time-invariant is sometimes called, to emphasize the fact, a *time-varying* system. Sets of (scalar) state equations describing a linear (time-invariant) control system are the easiest to manage analytically and numerically, and the first model of a situation is often constructed to be linear for this reason.

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2.1 Solution of Uncontrolled System

To begin with we shall consider dynamical systems (i.e. systems without the presence of control variables). We may also refer to such systems as *uncontrolled* (or unforced) systems.

We discuss methods of finding the *solution* (*state vector*) $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix} \in \mathbb{R}^{m\times 1}$ of the (initialized) **linear dynamical system** described by

$$\boxed{\dot{x} = Ax, \quad x(0) = x_0.} \tag{2.1}$$

Here $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in \mathbb{R}^{m\times m}$ ($x \mapsto Ax$ represents the linear dynamics F) and $x_0 \in \mathbb{R}^m$ is the **initial state**.

Note : We *identify* the column matrix (or vector) $\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}^{m\times 1}$ with the m-tuple (or point) $(a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$, whenever appropriate. However, we *do not* identify the row matrix (or covector) $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \in \mathbb{R}^{1\times n}$ with $(a_1, a_2, \ldots, a_n)$ (but rather with the linear *functional* $(x_1, x_2, \cdots, x_n) \mapsto a_1x_1 + a_2x_2 + \cdots + a_nx_n$).

We shall assume that *all the eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *of* A *are distinct.*

Note : In fact, in real-life situations, this is not too severe a restriction, since if A does have repeated eigenvalues, very small perturbations in a few of its elements (which will only be known to a certain degree of accuracy) will suffice to separate these equal eigenvalues.

Spectral form

If w_i is an eigenvector corresponding to λ_i, then $w_1, w_2, \ldots, w_m$ are *linearly independent* (see **Exercise 4**), so we can express the solution of (2.1) as

$$x(t) = c_1(t)w_1 + c_2(t)w_2 + \cdots + c_m(t)w_m \tag{2.2}$$

where $c_i = c_i(t)\,,\quad i = 1, 2, \ldots, m$ are scalar functions of time.

Differentiation of (2.2) and substitution into (2.1) gives :

$$\sum_{i=1}^{m} \dot{c}_i(t)w_i = A\sum_{i=1}^{m} c_i(t)w_i = \sum_{i=1}^{m} c_i(t)\lambda_i w_i.$$

Hence, by the independence of the $w_i \quad (i = 1, 2, \ldots, m)$

$$\dot{c}_i = \lambda_i\, c_i\,, \quad i = 1, 2, \ldots, m$$

and these equations have the solution

$$c_i(t) = c_i(0)\, e^{\lambda_i t}\,, \quad i = 1, 2, \ldots, m$$

giving

$$x(t) = \sum_{i=1}^{m} c_i(0)\, e^{\lambda_i t}\, w_i. \tag{2.3}$$

Let W denote the matrix whose columns are $w_1, w_2, \ldots, w_m$; that is,

$$W = \begin{bmatrix} w_1 & w_2 & \ldots & w_m \end{bmatrix}.$$

We shall denote by $v_1, v_2, \ldots, v_m$ the rows of the matrix W^{-1}; that is,

$$\begin{bmatrix} w_1 & w_2 & \ldots & w_m \end{bmatrix}^{-1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}.$$

Since we have

$$v_i w_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

multiplying (2.3) on the left by v_i and setting $t = 0$ in the resulting expression gives

$$v_i x(0) = c_i(0)\,, \quad i = 1, 2, \ldots, m.$$

Thus the solution of (2.1) is

$$\boxed{x(t) = \sum_{i=1}^{m} (v_i x(0))\, e^{\lambda_i t}\, w_i.} \tag{2.4}$$

Expression (2.4) depends *only* upon the initial condition and the eigenvalues and eigenvectors of A, and for this reason is referred to as the **spectral form solution**.

2.1.1 EXAMPLE. Find the general solution of the uncontrolled system :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

SOLUTION : The characteristic equation of A is

$$|\, \lambda I_2 - A \,| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} = 0 \iff \lambda^2 + 3\lambda + 2 = 0$$

which gives

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = -1\,.$$

(i) $\lambda = -2$.

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} 2w_{11} + w_{21} = 0 \\ -2w_{11} - w_{21} = 0 \end{cases}$$

which implies

$$w_{21} = -2w_{11}$$

and thus (we can choose)

$$w_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

(ii) $\lambda = -1$.

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} w_{12} \\ w_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} w_{12} + w_{22} = 0 \\ -2w_{12} - 2w_{22} = 0 \end{cases}$$

which implies

$$w_{22} = -w_{12}$$

and thus (we can choose)

$$w_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We have

$$W = \begin{bmatrix} w_1 & w_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \Rightarrow W^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix};$$

so

$$v_1 = \begin{bmatrix} -1 & -1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

Finally, we get

$$\begin{aligned} x(t) &= (v_1 x(0))\, e^{-2t}\, w_1 + (v_2 x(0))\, e^{-t}\, w_2 \\ &= (-x_1(0) - x_2(0))\, e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (2x_1(0) + x_2(0))\, e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

or

$$\begin{cases} x_1(t) = -\,(x_1(0) + x_2(0))\, e^{-2t} + (2x_1(0) + x_2(0))\, e^{-t} \\ x_2(t) = 2\,(x_1(0) + x_2(0))\, e^{-2t} - (2x_1(0) + x_2(0))\, e^{-t}. \end{cases}$$

Exponential form

We now present a different approach to solving equation (2.1) which avoids the need to calculate the eigenvectors of A.

Recall the definition of the *matrix exponential*

$$\boxed{\exp(tA) := I_m + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots} \tag{2.5}$$

2.1.2 Lemma. *Let* $A \in \mathbb{R}^{m\times m}$*. Then*

$$\frac{d}{dt}\left(\exp(tA)\right) = A\exp(tA) = \exp(tA)A.$$

Proof : We have

$$\begin{aligned}
\frac{d}{dt}\exp(tA) &= \lim_{h\to 0}\frac{1}{h}\left(\exp((t+h)A) - \exp(tA)\right)\\
&= \lim_{h\to 0}\frac{1}{h}\left(\exp(tA)\cdot\exp(hA) - \exp(tA)\right)\\
&= \exp(tA)\lim_{h\to 0}\frac{1}{h}\left(\exp(hA) - I_m\right)\\
&= \exp(tA)\lim_{h\to 0}\lim_{k\to\infty}\left(A + \frac{h}{2!}A + \cdots + \frac{h^{k-1}}{k!}A^k\right)\\
&= \exp(tA)A.
\end{aligned}$$

(Two convergent *limit processes* can be interchanged if one of them converges uniformly.) Observe that A commutes with each term of the (absolutely convergent) series for $\exp(tA)$, hence with $\exp(tA)$. This proves the lemma.□

By the preceding lemma, if $x(t) = \exp(tA)x_0$, then

$$\dot{x}(t) = \frac{d}{dt}\exp(tA)x_0 = A\exp(tA)x_0 = Ax(t)$$

for all $t \in \mathbb{R}$. Also,

$$x(0) = I_m x_0 = x_0.$$

Thus $x(t) = \exp(tA)x_0$ is *a* solution of (2.1).

To see that this is the *only* solution, let $x(\cdot)$ be any solution of (2.1) and set

$$y(t) = \exp(-tA)x(t).$$

Then (from the above lemma and the fact that $x(\cdot)$ is a solution of (2.1))

$$\begin{aligned}
\dot{y}(t) &= -A\exp(-tA)x(t) + \exp(-tA)\dot{x}(t)\\
&= -A\exp(-tA)x(t) + \exp(-tA)Ax(t)\\
&= 0
\end{aligned}$$

for all $t \in \mathbb{R}$ since $\exp(-tA)$ and A commute. Thus, $y(t)$ is a constant. Setting $t = 0$ shows that $y(t) = x_0$ and therefore *any solution of (2.1) is given by*

$$x(t) = \exp(tA)y(t) = \exp(tA)x_0.$$

Hence

$$\boxed{x(t) = \exp(tA)\, x_0} \tag{2.6}$$

does represent *the* solution of (2.1).

NOTE : In case the initial condition $x(0) = x_0$ is replaced by the slighty more general one $x(t_0) = x_0$, the solution is often written as

$$x(t) = \Phi(t, t_0)x_0. \tag{2.7}$$

One refers to (the matrix)

$$\boxed{\Phi(t, t_0) := \exp((t - t_0)A)} \tag{2.8}$$

as the **state transition matrix** (since it relates the state at any time t to the state at any other time t_0).

2.1.3 PROPOSITION. *The state transition matrix $\Phi(t, t_0)$ has the following properties :*

(a) $\dfrac{d}{dt}\Phi(t, t_0) = A\Phi(t, t_0)$.

(b) $\Phi(t_0, t_0) = I_m$.

(c) $\Phi(t_0, t) = \Phi^{-1}(t, t_0)$.

(d) $\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$.

PROOF : We have

(a) $\dfrac{d}{dt}\Phi(t, t_0) = \dfrac{d}{dt}\exp((t - t_0)A) = A\exp((t - t_0)A) = A\Phi(t, t_0)$.

(b) $\Phi(t, t) = \exp((t - t)A) = \exp(0) = I_m$.

(c) $\Phi^{-1}(t, t_0) = (\exp((t - t_0)A))^{-1} = \exp(-(t - t_0)A) = \exp((t_0 - t)A) = \Phi(t_0, t)$.

(d) $\Phi(t,t_1)\Phi(t_1,t_0) = \exp((t-t_1)A)\cdot\exp((t_1-t_0)A) = \exp\left((t-t_1+t_1-t_0)A\right) = \exp((t-t_0)A) = \Phi(t,t_0)$.

$\square$

Note : The matrix-valued mapping $X(t) = \Phi(t,t_0)$ (or *curve* in $\mathbb{R}^{m\times m}$) is the *unique* solution of the matrix differential equation

$$\dot{X} = AX, \quad X \in \mathbb{R}^{m\times m}$$

subject to the initial condition $X(t_0) = I_m$.

2.1.4 Example. (Simple harmonic motion) Consider a unit mass connected to a support through a spring whose spring constant is unity. If z measures the displacement of the mass from equilibrium, then

$$\ddot{z} + z = 0.$$

Letting $x_1 = z$ and $x_2 = \dot{z}$ gives

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Note : The associated transition matrix $\Phi(t,t_0)$ has the form

$$\Phi(t,t_0) = \begin{bmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) \\ \phi_{21}(t,t_0) & \phi_{22}(t,t_0) \end{bmatrix}$$

and therefore satisfies

$$\begin{bmatrix} \dot{\phi}_{11}(t,t_0) & \dot{\phi}_{12}(t,t_0) \\ \dot{\phi}_{21}(t,t_0) & \dot{\phi}_{22}(t,t_0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_{11}(t,t_0) & \phi_{12}(t,t_0) \\ \phi_{21}(t,t_0) & \phi_{22}(t,t_0) \end{bmatrix}$$

with the initial condition

$$\Phi(t_0,t_0) = I_m.$$

What is the physical interpretation of $\Phi(t,t_0)$ in this case ? The first column of $\Phi(t,t_0)$ has as its first entry the position as a function of time which results when the mass is displaced by one unit and released at t_0 with zero velocity. The second entry in the first column is the corresponding velocity. The second column of $\Phi(t, t_0)$ has as its first entry the position as a function of time which results when the mass is started from zero displacement but with unit velocity at $t = t_0$. The second entry in the second column is the corresponding velocity.

The series for computing $\Phi(t, t_0)$ in this case is easily summed because

$$A^k = \begin{cases} A & \text{if } k = 4p + 1 \\ -I_2 & \text{if } k = 4p + 2 \\ -A & \text{if } k = 4p + 3 \\ I_2 & \text{if } k = 4p. \end{cases}$$

A short calculation gives

$$\Phi(t, t_0) = \begin{bmatrix} \cos(t - t_0) & \sin(t - t_0) \\ -\sin(t - t_0) & \cos(t - t_0) \end{bmatrix}.$$

Exercise 21 Work out the preceding computation.

2.1.5 EXAMPLE. (Satellite problem) In section **1.3** we introduced the equations of a unit mass in an inverse square law force field. These were then *linearized* about a circular orbit to get

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ u_1(t) \\ 0 \\ u_2(t) \end{bmatrix}.$$

The series for computing $\Phi(t,0)$ can be summed to get

$$\Phi(t,0) = \begin{bmatrix} 4-3\cos\omega t & \dfrac{\sin\omega t}{\omega} & 0 & \dfrac{2(1-\cos\omega t)}{\omega} \\ 3\omega\sin\omega t & \cos\omega t & 0 & 2\sin\omega t \\ 6(-\omega t+\sin\omega t) & -\dfrac{2(1-\cos\omega t)}{\omega} & 1 & \dfrac{-3\omega t+4\sin\omega t}{\omega} \\ 6\omega(-1+\cos\omega t) & -2\sin\omega t & 0 & -3+4\cos\omega t \end{bmatrix}.$$

Evaluation of the matrix exponential

Evaluation of $\exp(tA)$, when *all the eigenvalues* $\lambda_1, \lambda_2, \dots, \lambda_m$ *are distinct*, can be achieved by **Sylvester's formula** which gives

$$\boxed{\exp(tA) = \sum_{k=1}^{m} e^{\lambda_k t} Z_k} \tag{2.9}$$

where

$$Z_k = \prod_{\substack{i=1 \\ i\neq k}}^{m} \frac{A-\lambda_i I_m}{\lambda_k-\lambda_i}, \quad k = 1,2,\dots,m.$$

NOTE : Since the Z_k $(k = 1,2,\dots,m)$ in (2.9) are constant matrices depending only on A and its eigenvalues, the solution in the form given in (2.9) requires calculation of only the eigenvalues of A.

2.1.6 EXAMPLE. Consider again the uncontrolled system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The solution is

$$x(t) = \exp(tA)\, x_0 = \left(\sum_{k=1}^{2} e^{\lambda_k t} Z_k \right) x_0$$

where

$$Z_k = \prod_{\substack{i=1 \\ i\neq k}}^{2} \frac{A - \lambda_i I_2}{\lambda_k - \lambda_i}\,, \quad k = 1, 2.$$

We have

$$Z_1 = \frac{A-(-1)I_2}{-2-(-1)} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}; \quad Z_2 = \frac{A-(-2)I_2}{-1-(-2)} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}.$$

Hence,

$$\begin{aligned} x(t) &= \left(e^{-2t} Z_1 + e^{-t} Z_2\right) x_0 \\ &= \left(e^{-2t}\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} + e^{-t}\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\right)\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \end{aligned}$$

or

$$\begin{cases} x_1(t) = -\left(x_1(0) + x_2(0)\right) e^{-2t} + \left(2x_1(0) + x_2(0)\right) e^{-t} \\ \\ x_2(t) = 2\left(x_1(0) + x_2(0)\right) e^{-2t} - \left(2x_1(0) + x_2(0)\right) e^{-t}. \end{cases}$$

An alternative way of evaluating $\exp(tA)$ (again, when *all the eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *are distinct*), is as follows.

We can write

$$e^{t\lambda} = q(\lambda)\cdot \mathrm{char}_A(\lambda) + r(\lambda) \tag{2.10}$$

where $\deg(r) < m$. Since (2.10) is an identity, we have

$$\exp(tA) \equiv q(A)\cdot \mathrm{char}_A(A) + r(A)$$

which by the **Cayley-Hamilton Theorem** reduces to

$$\exp(tA) \equiv r(A)$$

showing that $\exp(tA)$ *can be represented by a finite sum of powers of* A *of degree not exceeding* $m-1$. Then m coefficients of $r(\lambda)$ are functions of t obtained from the solution of the system of m linear equations

$$e^{\lambda_i t} = r(\lambda_i)\,, \quad i = 1, 2, \ldots, m.$$

2.1.7 Example. Consider once again the uncontrolled system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Since $m = 2$, the polynomial $r(\lambda)$ can be written

$$r(\lambda) = r_0\lambda + r_1$$

and so we have

$$\begin{cases} e^{-t} = r_1 - r_0 \\ \\ e^{-2t} = r_1 - 2r_0 \end{cases}$$

which gives $r_0 = e^{-t} - e^{-2t}$ and $r_1 = 2e^{-t} - e^{-2t}$. Hence, the solution is

$$\begin{aligned} x(t) &= \exp(tA)\, x_0 = (r_0 A + r_1 I_2) x_0 \\ &= \left((e^{-t} - e^{-2t}) \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + (2e^{-t} - e^{-2t}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \end{aligned}$$

or

$$\begin{cases} x_1(t) = -\left(x_1(0) + x_2(0)\right) e^{-2t} + \left(2x_1(0) + x_2(0)\right) e^{-t} \\ \\ x_2(t) = 2\left(x_1(0) + x_2(0)\right) e^{-2t} - \left(2x_1(0) + x_2(0)\right) e^{-t}. \end{cases}$$

2.2 Solution of Controlled System

Consider the (initialized) linear control system, written in *state space form*,

$$\boxed{\dot{x} = Ax + Bu(t), \quad x(0) = x_0} \tag{2.11}$$

where $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times \ell}$, and $\ell \le m$.

After multiplication of both sides of (2.11), on the left, by $\exp(-tA)$, the equation can be written

$$\frac{d}{dt}\left(\exp(-tA)x\right) = \exp(-tA)Bu \tag{2.12}$$

which produces

$$x(t) = \exp(tA)\left[x_0 + \int_0^t \exp(-\tau A)Bu(\tau)\,d\tau\right]. \tag{2.13}$$

If the initial condition is $x(t_0) = x_0$, then integration of (2.12) from t_0 to t and use of the definition of Φ gives

$$\boxed{x(t) = \Phi(t,t_0)\left[x_0 + \int_{t_0}^t \Phi(t_0,\tau)Bu(\tau)\,d\tau\right].} \tag{2.14}$$

NOTE : If $u(t)$ is known for $t \geq t_0$, then $x(t)$ can be determined by finding the state transition matrix and carrying out the integration in (2.14).

2.2.1 EXAMPLE. Consider the *equation of motion*

$$\ddot{z} = u(t)$$

of a unit mass moving in a straight line, subject to an external force $u(t)$, $z(t)$ being the displacement from some fixed point. In state space form, taking

$$x_1 = z \quad \text{and} \quad x_2 = \dot{z}$$

as state variables, this becomes

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) = Ax + Bu(t).$$

Since here we have $A^2 = 0$, $\exp(tA) = I_2 + tA$, and so

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\int_0^t \begin{bmatrix} 1 & -\tau \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau)\,d\tau.$$

Solving for $x_1(t)$ leads to

$$z(t) = z(0) + t\dot{z}(0) + \int_o^t (t-\tau)u(\tau)\,d\tau$$

where $\dot{z}(0)$ denotes the initial velocity of the mass.

2.2.2 Example. We are now in a position to express the solution of the linearized equations describing the motion of a satellite in a near circular orbit. We have

$$\begin{bmatrix} x_1(t) \\ \\ x_2(t) \\ \\ x_3(t) \\ \\ x_4(t) \end{bmatrix} = \begin{bmatrix} 4-3\cos\omega t & \frac{\sin\omega t}{\omega} & 0 & \frac{2(1-\cos\omega t)}{\omega} \\ \\ 3\omega\sin\omega t & \cos\omega t & 0 & 2\sin\omega t \\ \\ 6(-\omega t+\sin\omega t) & -\frac{2(1-\cos\omega t)}{\omega} & 1 & \frac{-3\omega t+4\sin\omega t}{\omega} \\ \\ 6\omega(-1+\cos\omega t) & -2\sin\omega t & 0 & -3+4\cos\omega t \end{bmatrix} \begin{bmatrix} x_1(0) \\ \\ x_2(0) \\ \\ x_3(0) \\ \\ x_4(0) \end{bmatrix} +$$

$$+\int_0^t \left(\begin{bmatrix} \frac{\sin\omega(t-\tau)}{\omega} \\ \\ \cos\omega(t-\tau) \\ \\ -\frac{2(1-\cos\omega(t-\tau))}{\omega} \\ \\ -2\sin\omega(t-\tau) \end{bmatrix} u_1(\tau) + \begin{bmatrix} \frac{2(1-\cos\omega(t-\tau))}{\omega} \\ \\ 2\sin\omega(t-\tau) \\ \\ \frac{-3\omega(t-\tau)+4\sin\omega(t-\tau)}{\omega} \\ \\ -3+4\cos\omega(t-\tau) \end{bmatrix} u_2(\tau) \right) d\tau.$$

2.3 Time-varying Systems

Of considerable importance in many applications are linear systems in which the elements of A and B are (continuous) functions of time for $t \geq 0$.

Note : In general, it will *not* be possible to give explicit expressions for solutions and we shall content ourselves with obtaining some general properties.

We first consider the *uncontrolled* case

$$\boxed{\dot{x} = A(t)x\,, \quad x(0) = x_0.} \tag{2.15}$$

2.3.1 Theorem. (Existence and Uniqueness Theorem) *If the matrix-valued mapping*

$$A : [0,\infty) \to \mathbb{R}^{m\times m}, \quad t \mapsto A(t)$$

is continuous, then (2.15) has a unique solution

$$x(t) = X(t)x_0, \qquad t \geq 0$$

where $X(\cdot)$ *is the unique matrix-valued mapping (or curve in* $\mathbb{R}^{m\times m}$*) satisfying*

$$\dot{X} = A(t)X, \quad X(0) = I_m. \tag{2.16}$$

Proof : We shall use the *method of successive approximations* to establish the existence of a solution of (2.16). In place of (2.16), we consider the *integral equation*

$$X = I_m + \int_0^t A(\tau)X\,d\tau. \tag{2.17}$$

Define the sequence $(X_k)_{k\geq 0}$ of matrices (in fact, of matrix-valued mappings) as follows :

$$\begin{aligned} X_0 &= I_m \\ X_{k+1} &= I_m + \int_0^t A(\tau)X_k\,d\tau\,, \quad k = 0,1,2,\dots \end{aligned}$$

Then we have

$$X_{k+1} - X_k = \int_0^t A(\tau)(X_k - X_{k-1})\,d\tau\,, \quad k = 1,2,\dots$$

Let

$$\nu = \max_{0\leq t\leq t_1} \|A(t)\|$$

where

$$\|A(t)\| := \sum_{i,j=1}^m |a_{ij}(t)|.$$

Note : *Any* matrix norm (on $\mathbb{R}^{m\times m}$) will do.

We have

$$\begin{aligned} \|X_{k+1} - X_k\| &= \|\int_0^t A(\tau)(X_k - X_{k-1})\,d\tau\| \\ &\leq \int_0^t \|A(\tau)\|\|X_k - X_{k-1}\|\,d\tau \\ &\leq \nu\int_0^t \|X_k - X_{k-1}\|\,d\tau \end{aligned}$$

for $0 \leq t \leq t_1$. Since, in this same interval,

$$\|X_1 - X_0\| \leq \int_0^t \|A(\tau)\| \, d\tau \leq \nu t$$

we have *inductively*

$$\|X_{k+1} - X_k\| \leq M_{k+1} := \frac{\nu^{k+1} t_1^{k+1}}{(k+1)!} \quad \text{for } \ 0 \leq t \leq t_1.$$

NOTE : The WEIERSTRASS M-TEST states that *if* $\xi_k : [t_0, t_1] \to \mathbb{R}^{m \times m}$ *are continuous and*

- $\|\xi_k(t)\| \leq M_k$ *for every* k
- $\displaystyle\sum_{k=0}^{\infty} M_k < \infty$

then the series $\displaystyle\sum_{k \geq 0} \xi_k(t)$ *converges uniformly and absolutely on the interval* $[t_0, t_1]$.

Hence, the (matrix-valued mapping) series

$$X_0 + \sum_{k \geq 0} (X_{k+1} - X_k)$$

converges uniformly for $0 \leq t \leq t_1$. Consequently, (X_k) converges *uniformly* and *absolutely* to a matrix-valued mapping $X(\cdot)$, which satisfies (2.17), and thus (2.16).

Since, by assumption, $A(\cdot)$ is continuous for $t \geq 0$, *we must take* t_1 *arbitrarily large.* We thus obtain a solution valid for $t \geq 0$.

It is easy to verify that $x(t) = X(t)x_0$ is a solution of (2.15), satisfying the required initial condition.

Let us now establish uniqueness of this solution. Let Y be another solution of (2.16). Then Y satisfies (2.17), and thus we have the relation

$$X - Y = \int_0^t A(\tau)(X(\tau) - Y(\tau)) \, d\tau.$$

Hence

$$\|X - Y\| \leq \int_0^t \|A(\tau)\| \|X(\tau) - Y(\tau)\| \, d\tau.$$

Since Y is differentiable, hence continuous, define

$$\nu_1 := \max_{0\leq t\leq t_1} \|X(t) - Y(t)\|.$$

We obtain

$$\|X - Y\| \leq \nu_1 \int_0^t \|A(\tau)\|\, d\tau, \quad 0 \leq t \leq t_1.$$

Using this bound, we obtain

$$\begin{aligned}\|X - Y\| &\leq \nu_1 \int_0^t \|A(\tau)\| \left(\int_0^\tau \|A(\sigma)\|\, d\sigma \right) d\tau \\ &\leq \frac{\nu_1 \left(\int_0^t \|A(\tau)\|\, d\tau \right)^2}{2}.\end{aligned}$$

Iterating, we get

$$\|X - Y\| \leq \frac{\nu_1^k \left(\int_0^t \|A(\tau)\|\, d\tau \right)^{k+1}}{(k+1)!}.$$

Letting $k \to \infty$, we see that $\|X - Y\| \leq 0$. Hence $X \equiv Y$.

Exercise 22 Show that

$$\lim_{k\to\infty} \frac{\alpha^k}{(k+1)!} = 0 \qquad (\alpha > 0).$$

Having obtained the matrix X, it is easy to see that $x(t) = X(t)x_0$ is a solution of (2.15). Since the uniqueness of solutions of (2.15) is readily established by means of the same argument as above, it is easy to see that $x(t) = X(t)x_0$ is *the* solution.

□

NOTE : We can no longer define a matrix exponential, but there is a result corresponding to the fact that $\exp(tA)$ is nonsingular when A is constant. We can write $x(t) = \Phi(t, 0)x_0$, where $\Phi(t, 0)$ has the form

$$I_m + \int_0^t A(\tau)\, d\tau + \int_0^t A(\tau_1) \int_0^{\tau_1} A(\tau_2)\, d\tau_2\, d\tau_1 + \int_0^t A(\tau_1) \int_0^{\tau_1} A(\tau_2) \int_0^{\tau_2} A(\tau_3)\, d\tau_3\, d\tau_2\, d\tau_1 \cdots$$

(the PEANO-BAKER SERIES).

Some remarks and corollaries

2.3.2 PROPOSITION. *In the* EXISTENCE AND UNIQUENESS THEOREM *the matrix* $X(t)$ *is nonsingular (for every* $t \geq 0$*).*

PROOF : Define a matrix-valued mapping $Y(\cdot)$ as the solution of

$$\dot{Y} = -YA(t), \quad Y(0) = I_m. \tag{2.18}$$

(Such a mapping exists and is unique by an argument virtually identical to that which is used in the proof of the EXISTENCE AND UNIQUENESS THEOREM.)

Now

$$\frac{d}{dt}(YX) = \dot{Y}X + Y\dot{X} = -YAX + YAX = 0$$

so $Y(t)X(t)$ is equal to a constant matrix, which must be the unit matrix because of condition at $t = 0$.

Exercise 23 Show that (for every $t \geq 0$)

$$\det\,(X(t)) \neq 0.$$

Hence $X(t)$ is nonsingular (and its inverse is in fact $Y(t)$). □

We can also generalize the idea of *state transition matrix* by writing

$$\boxed{\Phi(t, t_0) := X(t)X^{-1}(t_0)} \tag{2.19}$$

which exists for all $t, t_0 \geq 0$. It is easy to verify that

$$x(t) = \Phi(t, t_0)x_0 \tag{2.20}$$

is the solution of (2.15). Also, $\Phi(t, t_0)^{-1} = \Phi(t_0, t)$.

NOTE : The expression (2.20) has the same form as that for the time invariant case. However, it is most interesting that although, in general, it is *not* possible to obtain an analytic expression for the solution of (2.16), and therefore for $\Phi(t, t_0)$ in (2.20), this latter matrix possesses precisely the same properties as those for the constant case.

When $m = 1$, we can see that

$$\Phi(t, t_0) = \exp\left(\int_{t_0}^{t} A(\tau)\, d\tau\right).$$

Note that the formula above is *generally not true.* However, one can show that it does hold if

$$A(t) \int_{t_0}^{t} A(\tau)\, d\tau = \left(\int_{t_0}^{t} A(\tau)\, d\tau\right) A(t) \quad \text{for all } t.$$

Otherwise this is not necessarily true and *the state transition matrix is not necessarily the exponential of the integral of* A.

The following result is interesting. We shall omit the proof.

2.3.3 Proposition. *If* $\Phi(t, t_0)$ *is the state transition matrix for*

$$\dot{x} = A(t)x\,, \quad x(t_0) = x_0$$

then

$$\det\,(\Phi(t, t_0)) = e^{\int_{t_0}^{t} \mathrm{tr}\,(A(\tau))\, d\tau}.$$

A further correspondence with the time-invariant case is the following result.

2.3.4 Proposition. *The solution of*

$$\dot{x} = A(t)x + B(t)u(t)\,, \quad x(t_0) = x_0 \tag{2.21}$$

is given by

$$\boxed{x(t) = \Phi(t, t_0)\left[x_0 + \int_{t_0}^{t} \Phi(t_0, \tau)B(\tau)u(\tau)\, d\tau\right]} \tag{2.22}$$

where $\Phi(t, t_0)$ *is defined in (2.19).*

Proof : Put

$$x = X(t)w \iff X^{-1}(t)x = w$$

where $X(t)$ is defined in (2.16). Substitution into (2.21) produces

$$\dot{x} = AXw + X\dot{w} = AXw + Bu(t).$$

Hence $X(t)\dot{w} = Bu(t)$ and so $\dot{w} = X^{-1}(t)Bu(t)$, which gives

$$w(t) = w(t_0) + \int_{t_0}^{t} X^{-1}(\tau)B(\tau)u(\tau)\,d\tau.$$

The desired expression then follows using $x_0 = X(t_0)w(t_0)$ and (2.19). Indeed,

$$x_0 = x(t_0) = X(t_0)w(t_0) \Rightarrow w(t_0) = X^{-1}(t_0)x_0$$

and we have

$$\begin{aligned} x(t) = X(t)w &= X(t)\left[X^{-1}(t_0)x_0 + \int_{t_0}^{t} X^{-1}(\tau)B(\tau)u(\tau)\,d\tau\right] \\ &= X(t)X^{-1}(t_0)\left[x_0 + \int_{t_0}^{t} X(t_0)X^{-1}(\tau)B(\tau)u(\tau)\,d\tau\right] \\ &= \Phi(t,t_0)\left[x_0 + \int_{t_0}^{t} \Phi(t_0,\tau)B(\tau)u(\tau)\,d\tau\right]. \end{aligned}$$

□

2.4 Relationship between State Space and Classical Forms

Classical linear control theory deals with scalar ODEs of the form

$$z^{(m)} + k_1 z^{(m-1)} + \cdots + k_{m-1} z^{(1)} + k_m z = \beta_0 u^{(\ell)} + \beta_1 u^{(\ell-1)} + \cdots + \beta_\ell u$$

where $k_1, k_2, \ldots, k_m$ and $\beta_0, \beta_1, \ldots, \beta_\ell$ are constants; it is assumed that $\ell < m$.

We shall consider a simplified form

$$\boxed{z^{(m)} + k_1 z^{(m-1)} + \cdots + k_m z = u(t)} \tag{2.23}$$

where $u(\cdot)$ is the single control variable.

It is easy to write (2.23) in matrix form by taking as state variables

$$w_1 = z, \quad w_2 = z^{(1)}, \quad \ldots, \quad w_m = z^{(m-1)}. \tag{2.24}$$

Since

$$\dot{w}_i = w_{i+1}\,, \quad i = 1, 2, \ldots, m-1$$

(2.23) and (2.24) lead to the *state space form*

$$\boxed{\dot{w} = Cw + du(t)} \tag{2.25}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -k_m & -k_{m-1} & -k_{m-2} & \ldots & -k_1 \end{bmatrix}, \; w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}, \; d = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{2.26}$$

The matrix C is called the **companion form matrix**.

Exercise 24 Show that the characteristic polynomial of C is

$$\mathrm{char}_C(\lambda) = \lambda^m + k_1\lambda^{m-1} + k_2\lambda^{m-2} + \cdots + k_m \tag{2.27}$$

which has the same coefficients as those in (2.23).

The state space form (2.25) is very special; we call it the **canonical form**.

NOTE : The classical form (2.23) and the canonical form are equivalent.

Having seen that (2.23) can be put into matrix form, a natural question is to ask whether the converse hold : *can any linear system in state space form with a single control variable*

$$\dot{x} = Ax + bu(t)$$

be put into the classical form (2.23) ?

2.4.1 Example. Consider the linear system in state space form with a single control variable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t). \tag{2.28}$$

The (state) equations describing the system are

$$\begin{cases} \dot{x}_1 = -2x_1 + 2x_2 + u(t) \\ \\ \dot{x}_2 = x_1 - x_2. \end{cases}$$

Differentiating the second equation we get

$$\ddot{x}_2 = \dot{x}_1 - \dot{x}_2 = (-2x_1 + 2x_2 + u(t)) - (x_1 - x_2) = -3x_1 + 3x_2 + u(t).$$

Hence

$$\ddot{x}_2 + 3\dot{x}_2 = u(t).$$

This second-order ODE for x_2 has the form (2.23) for $m = 2$. Its associated canonical form is

$$\dot{w} = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)\,. \tag{2.29}$$

We expect that there is a transformation on $\mathbb{R}^2$ that transforms our original control system (2.28) to the canonical form (2.29). Since the differential equations are linear, we expect that the transformation is linear, say $w = Tx$. Differentiation than gives

$$\dot{w} = TAT^{-1}w + Tbu(t).$$

If we set

$$w_1 = x_2 \quad \text{and} \quad w_2 = \dot{x}_2$$

then

$$\begin{cases} w_1 = x_2 \\ \\ w_2 = x_1 - x_2 \end{cases}$$

so a transformation transforming (2.28) into (2.29) is given by the matrix

$$T = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

2.4.2 EXAMPLE. *The control system* in state space form

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + bu(t)$$

(where $b \in \mathbb{R}^{2\times 1}$) *cannot be transformed to the canonical form* (or, equivalently, to the classical form).

Exercise 25 Prove the preceding statement.

We would like to determine *when* and *how* such a procedure can be carried out in general. Thus the natural questions concerning existence, uniqueness, and computation of T arise. The answer to these questions is provided by the following result.

2.4.3 THEOREM. *A linear control system in state space form*

$$\dot{x} = Ax + bu(t)$$

(where $A \in \mathbb{R}^{m\times m}$ and $0 \neq b \in \mathbb{R}^{m\times 1}$) can be transformed by a linear transformation (i.e. invertible linear mapping)

$$w = Tx$$

into the canonical form

$$\dot{w} = Cw + du(t)$$

where C and d are given by (2.26), provided

$$\boxed{\operatorname{rank}\begin{bmatrix} b & Ab & A^2b & \ldots & A^{m-1}b \end{bmatrix} = m.} \tag{2.30}$$

Conversely, if such a transformation T exists, then (2.30) holds.

Proof : ($\Leftarrow$) *Sufficiency.* Substitution of $w = Tx$ into

$$\dot{x} = Ax + bu$$

produces

$$\dot{w} = TAT^{-1}w + Tbu.$$

We take

$$T = \begin{bmatrix} \tau \\ \tau A \\ \tau A^2 \\ \vdots \\ \tau A^{m-1} \end{bmatrix}$$

where τ is any row m-vector such that T is nonsingular, *assuming for the present that at least one suitable τ exists.*

Denote the columns of T^{-1} by $s_1, s_2, \dots, s_m$ and consider

$$TAT^{-1} = \begin{bmatrix} \tau A s_1 & \tau A s_2 & \dots & \tau A s_m \\ \tau A^2 s_1 & \tau A^2 s_2 & \dots & \tau A^2 s_m \\ \vdots & \vdots & & \vdots \\ \tau A^m s_1 & \tau A^m s_2 & \dots & \tau A^m s_m \end{bmatrix}.$$

Comparison with the identity $TT^{-1} = I_m$ (that is,

$$\begin{bmatrix} \tau s_1 & \tau s_2 & \dots & \tau s_m \\ \tau A s_1 & \tau A s_2 & \dots & \tau A s_m \\ \vdots & \vdots & & \vdots \\ \tau A^{m-1} s_1 & \tau A^{m-1} s_2 & \dots & \tau A^{m-1} s_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix})$$

establishes that the i^{th} row of TAT^{-1} is the $(i+1)^{th}$ row of I_m $(1 = 1, 2, \dots, m-1)$, so TAT^{-1} has the same form as C in (2.26), with last row given by

$$k_i = -\tau A^m s_{m-i+1}\,, \quad i = 1, 2, \dots, m.$$

For

$$\dot{w} = TAT^{-1}w + Tbu$$

to be identical to

$$\dot{w} = Cw + du$$

we must also have

$$Tb = d$$

and substitution of T into this relation gives

$$\tau b = 0, \quad \tau Ab = 0, \quad \ldots, \quad \tau A^{m-2}b = 0, \quad \tau A^{m-1}b = 1$$

or

$$\tau \begin{bmatrix} b & Ab & A^2b & \ldots & A^{m-1}b \end{bmatrix} = d^T$$

which has a unique solution for τ (in view of condition (2.30)).

It remains to prove that (the matrix) T is *nonsingular*; we shall show that its rows are linearly independent. Suppose that

$$\alpha_1\tau + \alpha_2\tau A + \cdots + \alpha_m \tau A^{m-1} = 0$$

for some scalars α_i, $i = 1, 2, \ldots, m$. Multiplying this relation on the right by b gives $\alpha_m = 0$. Similarly, multiplying on the right successively by Ab, $A^2b, \ldots, A^{m-1}b$ gives $\alpha_{m-1} = 0, \ldots, \alpha_1 = 0$. Thus, the rows of T are linearly independent.

(⇒) *Necessity.* Conversely, if such a transformation T exists, then

$$\begin{aligned}
\operatorname{rank} \begin{bmatrix} b & Ab & \ldots & A^{m-1}b \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} Tb & TAb & TA^2b & \ldots & TA^{m-1}b \end{bmatrix} \\
&= \operatorname{rank} \begin{bmatrix} Tb & (TAT^{-1})Tb & \ldots & (TAT^{-1})^{m-1}Tb \end{bmatrix} \\
&= \operatorname{rank} \begin{bmatrix} d & Cd & C^2d & \ldots & C^{m-1}d \end{bmatrix}.
\end{aligned}$$

It is easy to verify that this last matrix has the triangular form

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & \theta_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \dots & \theta_{m-4} & \theta_{m-3} \\ 0 & 1 & \theta_1 & \dots & \theta_{m-3} & \theta_{m-2} \\ 1 & \theta_1 & \theta_2 & \dots & \theta_{m-2} & \theta_{m-1} \end{bmatrix}$$

and therefore has full rank. This completes the proof. $\square$

Note : T can be constructed using

$$T = \begin{bmatrix} \tau \\ \tau A \\ \vdots \\ \tau A^{m-1} \end{bmatrix} \quad \text{and} \quad \tau \begin{bmatrix} b & Ab & A^2 b & \dots & A^{m-1} b \end{bmatrix} = d^T.$$

However, we can also give an *explicit expression for the matrix in the transformation* $x = T^{-1} w$. We have seen that

$$T \begin{bmatrix} b & Ab & A^2 b & \dots & A^{m-1} b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & \theta_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \dots & \theta_{m-4} & \theta_{m-3} \\ 0 & 1 & \theta_1 & \dots & \theta_{m-3} & \theta_{m-2} \\ 1 & \theta_1 & \theta_2 & \dots & \theta_{m-2} & \theta_{m-1} \end{bmatrix}.$$

This latter matrix has elements given by

$$\theta_i = -k_1 \theta_{i-1} - k_2 \theta_{i-2} - \cdots - k_i\,, \quad i = 1, 2, \dots, m-1.$$

It is straightforward to verify that

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & \theta_1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 1 & \dots & \theta_{m-4} & \theta_{m-3} \\ 0 & 1 & \theta_1 & \dots & \theta_{m-3} & \theta_{m-2} \\ 1 & \theta_1 & \theta_2 & \dots & \theta_{m-2} & \theta_{m-1} \end{bmatrix}^{-1} = \begin{bmatrix} k_{m-1} & k_{m-2} & \dots & k_1 & 1 \\ k_{m-2} & k_{m-3} & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ k_1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Finally,

$$\boxed{T^{-1} = \begin{bmatrix} b & Ab & A^2b & \dots & A^{m-1}b \end{bmatrix} \begin{bmatrix} k_{m-1} & k_{m-2} & \dots & k_1 & 1 \\ k_{m-2} & k_{m-3} & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ k_1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.} \tag{2.31}$$

($k_1, k_2, \dots, k_{m-1}$ are the coefficients in the characteristic equation of C.)

2.4.4 Example. Consider a system in the form

$$\dot{x} = Ax + bu(t)$$

where

$$A = \begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Find the matrix T and the transformed system.

Solution : From

$$\tau b = 0\,, \quad \tau Ab = 1$$

with $\tau = \begin{bmatrix} \tau_1 & \tau_2 \end{bmatrix}$ we have

$$\begin{cases} \tau_1 + \tau_2 = 0 \\ -2\tau_1 + 6\tau_2 = 1 \end{cases}$$

whence

$$\tau_1 = -\frac{1}{8} \quad \text{and} \quad \tau_2 = \frac{1}{8}\,\cdot$$

Now

$$T = \begin{bmatrix} \tau \\ \tau A \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & 1 \\ 3 & 5 \end{bmatrix}$$

and

$$T^{-1} = \begin{bmatrix} -5 & 1 \\ 3 & 1 \end{bmatrix}.$$

Then

$$TAT^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -5 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -14 & 3 \end{bmatrix}.$$

Thus the transformed system is

$$\dot{w} = TAT^{-1}w + Tbu = \begin{bmatrix} 0 & 1 \\ -14 & 3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} -1 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}u;$$

that is,

$$\begin{cases} \dot{w}_1 = w_2 \\ \dot{w}_2 = -14w_1 + 3w_2 + u \end{cases}$$

or (for $w_1 = z,\ w_2 = \dot{z}$)

$$\ddot{z} - 3\dot{z} + 14z = u\,.$$

A result similar to THEOREM 2.4.3 can be obtained for systems having zero input and scalar output, so that the system equations are

$$\begin{cases} \dot{x} = Ax \\ \\ y = cx \end{cases} \tag{2.32}$$

where $A \in \mathbb{R}^{m\times m}$, $c \in \mathbb{R}^{1\times m}$, and $y(\cdot)$ is the output variable.

2.4.5 THEOREM. *Any system described by (2.32) can be transformed by $x = Sv$, with S nonsingular, into the canonical form*

$$\dot{v} = Ev, \quad y = fv \tag{2.33}$$

where

$$E = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -e_m \\ 1 & 0 & 0 & \dots & 0 & -e_{m-1} \\ 0 & 1 & 0 & \dots & 0 & -e_{m-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -e_1 \end{bmatrix} \quad \textit{and} \quad f = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix} \tag{2.34}$$

provided that

$$\boxed{\operatorname{rank} \begin{bmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{m-1} \end{bmatrix} = m.} \tag{2.35}$$

Conversely, if such a transformation S exists, then condition (2.35) holds.

The proof is very similar to that of THEOREM 2.4.3 and will be omitted.

NOTE : E is also a companion matrix because its characteristic polynomial is

$$\operatorname{char}_E(\lambda) = \det\,(\lambda I_m - E) = \lambda^m + e_1\lambda^{m-1} + \cdots + e_m$$

which again is identical to the characteristic polynomial of A.

2.5 Exercises

Exercise 26 Find the general solution, in *spectral form*, of the (initialized) uncontrolled system

$$\dot{x} = Ax\,, \quad x(0) = x_0$$

in each of the following cases :

(a) $A = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

(c) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(d) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

(e) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Exercise 27 Find the general solution, in *exponential form*, of the (initialized) uncontrolled system

$$\dot{x} = Ax\,, \quad x(0) = x_0$$

in each of the cases given in **Exercise 26**.

Exercise 28 Consider the equation of *simple harmonic motion*

$$\ddot{z} + \omega^2 z = 0.$$

Take as state variables $x_1 = z$ and $x_2 = \frac{1}{\omega}\dot{z}$, and find the state transition matrix $\Phi(t, 0)$.

Exercise 29 Use the exponential matrix to solve the *rabbit-fox environment problem*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_1 & -a_2 \\ a_3 & -a_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (a_1, a_2, a_3, a_4 > 0)$$

subject to the condition

$$\frac{a_1}{a_3} = \frac{a_2}{a_4}\,\cdot$$

Show that for arbitrary initial conditions, the populations will attain a steady state as $t \to \infty$ only if $a_1 - a_4 < 0$, and give an expression for the ultimate size of the rabbit population in this case. Finally, deduce that if the environment is to reach a steady state in which both rabbits and foxes are present, then $x_1(0) > \frac{a_1}{a_3} x_2(0)$.

Exercise 30 A linear control system is described by the equations

$$\begin{cases} \dot{x}_1 = x_1 + 4x_2 + u(t) \\ \\ \dot{x}_2 = 3x_1 + 2x_2 . \end{cases}$$

Determine the *state transition matrix* and write down the general solution.

Exercise 31 A linear control system is described by

$$\ddot{z} + 3\dot{z} + 2z = u \,, \quad z(0) = \dot{z}(0) = 0$$

where

$$u(t) = \begin{cases} 1 & \text{if } \ 0 \leq t < 1 \\ \\ 0 & \text{if } \ t \geq 1. \end{cases}$$

Calculate the *state transition matrix* and determine $z(2)$.

Exercise 32 Verify that *the* solution of the matrix differential equation

$$\dot{W} = AW + WB, \quad W(0) = C$$

(where $A, B \in \mathbb{R}^{m \times m}$) is

$$W(t) = \exp(tA)C\exp(tB).$$

Exercise 33 Consider the linear control system

$$\dot{x} = Ax + bu(t)$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and take

$$u(t) = \begin{cases} 1 & \text{if } \ t \geq 0 \\ \\ 0 & \text{if } \ t < 0. \end{cases}$$

Evaluate $\exp(tA)$ and show that the solution of this problem, subject to

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is

$$x(t) = \begin{bmatrix} 1 + 3t + \frac{5}{2}t^2 + \frac{7}{6}t^3 + \cdots \\ \\ t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \cdots \end{bmatrix}.$$

Exercise 34 Consider the matrix differential equation

$$\dot{X} = A(t)X, \quad X(0) = I_m.$$

Show that, when $m = 2$,

$$\frac{d}{dt}\det\left(X(t)\right) = \operatorname{tr}\left(A(t)\right) \cdot \det\left(X(t)\right)$$

and hence deduce that $X(t)$ is nonsingular, $t \geq 0$.

Exercise 35 Verify that the properties of the *state transition matrix*

(a) $\frac{d}{dt}\Phi(t, t_0) = A\Phi(t, t_0)$.

(b) $\Phi(t_0, t_0) = I_m$.

(c) $\Phi(t_0, t) = \Phi^{-1}(t, t_0)$.

(d) $\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$

do carry over to the time varying case.

Exercise 36 Consider the (initialized) *uncontrolled* system

$$\dot{x} = A(t)x, \quad x(0) = x_0.$$

If $B(t) = \int_0^t A(\tau)\, d\tau$, show that *the* solution in this case is

$$x(t) = \exp(B(t))x_0$$

provided $B(t)$ and $A(t)$ commute with each other (for all $t \geq 0$).

Exercise 37 Verify that *the* solution of the matrix differential equation

$$\dot{W} = A(t)W + WA^T(t), \quad W(t_0) = C$$

is

$$W(t) = \Phi(t, t_0)C\Phi^T(t, t_0).$$

Exercise 38 For the linear control system

$$\dot{x} = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

determine $\Phi(t, 0)$. If

$$x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad u(t) = e^{2t}, \;\; t \geq 0$$

use formula

$$x(t) = \Phi(t, 0) \left[x_0 + \int_0^t \Phi(0, \tau) B u(\tau)\, d\tau \right]$$

to obtain the expression for $x(t)$.

Exercise 39 Consider a single-input control system, written in state space form,

$$\dot{x} = Ax + bu(t).$$

Find the matrix T of the linear tansformation $w = Tx$ and the transformed system (the system put into *canonical form*)

$$\dot{w} = Cw + du(t)$$

for each of the following cases :

(a) $A = \begin{bmatrix} -1 & -1 \\ 2 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$

(b) $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$

Exercise 40 Consider a single-output (uncontrolled) system, written in state space form,

$$\dot{x} = Ax, \quad y = cx.$$

Find the matrix P of the linear transformation $x = Pv$ and the transformed system (the system put into the *canonical form*)

$$\dot{v} = Ev, \quad y = fv$$

when

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}.$$