## Chapter 3

## Linear Control Systems

## Topics :

1. Controllability
2. Observability
3. Linear Feedback
4. Realization Theory

Copyright © Claudiu C. Remsing, 2006.
All rights reserved.

-

Intuitively, a control system should be designed so that the input $u(\cdot)$ "controls" all the states; and also so that all states can be "observed" from the output $y(\cdot)$. The concepts of (complete) controllability and observability formalize these ideas.

Another two fundamental concepts of control theory - feedback and realization are introduced. Using (linear) feedback it is possible to exert a considerable influence on the behaviour of a (linear) control system.


0

### 3.1 Controllability

An essential first step in dealing with many control problems is to determine whether a desired objective can be achieved by manipulating the chosen control variables. If not, then either the objective will have to be modified or control will have to be applied in some different fashion.

We shall discuss the general property of being able to transfer (or steer) a control system from any given state to any other by means of a suitable choice of control functions.
3.1.1 Definition. The linear control system $\Sigma$ defined by

$$
\begin{equation*}
\dot{x}=A(t) x+B(t) u(t) \tag{3.1}
\end{equation*}
$$

where $A(t) \in \mathbb{R}^{m \times m}$ and $B(t) \in \mathbb{R}^{m \times \ell}$, is said to be completely controllable (c.c.) if for any $t_{0}$, any initial state $x\left(t_{0}\right)=x_{0}$, and any given final state $x_{f}$, there exist a finite time $t_{1}>t_{0}$ and a control $u:\left[t_{0}, t_{1}\right] \rightarrow \mathbb{R}^{\ell}$ such that $x\left(t_{1}\right)=x_{f}$.

Note : (1) The qualifying term "completely" implies that the definition holds for all $x_{0}$ and $x_{f}$, and several other types of controllability can be defined.
(2) The control $u(\cdot)$ is assumed piecewise-continuous in the interval $\left[t_{0}, t_{1}\right]$.
3.1.2 Example. Consider the control system described by

$$
\left\{\begin{array}{l}
\dot{x}_{1}=a_{1} x_{1}+a_{2} x_{2}+u(t) \\
\\
\dot{x}_{2}=x_{2}
\end{array}\right.
$$

Clearly, by inspection, this is not completely controllable (c.c.) since $u(\cdot)$ has no influence on $x_{2}$, which is entirely determined by the second equation and $x_{2}\left(t_{0}\right)$.

We have

$$
x_{f}=\Phi\left(t_{1}, t_{0}\right)\left[x_{0}+\int_{t_{0}}^{t_{1}} \Phi\left(t_{0}, \tau\right) B(\tau) u(\tau) d \tau\right]
$$

or

$$
0=\Phi\left(t_{1}, t_{0}\right)\left[x_{0}-\Phi\left(t_{0}, t_{1}\right) x_{f}+\int_{t_{0}}^{t_{1}} \Phi\left(t_{0}, \tau\right) B(\tau) u(\tau) d \tau\right]
$$

Since $\Phi\left(t_{1}, t_{0}\right)$ is nonsingular it follows that if $u(\cdot)$ transfers $x_{0}$ to $x_{f}$, it also transfers $x_{0}-\Phi\left(t_{0}, t_{1}\right) x_{f}$ to the origin in the same time interval. Since $x_{0}$ and $x_{f}$ are arbitrary, it therefore follows that - in the controllability definition the given final state can be taken to be the zero vector without loss of generality.

Note : For time-invariant control systems - in the controllability definition - the initial time $t_{0}$ can be set equal to zero.

## The Kalman rank condition

For linear time-invariant control systems a general algebraic criterion (for complete controllability) can be derived.
3.1.3 ThEOREM. The linear time-invariant control system

$$
\begin{equation*}
\dot{x}=A x+B u(t) \tag{3.2}
\end{equation*}
$$

(or the pair $(A, B))$ is c.c. if and only if the (Kalman) controllability matrix

$$
\mathcal{C}=\mathcal{C}(A, B):=\left[\begin{array}{lllll}
B & A B & A^{2} B & \ldots & A^{m-1} B
\end{array}\right] \in \mathbb{R}^{m \times m \ell}
$$

has rank $m$.
Proof : $\quad(\Rightarrow)$ We suppose the system is c.c. and wish to prove that $\operatorname{rank}(\mathcal{C})=m$. This is done by assuming $\operatorname{rank}(\mathcal{C})<m$, which leads to a contradiction.

Then there exists a constant row $m$-vector $q \neq 0$ such that

$$
q B=0, \quad q A B=0, \quad \ldots, \quad q A^{m-1} B=0
$$

In the expression

$$
x(t)=\exp (t A)\left[x_{0}+\int_{0}^{t} \exp (-\tau A) B u(\tau) d \tau\right]
$$

for the solution of (3.2) subject to $x(0)=x_{0}$, set $t=t_{1}, x\left(t_{1}\right)=0$ to obtain (since $\exp \left(t_{1} A\right)$ is nonsingular)

$$
-x_{0}=\int_{0}^{t_{1}} \exp (-\tau A) B u(\tau) d \tau
$$

Now, $\exp (-\tau A)$ can be expressed as some polynomial $r(A)$ in $A$ having degree at most $m-1$, so we get

$$
-x_{0}=\int_{0}^{t_{1}}\left(r_{0} I_{m}+r_{1} A+\cdots+r_{m-1} A^{m-1}\right) B u(\tau) d \tau
$$

Multiplying this relation on the left by $q$ gives

$$
q x_{0}=0
$$

Since the system is c.c., this must hold for any vector $x_{0}$, which implies $q=0$, contradiction.
$(\Leftarrow)$ We asume $\operatorname{rank}(\mathcal{C})=m$, and wish to show that for any $x_{0}$ there is a function $u:\left[0, t_{1}\right] \rightarrow \mathbb{R}^{\ell}$, which when substituted into

$$
\begin{equation*}
x(t)=\exp (t A)\left[x_{0}+\int_{0}^{t} \exp (-\tau A) B u(\tau) d \tau\right] \tag{3.3}
\end{equation*}
$$

produces

$$
x\left(t_{1}\right)=0
$$

Consider the symmetric matrix

$$
W_{c}:=\int_{0}^{t_{1}} \exp (-\tau A) B B^{T} \exp \left(-\tau A^{T}\right) d \tau
$$

One can show that $W_{c}$ is nonsingular. Indeed, consider the quadratic form associated to $W_{c}$

$$
\begin{aligned}
\alpha^{T} W_{c} \alpha & =\int_{0}^{t_{1}} \psi(\tau) \psi^{T}(\tau) d \tau \\
& =\int_{0}^{t_{1}}\|\psi(\tau)\|_{e}^{2} d \tau \geq 0
\end{aligned}
$$

where $\alpha \in \mathbb{R}^{m \times 1}$ is an arbitrary column vector and $\psi(\tau):=\alpha^{T} \exp (-\tau A) B$. It is clear that $W_{c}$ is positive semi-definite, and will be singular only if there
exists an $\bar{\alpha} \neq 0$ such that $\bar{\alpha}^{T} W_{c} \bar{\alpha}=0$. However, in this case, it follows (using the properties of the norm) that $\psi(\tau) \equiv 0, \quad 0 \leq \tau \leq t_{1}$. Hence, we have

$$
\bar{\alpha}^{T}\left(I_{m}-\tau A+\frac{\tau^{2}}{2!} A^{2}-\frac{\tau^{3}}{3!} A^{3}+\cdots\right) B=0, \quad 0 \leq \tau \leq t_{1}
$$

from which it follows that

$$
\bar{\alpha}^{T} B=0, \quad \bar{\alpha}^{T} A B=0, \quad \bar{\alpha}^{T} A^{2} B=0, \quad \cdots
$$

This implies that $\bar{\alpha}^{T} \mathcal{C}=0$. Since by assumption $\mathcal{C}$ has rank $m$, it follows that such a nonzero vector $\bar{\alpha}$ cannot exist, so $W_{c}$ is nonsingular.

Now, if we choose as the control vector

$$
u(t)=-B^{T} \exp \left(-t A^{T}\right) W_{c}^{-1} x_{0}, \quad t \in\left[0, t_{1}\right]
$$

then substitution into (3.3) gives

$$
\begin{aligned}
x\left(t_{1}\right) & =\exp \left(t_{1} A\right)\left[x_{0}-\int_{0}^{t_{1}} \exp (-\tau A) B B^{T} \exp \left(-\tau A^{T}\right) d \tau \cdot\left(W_{c}^{-1} x_{0}\right)\right] \\
& =\exp \left(t_{1} A\right)\left[x_{0}-W_{c} W_{c}^{-1} x_{0}\right]=0
\end{aligned}
$$

as required.
3.1.4 Corollary. If $\operatorname{rank}(B)=r$, then the condition in Theorem 3.1.3 reduces to

$$
\operatorname{rank}\left[\begin{array}{llll}
B & A B & \ldots & A^{m-r} B
\end{array}\right]=m
$$

Proof: Define the matrix

$$
\mathcal{C}_{k}:=\left[\begin{array}{llll}
B & A B & \cdots & A^{k} B
\end{array}\right], \quad k=0,1,2 \ldots
$$

If $\operatorname{rank}\left(\mathcal{C}_{j}\right)=\operatorname{rank}\left(\mathcal{C}_{j+1}\right)$ it follows that all the columns of $A^{j+1} B$ must be linearly dependent on those of $\mathcal{C}_{j}$. This then implies that all the columns of $A^{j+2} B, A^{j+3} B, \ldots$ must also be linearly dependent on those of $\mathcal{C}_{j}$, so that

$$
\operatorname{rank}\left(\mathcal{C}_{j}\right)=\operatorname{rank}\left(\mathcal{C}_{j+1}\right)=\operatorname{rank}\left(\mathcal{C}_{j+2}\right)=\cdots
$$

Hence the rank of $\mathcal{C}_{k}$ increases by at least one when the index $k$ is increased by one, until the maximum value of $\operatorname{rank}\left(\mathcal{C}_{k}\right)$ is attained when $k=j$. Since $\operatorname{rank}\left(\mathcal{C}_{0}\right)=\operatorname{rank}(B)=r$ and $\operatorname{rank}\left(\mathcal{C}_{k}\right) \leq m$ it follows that $r+j \leq m$, giving $j \leq m-r$ as required.
3.1.5 Example. Consider the linear control system $\Sigma$ described by

$$
\dot{x}=\left[\begin{array}{rr}
-2 & 2 \\
1 & -1
\end{array}\right] x+\left[\begin{array}{l}
1 \\
0
\end{array}\right] u(t) .
$$

The (Kalman) controllability matrix is

$$
\mathcal{C}=\mathcal{C}^{\Sigma}=\left[\begin{array}{rr}
1 & -2 \\
0 & 1
\end{array}\right]
$$

which has rank 2, so the control system $\Sigma$ is c.c.
Note: When $\ell=1, B$ reduces to a column vector $b$ and Theorem 2.4.3 can be restated as: A linear control system in the form

$$
\dot{x}=A x+b u(t)
$$

can be transformed into the canonical form

$$
\dot{w}=C w+d u(t)
$$

if and only if it is c.c.

## Controllability criterion

We now give a general criterion for (complete) controllability of control systems (time-invariant or time-varying) as well as an explicit expression for a control vector which carry out a required alteration of states.
3.1.6 Theorem. The linear control system $\Sigma$ defined by

$$
\dot{x}=A(t) x+B(t) u(t)
$$

is c.c. if and only if the symmetric matrix, called the controllability Gramian,

$$
\begin{equation*}
W_{c}\left(t_{0}, t_{1}\right):=\int_{t_{0}}^{t_{1}} \Phi\left(t_{0}, \tau\right) B(\tau) B^{T}(\tau) \Phi^{T}\left(t_{0}, \tau\right) d \tau \in \mathbb{R}^{m \times m} \tag{3.4}
\end{equation*}
$$

is nonsingular. In this case the control

$$
u^{*}(t)=-B^{T}(t) \Phi^{T}\left(t_{0}, t\right) W_{c}\left(t_{0}, t_{1}\right)^{-1}\left[x_{0}-\Phi\left(t_{0}, t_{1}\right) x_{f}\right], \quad t \in\left[t_{0}, t_{1}\right]
$$

transfers $x\left(t_{0}\right)=x_{0}$ to $x\left(t_{1}\right)=x_{f}$.
Proof : $\quad(\Leftarrow)$ Sufficiency. If $W_{c}\left(t_{0}, t_{1}\right)$ is assumed nonsingular, then the control defined by

$$
u^{*}(t)=-B^{T}(t) \Phi^{T}\left(t_{0}, t\right) W_{c}\left(t_{0}, t_{1}\right)^{-1}\left[x_{0}-\Phi\left(t_{0}, t_{1}\right) x_{f}\right], \quad t \in\left[t_{0}, t_{1}\right]
$$

exists. Now, substitution of the above expression into the solution

$$
x(t)=\Phi\left(t, t_{0}\right)\left[x_{0}+\int_{t_{0}}^{t} \Phi\left(t_{0}, \tau\right) B(\tau) u(\tau) d \tau\right]
$$

of

$$
\dot{x}=A(t) x+B(t) u(t)
$$

gives

$$
\begin{aligned}
x\left(t_{1}\right)= & \Phi\left(t_{1}, t_{0}\right)\left[x_{0}+\int_{t_{0}}^{t_{1}} \Phi\left(t_{0}, \tau\right) B(\tau)\left(-B^{T}(\tau) \Phi^{T}\left(t_{0}, \tau\right) W_{c}\left(t_{0}, t_{1}\right)^{-1}\right.\right. \\
& {\left.\left.\left[x_{0}-\Phi\left(t_{0}, t_{1}\right) x_{f}\right]\right) d \tau\right] } \\
= & \Phi\left(t_{1}, t_{0}\right)\left[x_{0}-W_{c}\left(t_{0}, t_{1}\right) W_{c}\left(t_{0}, t_{1}\right)^{-1}\left[x_{0}-\Phi\left(t_{0}, t_{1}\right) x_{f}\right]\right] \\
= & \Phi\left(t_{1}, t_{0}\right)\left[x_{0}-x_{0}+\Phi\left(t_{0}, t_{1}\right) x_{f}\right] \\
= & \Phi\left(t_{1}, t_{0}\right) \Phi\left(t_{0}, t_{1}\right) x_{f} \\
= & x_{f}
\end{aligned}
$$

$(\Rightarrow)$ Necessity. We need to show that if $\Sigma$ is c.c., then $W_{c}\left(t_{0}, t_{1}\right)$ is nonsingular. First, notice that if $\alpha \in \mathbb{R}^{m \times 1}$ is an arbitrary column vector, then from
(3.4) since $W=W_{c}\left(t_{0}, t_{1}\right)$ is symmetric we can construct the quadratic form

$$
\begin{aligned}
\alpha^{T} W \alpha & =\int_{t_{0}}^{t_{1}} \theta^{T}\left(\tau, t_{0}\right) \theta\left(\tau, t_{0}\right) d \tau \\
& =\int_{t_{0}}^{t_{1}}\|\theta\|_{e}^{2} d \tau \geq 0
\end{aligned}
$$

where $\theta\left(\tau, t_{0}\right):=B^{T}(\tau) \Phi^{T}\left(t_{0}, \tau\right) \alpha$, so that $W_{c}\left(t_{0}, t_{1}\right)$ is positive semi-definite. Suppose that there exists some $\bar{\alpha} \neq 0$ such that $\bar{\alpha}^{T} W \bar{\alpha}=0$. Then we get $($ for $\bar{\theta}=\theta$ when $\alpha=\bar{\alpha}$ )

$$
\int_{t_{0}}^{t_{1}}\|\bar{\theta}\|_{e}^{2} d \tau=0
$$

which in turn implies (using the properties of the norm) that $\bar{\theta}\left(\tau, t_{0}\right) \equiv$ $0, \quad t_{0} \leq \tau \leq t_{1}$. However, by assumption $\Sigma$ is c.c. so there exists a control $v(\cdot)$ making $x\left(t_{1}\right)=0$ if $x\left(t_{0}\right)=\bar{\alpha}$. Hence

$$
\bar{\alpha}=-\int_{t_{0}}^{t_{1}} \Phi\left(t_{0}, \tau\right) B(\tau) v(\tau) d \tau
$$

Therefore

$$
\begin{aligned}
\|\bar{\alpha}\|_{e}^{2} & =\bar{\alpha}^{T} \bar{\alpha} \\
& =-\int_{t_{0}}^{t_{1}} v^{T}(\tau) B^{T}(\tau) \Phi^{T}\left(t_{0}, \tau\right) \bar{\alpha} d \tau \\
& =-\int_{t_{0}}^{t_{1}} v^{T}(\tau) \bar{\theta}\left(\tau, t_{0}\right) d \tau=0
\end{aligned}
$$

which contradicts the assumption that $\bar{\alpha} \neq 0$. Hence $W_{c}\left(t_{0}, t_{1}\right)$ is positive definite and is therefore nonsingular.
3.1.7 Example. The control system is

$$
\dot{x}=\left[\begin{array}{rr}
-2 & 2 \\
1 & -1
\end{array}\right] x+\left[\begin{array}{l}
1 \\
1
\end{array}\right] u(t)
$$

Observe that $\lambda=0$ is an eigenvalue of $A$, and $b=\left[\begin{array}{l}1 \\ 1\end{array}\right]$ is a corresponding eigenvector, so the controllability rank condition does not hold. However, $A$ is
similar to its companion matrix. Using the matrix $T=\left[\begin{array}{rr}0 & 1 \\ 1 & -1\end{array}\right]$ computed before (see Example 2.5.1) and $w=T x$ we have the system

$$
\dot{w}=\left[\begin{array}{rr}
0 & 1 \\
0 & -3
\end{array}\right] w+\left[\begin{array}{l}
1 \\
0
\end{array}\right] u .
$$

Differentiation of the $w_{1}$ equation and substitution produces a second-order ODE for $w_{1}$ :

$$
\ddot{w}_{1}+3 \dot{w}_{1}=3 u+\dot{u} .
$$

One integration produces a first-order ODE

$$
\dot{w}_{1}+3 w_{1}=3 \int u(\tau) d \tau+u
$$

which shows that the action of arbitrary inputs $u(\cdot)$ affects the dynamics in only a one-dimensional space. The original $x$ equations might lead us to think that $u(\cdot)$ can fully affect $x_{1}$ and $x_{2}$, but notice that the $w_{2}$ equation says that $u(\cdot)$ has no effect on the dynamics of the difference $x_{1}-x_{2}=w_{2}$. Only when the initial condition for $w$ involves $w_{2}(0)=0$ can $u(\cdot)$ be used to control a trajectory. That is, the inputs completely control only the states that lie in the subspace

$$
\operatorname{span}[b A b]=\operatorname{span}\{b\}=\operatorname{span}\left[\begin{array}{l}
1 \\
1
\end{array}\right] .
$$

Solutions starting with $x_{1}(0)=x_{2}(0)$ satisfy

$$
x_{1}(t)=x_{2}(t)=\int_{0}^{t} u(\tau) d \tau+x_{1}(0) .
$$

One can steer along the line $x_{1}=x_{2}$ from any initial point to any final point $x_{1}\left(t_{1}\right)=x_{2}\left(t_{1}\right)$ at any finite time $t_{1}$ by appropriate choice of $u(\cdot)$. On the other hand, if the initial condition lies off the line $x_{1}=x_{2}$, then the difference $w_{2}=x_{1}-x_{2}$ decays exponentially so there is no chance of steering to an arbitrarily given final state in finite time.

Note : The control function $u^{*}(\cdot)$ which transfers the system from $x_{0}=x\left(t_{0}\right)$ to $x_{f}=x\left(t_{1}\right)$ requires calculation of the state transition matrix $\Phi\left(\cdot, t_{0}\right)$ and the controllability Gramian $W_{c}\left(\cdot, \tau_{0}\right)$. However, this is not too dificult for linear timeinvariant control systems, although rather tedious. Of course, there will in general be many other suitable control vectors which achieve the same result.
3.1.8 Proposition. If $u(\cdot)$ is any other control taking $x_{0}=x\left(t_{0}\right)$ to $x_{f}=x\left(t_{1}\right)$, then

$$
\int_{t_{0}}^{t_{1}}\|u(\tau)\|_{e}^{2} d \tau>\int_{t_{0}}^{t_{1}}\left\|u^{*}(\tau)\right\|_{e}^{2} d \tau
$$

Proof : Since both $u^{*}$ and $u$ satisfy

$$
x_{f}=\Phi\left(t_{1}, t_{0}\right)\left[x_{0}+\int_{t_{0}}^{t_{1}} \Phi\left(t_{0}, \tau\right) B(\tau) u(\tau) d \tau\right]
$$

we obtain after subtraction

$$
0=\int_{t_{0}}^{t_{1}} \Phi\left(t_{0}, \tau\right) B(\tau)\left[u(\tau)-u^{*}(\tau)\right] d \tau
$$

Multiplication of this equation on the left by

$$
\left[x_{0}-\Phi\left(t_{0}, t_{1}\right) x_{f}\right]^{T}\left[W_{c}\left(t_{0}, t_{1}\right)^{-1}\right]^{T}
$$

gives

$$
\int_{t_{0}}^{t_{1}}\left(u^{*}\right)^{T}(\tau)\left[u^{*}(\tau)-u(\tau)\right] d \tau=0
$$

and thus

$$
\int_{t_{0}}^{t_{1}}\left\|u^{*}(\tau)\right\|_{e}^{2} d \tau=\int_{t_{0}}^{t_{1}}\left(u^{*}\right)^{T}(\tau) u(\tau) d \tau
$$

Therefore

$$
\begin{aligned}
0<\int_{t_{0}}^{t_{1}}\left\|u^{*}(\tau)-u(\tau)\right\|_{e}^{2} d \tau & =\int_{t_{0}}^{t_{1}}\left[u^{*}(\tau)-u(\tau)\right]^{T}\left[u^{*}(\tau)-u(\tau)\right] d \tau \\
& =\int_{t_{0}}^{t_{1}}\left(\|u(\tau)\|_{e}^{2}+\left\|u^{*}(\tau)\right\|_{e}^{2}-2\left(u^{*}\right)^{T}(\tau) u(\tau)\right) d \tau \\
& =\int_{t_{0}}^{t_{1}}\left(\|u(\tau)\|_{e}^{2}-\left\|u^{*}(\tau)\right\|_{e}^{2}\right) d \tau
\end{aligned}
$$

and so

$$
\int_{t_{0}}^{t_{1}}\|u(\tau)\|_{e}^{2} d \tau=\int_{t_{0}}^{t_{1}}\left(\left\|u^{*}(\tau)\right\|_{e}^{2}+\left\|u^{*}(\tau)-u(\tau)\right\|_{e}^{2}\right) d \tau>\int_{t_{0}}^{t_{1}}\left\|u^{*}(\tau)\right\|_{e}^{2} d \tau
$$

as required.
NOTE : This result can be interpreted as showing that the control

$$
u^{*}(t)=-B^{T}(t) \Phi^{T}\left(t_{0}, t\right) W_{c}\left(t_{0}, t_{1}\right)^{-1}\left[x_{0}-\Phi\left(t_{0}, t_{1}\right) x_{f}\right]
$$

is "optimal", in the sense that it minimizes the integral

$$
\int_{t_{0}}^{t_{1}}\|u(\tau)\|_{e}^{2} d \tau=\int_{t_{0}}^{t_{1}}\left(u_{1}^{2}(\tau)+u_{2}^{2}(\tau)+\cdots+u_{\ell}^{2}(\tau)\right) d \tau
$$

over the set of all (admissible) controls which transfer $x_{0}=x\left(t_{0}\right)$ to $x_{f}=x\left(t_{1}\right)$, and this integral can be thought of as a measure of control "energy" involved.

## Algebraic equivalence and decomposition of control systems

We now indicate a further aspect of controllability. Let $P(\cdot)$ be a matrixvalued mapping which is continuous and such that $P(t)$ is nonsingular for all $t \geq t_{0}$. (The continuous maping $P:\left[t_{0}, \infty\right) \rightarrow \mathrm{GL}(m, \mathbb{R})$ is a path in the general linear group $\operatorname{GL}(m, \mathbb{R})$.) Then the system $\widetilde{\Sigma}$ obtained from $\Sigma$ by the transformation

$$
\widetilde{x}=P(t) x
$$

is said to be algebraically equivalent to $\Sigma$.
3.1.9 Proposition. If $\Phi\left(t, t_{0}\right)$ is the state transition matrix for $\Sigma$, then

$$
P(t) \Phi\left(t, t_{0}\right) P^{-1}\left(t_{0}\right)=\widetilde{\Phi}\left(t, t_{0}\right)
$$

is the state transition matrix for $\widetilde{\Sigma}$.
Proof: We recall that $\Phi\left(t, t_{0}\right)$ is the unique matrix-valued mapping satisfying

$$
\dot{\Phi}\left(t, t_{0}\right)=A(t) \Phi\left(t, t_{0}\right), \quad \Phi\left(t_{0}, t_{0}\right)=I_{m}
$$

and is nonsingular. Clearly,

$$
\widetilde{\Phi}\left(t_{0}, t_{0}\right)=I_{m} ;
$$

differentiation of

$$
\widetilde{x}=P(t) x
$$

gives

$$
\begin{aligned}
\dot{\tilde{x}} & =\dot{P} x+P \dot{x} \\
& =(\dot{P}+P A) x+P B u \\
& =(\dot{P}+P A) P^{-1} \widetilde{x}+P B u .
\end{aligned}
$$

We need to show that $\widetilde{\Phi}$ is the state transition matrix for

$$
\dot{\tilde{x}}=(\dot{P}+P A) P^{-1} \widetilde{x}+P B u .
$$

We have

$$
\begin{aligned}
\dot{\tilde{\Phi}}\left(t, t_{0}\right) & =\frac{d}{d t}\left[P(t) \Phi\left(t, t_{0}\right) P^{-1}\left(t_{0}\right)\right] \\
& =\dot{P}(t) \Phi\left(t, t_{0}\right) P^{-1}\left(t_{0}\right)+P(t) \dot{\Phi}\left(t, t_{0}\right) P^{-1}\left(t_{0}\right) \\
& =\left[(\dot{P}(t)+P(t) A(t)) P^{-1}(t)\right] P(t) \Phi\left(t, t_{0}\right) P^{-1}\left(t_{0}\right) \\
& =\left[(\dot{P}(t)+P(t) A(t)) P^{-1}(t)\right] \widetilde{\Phi}\left(t, t_{0}\right) .
\end{aligned}
$$

3.1.10 Proposition. If $\Sigma$ is c.c., then so is $\widetilde{\Sigma}$.

Proof: The system matrices for $\widetilde{\Sigma}$ are

$$
\widetilde{A}=(\dot{P}+P A) P^{-1} \quad \text { and } \quad \widetilde{B}=P B
$$

so the controllability matrix for $\widetilde{\Sigma}$ is

$$
\begin{aligned}
\widetilde{W} & =\int_{t_{0}}^{t_{1}} \widetilde{\Phi}\left(t_{0}, \tau\right) \widetilde{B}(\tau) \widetilde{B}^{T}(\tau) \widetilde{\Phi}^{T}\left(t_{0}, \tau\right) d \tau \\
& =\int_{t_{0}}^{t_{1}} P\left(t_{0}\right) \Phi\left(t_{0}, \tau\right) P^{-1}(\tau) P(\tau) B(\tau) B^{T}(\tau) P^{T}(\tau)\left(P^{-1}(\tau)\right)^{T} \Phi^{T}\left(t_{0}, \tau\right) P^{T}\left(t_{0}\right) d \tau \\
& =P\left(t_{0}\right) W_{c}\left(t_{0}, t_{1}\right) P^{T}\left(t_{0}\right) .
\end{aligned}
$$

Thus the matrix $\widetilde{W}=\widetilde{W}_{c}\left(t_{0}, t_{1}\right)$ is nonsingular since the matrices $W_{c}\left(t_{0}, t_{1}\right)$ and $P\left(t_{0}\right)$ each have rank $m$.

The following important result on system decomposition then holds :
3.1.11 ThEOREM. When the linear control system $\Sigma$ is time-invariant then if the controllability matrix $\mathcal{C}^{\Sigma}$ has rank $m_{1}<m$ there exists a control system, algebraically equivalent to $\Sigma$, having the form

$$
\begin{aligned}
{\left[\begin{array}{l}
\dot{x}_{(1)} \\
\dot{x}_{(2)}
\end{array}\right] } & =\left[\begin{array}{cc}
A_{1} & A_{2} \\
0 & A_{3}
\end{array}\right]\left[\begin{array}{l}
x_{(1)} \\
x_{(2)}
\end{array}\right]+\left[\begin{array}{c}
B_{1} \\
0
\end{array}\right] u(t) \\
y & =\left[\begin{array}{ll}
C_{1} & C_{2}
\end{array}\right] x
\end{aligned}
$$

where $x_{(1)}$ and $x_{(2)}$ have orders $m_{1}$ and $m-m_{1}$, respectively, and $\left(A_{1}, B_{1}\right)$ is c.c.

We shall postpone the proof of this until a later section (see the proof of THEOREM 3.4.5) where an explicit formula for the transformation matrix will also be given.

Note : It is clear that the vector $x_{(2)}$ is completely unaffected by the control $u(\cdot)$. Thus the state space has been divided into two parts, one being c.c. and the other uncontrollable.

### 3.2 Observability

Closely linked to the idea of controllability is that of observability, which in general terms means that it is possible to determine the state of a system by measuring only the output.
3.2.1 Definition. The linear control system (with outputs) $\Sigma$ described by

$$
\left\{\begin{array}{l}
\dot{x}=A(t) x+B(t) u(t)  \tag{3.5}\\
y=C(t) x
\end{array}\right.
$$

is said to be completely observable (c.o.) if for any $t_{0}$ and any initial state $x\left(t_{0}\right)=x_{0}$, there exists a finite time $t_{1}>t_{0}$ such that knowledge of $u(\cdot)$ and $y(\cdot)$ for $t \in\left[t_{0}, t_{1}\right]$ suffices to determine $x_{0}$ uniquely.

Note : There is in fact no loss of generality in assuming $u(\cdot)$ is identically zero throughout the interval. Indeed, for any input $u:\left[t_{0}, t_{1}\right] \rightarrow \mathbb{R}^{\ell}$ and initial state $x_{0}$, we have

$$
y(t)-\int_{t_{0}}^{t} C(t) \Phi(t, \tau) B(\tau) u(\tau) d \tau=C(t) \Phi\left(t, t_{0}\right) x_{0}
$$

Defining

$$
\widehat{y}(t):=y(t)-\int_{t_{0}}^{t} C(t) \Phi(t, \tau) B(\tau) u(\tau) d \tau
$$

we get

$$
\widehat{y}(t)=C(t) \Phi\left(t, t_{0}\right) x_{0}
$$

Thus a linear control system is c.o. if and only if knowledge of the output $\widehat{y}(\cdot)$ with zero input on the interval $\left[t_{0}, t_{1}\right]$ allows the initial state $x_{0}$ to be determined.
3.2.2 Example. Consider the linear control system described by

$$
\left\{\begin{array}{l}
\dot{x}_{1}=a_{1} x_{1}+b_{1} u(t) \\
\dot{x}_{2}=a_{2} x_{2}+b_{2} u(t) \\
y=x_{1}
\end{array}\right.
$$

The first equation shows that $x_{1}(\cdot)(=y(\cdot))$ is completely determined by $u(\cdot)$ and $x_{1}\left(t_{0}\right)$. Thus it is impossible to determine $x_{2}\left(t_{0}\right)$ by measuring the output, so the system is not completely observable (c.o.).
3.2.3 ThEOREM. The linear control system $\Sigma$ is c.o. if and only if the symmetric matrix, called the observability Gramian,

$$
\begin{equation*}
W_{o}\left(t_{0}, t_{1}\right):=\int_{t_{0}}^{t_{1}} \Phi^{T}\left(\tau, t_{0}\right) C^{T}(\tau) C(\tau) \Phi\left(\tau, t_{0}\right) d \tau \in \mathbb{R}^{m \times m} \tag{3.6}
\end{equation*}
$$

is nonsingular.

Proof : $\quad(\Leftarrow)$ Sufficiency. Assuming $u(t) \equiv 0, t \in\left[t_{0}, t_{1}\right]$, we have

$$
y(t)=C(t) \Phi\left(t, t_{0}\right) x_{0} .
$$

Multiplying this relation on the left by $\Phi^{T}\left(t, t_{0}\right) C^{T}(t)$ and integrating produces

$$
\int_{t_{0}}^{t_{1}} \Phi^{T}\left(\tau, t_{0}\right) C^{T}(\tau) y(\tau) d \tau=W_{o}\left(t_{0}, t_{1}\right) x_{0}
$$

so that if $W_{o}\left(t_{0}, t_{1}\right)$ is nonsingular, the initial state is

$$
x_{0}=W_{o}\left(t_{0}, t_{1}\right)^{-1} \int_{t_{0}}^{t_{1}} \Phi^{T}\left(\tau, t_{0}\right) C^{T}(\tau) y(\tau) d \tau
$$

so $\Sigma$ is c.o.
$(\Rightarrow)$ Necessity. We now assume that $\Sigma$ is c.o. and prove that $W=W_{o}\left(t_{0}, t_{1}\right)$ is nonsingular. First, if $\alpha \in \mathbb{R}^{m \times 1}$ is an arbitrary column vector,

$$
\alpha^{T} W \alpha=\int_{t_{0}}^{t_{1}}\left(C(\tau) \Phi\left(\tau, t_{0}\right) \alpha\right)^{T} C(\tau) \Phi\left(\tau, t_{0}\right) \alpha d \tau \geq 0
$$

so $W_{o}\left(t_{0}, t_{1}\right)$ is positive semi-definite. Next, suppose there exists an $\bar{\alpha} \neq 0$ such that $\bar{\alpha}^{T} W \bar{\alpha}=0$. It then follows that

$$
C(\tau) \Phi\left(\tau, t_{0}\right) \bar{\alpha} \equiv 0, \quad t_{0} \leq \tau \leq t_{1}
$$

This implies that when $x_{0}=\bar{\alpha}$ the output is identically zero throughout the time interval, so that $x_{0}$ cannot be determined in this case from the knowledge of $y(\cdot)$. This contradicts the assumption that $\Sigma$ is c.o., hence $W_{o}\left(t_{0}, t_{1}\right)$ is positive definite, and therefore nonsingular.

Note : Since the observability of $\Sigma$ is independent of $B$, we may refer to the observability of the pair $(A, C)$.

## Duality

3.2.4 Theorem. The linear control system (with outputs) $\Sigma$ defined by

$$
\left\{\begin{array}{l}
\dot{x}=A(t) x+B(t) u(t) \\
y=C(t) x
\end{array}\right.
$$

is c.c. if and only if the dual system $\Sigma^{\circ}$ defined by

$$
\left\{\begin{array}{l}
\dot{x}=-A^{T}(t) x+C^{T}(t) u(t) \\
y=B^{T}(t) x
\end{array}\right.
$$

is c.o.; and conversely.
Proof: We can see that if $\Phi\left(t, t_{0}\right)$ is the state transition matrix for the system $\Sigma$, then $\Phi^{T}\left(t_{0}, t\right)$ is the state transition matrix for the dual system $\Sigma^{\circ}$. Indeed, differentiate $I_{m}=\Phi\left(t, t_{0}\right) \Phi\left(t, t_{0}\right)^{-1}$ to get

$$
\begin{aligned}
0=\frac{d}{d t} I_{m} & =\dot{\Phi}\left(t, t_{0}\right) \Phi\left(t, t_{0}\right)^{-1}+\Phi\left(t, t_{0}\right) \dot{\Phi}\left(t_{0}, t\right) \\
& =A(t) \Phi\left(t, t_{0}\right) \Phi\left(t, t_{0}\right)^{-1}+\Phi\left(t, t_{0}\right) \dot{\Phi}\left(t_{0}, t\right) \\
& =A(t)+\Phi\left(t, t_{0}\right) \dot{\Phi}\left(t_{0}, t\right) .
\end{aligned}
$$

This implies

$$
\dot{\Phi}\left(t_{0}, t\right)=-\Phi\left(t_{0}, t\right) A(t)
$$

or

$$
\dot{\Phi}^{T}\left(t_{0}, t\right)=-A^{T}(t) \Phi^{T}\left(t_{0}, t\right) .
$$

Furthermore, the controllability matrix

$$
W_{c}^{\Sigma}\left(t_{0}, t_{1}\right)=\int_{t_{0}}^{t_{1}} \Phi\left(t_{0}, \tau\right) B(\tau) B^{T}(\tau) \Phi^{T}\left(t_{0}, \tau\right) d \tau
$$

(associated with $\Sigma$ ) is identical to the observability matrix $W_{o}^{\Sigma}\left(t_{0}, t_{1}\right)$ (associated with $\Sigma^{\circ}$ ).

Conversely, the observability matrix

$$
W_{o}^{\Sigma}\left(t_{0}, t_{1}\right)=\int_{t_{0}}^{t_{1}} \Phi^{T}\left(\tau, t_{0}\right) C^{T}(\tau) C(\tau) \Phi\left(\tau, t_{0}\right) d \tau
$$

(associated with $\Sigma$ ) is identical to the controllability matrix $W_{c}^{\Sigma^{\circ}}\left(t_{0}, t_{1}\right)$ (associated with $\Sigma^{\circ}$ ).

Note : This duality theorem is extremely useful, since it enables us to deduce immediately from a controllability result the corresponding one on observability (and conversely). For example, to obtain the observability criterion for the time-invariant case, we simply apply Theorem 3.1 .3 to $\Sigma^{\circ}$ to obtain the following result.
3.2.5 Theorem. The linear time-invariant control system

$$
\left\{\begin{array}{l}
\dot{x}=A x+B u(t)  \tag{3.7}\\
y=C x
\end{array}\right.
$$

(or the pair $(A, C)$ ) is c.o. if and only if the (Kalman) observability matrix

$$
\mathcal{O}=\mathcal{O}(A, C):=\left[\begin{array}{c}
C \\
C A \\
C A^{2} \\
\vdots \\
C A^{m-1}
\end{array}\right] \in \mathbb{R}^{m n \times m}
$$

has rank $m$.
3.2.6 Example. Consider the linear control system $\Sigma$ described by

$$
\left\{\begin{array}{l}
\dot{x}=\left[\begin{array}{rr}
-2 & 2 \\
1 & -1
\end{array}\right] x+\left[\begin{array}{l}
1 \\
0
\end{array}\right] u(t) \\
y=x_{1} .
\end{array}\right.
$$

The (Kalman) observability matrix is

$$
\mathcal{O}=\mathcal{O}^{\Sigma}=\left[\begin{array}{rr}
1 & 0 \\
-2 & 2
\end{array}\right]
$$

which has rank 2. Thus the control system $\Sigma$ is c.o.

In the single-output case (i.e. $n=1$ ), if $u(\cdot)=0$ and $y(\cdot)$ is known in the form

$$
\gamma_{1} e^{\lambda_{1} t}+\gamma_{2} e^{\lambda_{2} t}+\cdots+\gamma_{m} e^{\lambda_{m} t}
$$

assuming that all the eigenvalues $\lambda_{i}$ of $A$ are distinct, then $x_{0}$ can be obtained more easily than by using

$$
x_{0}=W_{o}\left(t_{0}, t_{1}\right)^{-1} \int_{t_{0}}^{t_{1}} \Phi^{T}\left(\tau, t_{0}\right) C^{T}(\tau) y(\tau) d \tau
$$

For suppose that $t_{0}=0$ and consider the solution of $\dot{x}=A x$ in the spectral form, namely

$$
x(t)=\left(v_{1} x(0)\right) e^{\lambda_{1} t} w_{1}+\left(v_{2} x(0)\right) e^{\lambda_{2} t} w_{2}+\cdots+\left(v_{m} x(0)\right) e^{\lambda_{m} t} w_{m}
$$

We have

$$
y(t)=\left(v_{1} x(0)\right)\left(c w_{1}\right) e^{\lambda_{1} t}+\left(v_{2} x(0)\right)\left(c w_{2}\right) e^{\lambda_{2} t}+\cdots+\left(v_{m} x(0)\right)\left(c w_{m}\right) e^{\lambda_{m} t}
$$

and equating coefficients of the exponential terms gives

$$
v_{i} x(0)=\frac{\gamma_{i}}{c w_{i}} \quad(i=1,2, \ldots, m)
$$

This represents $m$ linear equations for the $m$ unknown components of $x(0)$ in terms of $\gamma_{i}, v_{i}$ and $w_{i}(i=1,2, \ldots, m)$.

Again, in the single-output case, $C$ reduces to a row matrix $c$ and TheOREM 3.2 .5 can be restated as :

A linear system (with outputs) in the form

$$
\left\{\begin{array}{l}
\dot{x}=A x \\
y=c x
\end{array}\right.
$$

can be transformed into the canonical form

$$
\left\{\begin{array}{l}
\dot{v}=E v \\
y=f v
\end{array}\right.
$$

if and only if it is c.o.

## Decomposition of control systems

By duality, the result corresponding to THEOREM 3.1.11 is :
3.2.7 ThEOREM. When the linear control system $\Sigma$ is time-invariant then if the observability matrix $\mathcal{O}^{\Sigma}$ has rank $m_{1}<m$ there exists a control system, algebraically equivalent to $\Sigma$, having the form

$$
\begin{aligned}
{\left[\begin{array}{l}
\dot{x}_{(1)} \\
\dot{x}_{(2)}
\end{array}\right] } & =\left[\begin{array}{cc}
A_{1} & 0 \\
A_{2} & A_{3}
\end{array}\right]\left[\begin{array}{l}
x_{(1)} \\
x_{(2)}
\end{array}\right]+\left[\begin{array}{l}
B_{1} \\
B_{2}
\end{array}\right] u(t) \\
y & =C_{1} x_{(1)}
\end{aligned}
$$

where $x_{(1)}$ and $x_{(2)}$ have orders $m_{1}$ and $m-m_{1}$, respectively and $\left(A_{1}, C_{1}\right)$ is c.o.

We close this section with a decomposition result which effectively combines together Theorems 3.1 .11 and 3.2 .7 to show that a linear time-invariant control system can split up into four mutually exclusive parts, respectively

- c.c. but unobservable
- c.c. and c.o.
- uncontrollable and unobservable
- c.o. but uncontrollable.
3.2.8 Theorem. When the linear control system $\Sigma$ is time-invariant it is algebraically equivalent to

$$
\begin{aligned}
{\left[\begin{array}{c}
\dot{x}_{(1)} \\
\dot{x}_{(2)} \\
\dot{x}_{(3)} \\
\dot{x}_{(4)}
\end{array}\right] } & =\left[\begin{array}{cccc}
A_{11} & A_{12} & A_{13} & A_{14} \\
0 & A_{22} & 0 & A_{24} \\
0 & 0 & A_{33} & A_{34} \\
0 & 0 & 0 & A_{44}
\end{array}\right]\left[\begin{array}{c}
x_{(1)} \\
x_{(2)} \\
x_{(3)} \\
x_{(4)}
\end{array}\right]+\left[\begin{array}{c}
B_{1} \\
B_{2} \\
0 \\
0
\end{array}\right] u(t) \\
y & =C_{2} x_{(2)}+C_{4} x_{(4)}
\end{aligned}
$$

where the subscripts refer to the stated classification.

### 3.3 Linear Feedback

Consider a linear control system $\Sigma$ defined by

$$
\begin{equation*}
\dot{x}=A x+B u(t) \tag{3.8}
\end{equation*}
$$

where $A \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{m \times \ell}$. Suppose that we apply a (linear) feedback, that is each control variable is a linear combination of the state variables, so that

$$
u(t)=K x(t)
$$

where $K \in \mathbb{R}^{\ell \times m}$ is a feedback matrix. The resulting closed loop system is

$$
\begin{equation*}
\dot{x}=(A+B K) x . \tag{3.9}
\end{equation*}
$$

## The pole-shifting theorem

We ask the question whether it is possible to exert some influence on the behaviour of the closed loop system and, if so, to what extent. A somewhat surprising result, called the Spectrum Assignment Theorem, says in essence that for almost any linear control system $\Sigma$ it is possible to obtain arbitrary eigenvalues for the matrix $A+B K$ (and hence arbitrary asymptotic behaviour) using suitable feedback laws (matrices) $K$, subject only to the obvious constraint that complex eigenvalues must appear in pairs. "Almost any" means that this will be true for (completely) controllable systems.

Note : This theorem is most often referred to as the Pole-Shifting Theorem, a terminology that is due to the fact that the eigenvalues of $A+B K$ are also the poles of the (complex) function

$$
z \mapsto \frac{1}{\operatorname{det}\left(z I_{n}-A-B K\right)}
$$

This function appears often in classical control design.
The Pole-Shifting Theorem is central to linear control systems theory and is itself the starting point for more interesting analysis. Once we know that arbitrary sets of eigenvalues can be assigned, it becomes of interest to compare the performance
of different such sets. Also, one may ask what happens when certain entries of $K$ are restricted to vanish, which corresponds to constraints on what can be implemented.
3.3.1 Theorem. Let $\Lambda=\left\{\theta_{1}, \theta_{2}, \ldots, \theta_{m}\right\}$ be an arbitrary set of $m$ complex numbers (appearing in conjugate pairs). If the linear control system $\Sigma$ is c.c., then there exists a matrix $K \in \mathbb{R}^{\ell \times m}$ such that the eigenvalues of $A+B K$ are the set $\Lambda$.

Proof (when $\ell=1$ ): Since $\dot{x}=A x+B u(t)$ is c.c., it follows that there exists a (linear) transformation $w=T x$ such that the given system is transformed into

$$
\dot{w}=C w+d u(t)
$$

where

$$
C=\left[\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-k_{m} & -k_{m-1} & -k_{m-2} & \cdots & -k_{1}
\end{array}\right], \quad d=\left[\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
1
\end{array}\right]
$$

The feedback control $u=\underline{k} w$, where

$$
\underline{k}:=\left[\begin{array}{llll}
\underline{k}_{m} & \underline{k}_{m-1} & \cdots & \underline{k}_{1}
\end{array}\right]
$$

produces the closed loop matrix $C+d \underline{k}$, which has the same companion form as $C$ but with last row $-\left[\begin{array}{llll}\gamma_{m} & \gamma_{m-1} & \cdots & \gamma_{1}\end{array}\right]$, where

$$
\begin{equation*}
\underline{k}_{i}=k_{i}-\gamma_{i}, \quad i=1,2, \ldots, m \tag{3.10}
\end{equation*}
$$

Since

$$
C+d \underline{k}=T(A+b \underline{k} T) T^{-1}
$$

it follows that the desired matrix is

$$
K=\underline{k} T
$$

the entries $\underline{k}_{i}(i=1,2, \ldots, m)$ being given by (3.10).
In this equation $k_{i}(i=1,2, \ldots, m)$ are the coefficients in the characteristic polynomial of $A$; that is,

$$
\operatorname{det}\left(\lambda I_{m}-A\right)=\lambda^{m}+k_{1} \lambda^{m-1}+\cdots+k_{m}
$$

and $\gamma_{i}(i=1,2, \ldots, m)$ are obtained by equating coefficients of $\lambda$ in

$$
\lambda^{m}+\gamma_{1} \lambda^{m-1}+\cdots+\gamma_{m} \equiv\left(\lambda-\theta_{1}\right)\left(\lambda-\theta_{2}\right) \cdots\left(\lambda-\theta_{m}\right) .
$$

Note : The solution of (the closed loop system)

$$
\dot{x}=(A+B K) x
$$

depends on the eigenvalues of $A+B K$, so provided the control system $\Sigma$ is c.c., the theorem tells us that using linear feedback it is possible to exert a considerable influence on the time behaviour of the closed loop system by suitably choosing the numbers $\theta_{1}, \theta_{2}, \ldots, \theta_{m}$.
3.3.2 Corollary. If the linear time-invariant control system

$$
\left\{\begin{array}{l}
\dot{x}=A x+B u(t) \\
y=c x
\end{array}\right.
$$

is c.o., then there exists a matrix $L \in \mathbb{R}^{m \times 1}$ such that the eigenvalues of $A+L c$ are the set $\Lambda$.

This result can be deduced from Theorem 3.3.1 using the Duality TheOREM.
3.3.3 Example. Consider the linear control system

$$
\dot{x}=\left[\begin{array}{rr}
1 & -3 \\
4 & 2
\end{array}\right] x+\left[\begin{array}{l}
1 \\
1
\end{array}\right] u(t) .
$$

The characteristic equation of $A$ is

$$
\operatorname{char}_{A}(\lambda) \equiv \lambda^{2}-3 \lambda+14=0
$$

which has roots $\frac{3 \pm i \sqrt{47}}{2}$.
Suppose we wish the eigenvalues of the closed loop system to be -1 and -2 , so that the characteristic polynomial is

$$
\lambda^{2}+3 \lambda+2 .
$$

We have

$$
\begin{aligned}
& \underline{k}_{1}=k_{1}-\gamma_{1}=-3-3=-6 \\
& \underline{k}_{2}=k_{2}-\gamma_{2}=14-2=12 .
\end{aligned}
$$

Hence

$$
K=\underline{k} T=\frac{1}{8}\left[\begin{array}{ll}
12 & -6
\end{array}\right]\left[\begin{array}{rr}
-1 & 1 \\
3 & 5
\end{array}\right]=-\left[\begin{array}{ll}
\frac{15}{4} & \frac{9}{4}
\end{array}\right] .
$$

It is easy to verify that

$$
A+b K=\frac{1}{4}\left[\begin{array}{rr}
-11 & -21 \\
1 & -1
\end{array}\right]
$$

does have the desired eigenvalues.
3.3.4 Lemma. If the linear control system $\Sigma$ defined by

$$
\dot{x}=A x+B u(t)
$$

is c.c. and $B=\left[\begin{array}{llll}b_{1} & b_{2} & \cdots & b_{\ell}\end{array}\right]$ with $b_{i} \neq 0, \quad i=1,2, \ldots, \ell$, then there exist matrices $K_{i} \in \mathbb{R}^{\ell \times m}, \quad i=1,2, \ldots, \ell$ such that the systems

$$
\dot{x}=\left(A+B K_{i}\right) x+b_{i} u(t)
$$

are c.c.
Proof : For convenience consider the case $i=1$. Since the matrix

$$
\mathcal{C}=\left[\begin{array}{lllll}
B & A B & A^{2} B & \ldots & A^{m-1} B
\end{array}\right]
$$

has full rank, it is possible to select from its columns at least one set of $m$ vectors which are linearly independent. Define an $m \times m$ matrix $M$ by choosing such a set as follows :

$$
M=\left[\begin{array}{lllllllll}
b_{1} & A b_{1} & \ldots & A^{r_{1}-1} b_{1} & b_{2} & A b_{2} & \ldots & A^{r_{2}-1} b_{2} & \ldots
\end{array}\right]
$$

where $r_{i}$ is the smallest integer such that $A^{r_{i}} b_{i}$ is linearly dependent on all the preceding vectors, the process continuing until $m$ columns of $U$ are taken.

Define an $\ell \times m$ matrix $N$ having its $r_{1}^{t h}$ column equal to $e_{2}$, the second column of $I_{\ell}$, its $\left(r_{1}+r_{2}\right)^{\text {th }}$ column equal to $e_{3}$, its $\left(r_{1}+r_{2}+r_{3}\right)^{t h}$ column equal to $e_{4}$ and so on, all its other columns being zero.

It is then not difficult to show that the desired matrix in the statement of the Lemma is

$$
K_{1}=N M^{-1} .
$$

Proof of Theorem 3.3 .1 when $\ell>1$ :
Let $K_{1}$ be the matrix in the proof of Lemma 3.3.4 and define an $\ell \times m$ matrix $K^{\prime}$ having as its first row some vector $k$, and all its other rows zero. Then the control

$$
u=\left(K_{1}+K^{\prime}\right) x
$$

leads to the closed loop system

$$
\dot{x}=\left(A+B K_{1}\right) x+B K^{\prime} x=\left(A+B K_{1}\right) x+b_{1} k x
$$

where $b_{1}$ is the first column of $B$.
Since the system

$$
\dot{x}=\left(A+B K_{1}\right) x+b_{1} u
$$

is c.c., it now follows from the proof of the theorem when $\ell=1$, that $k$ can be chosen so that the eigenvalues of $A+B K_{1}+b_{1} k$ are the set $\Lambda$, so the desired feedback control is indeed $u=\left(K_{1}+K^{\prime}\right) x$.

If $y=C x$ is the output vector, then again by duality we can immediately deduce
3.3.5 Corollary. If the linear control system

$$
\left\{\begin{array}{l}
\dot{x}=A x+B u(t) \\
y=C x
\end{array}\right.
$$

is c.o., then there exists a matrix $L \in \mathbb{R}^{m \times n}$ such that the eigenvalues of $A+L C$ are the set $\Lambda$.

Algorithm for constructing a feedback matrix
The following method gives a practical way of constructing the feedback matrix $K$. Let all the eigenvalues $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{m}$ of $A$ be distinct and let

$$
W=\left[\begin{array}{llll}
w_{1} & w_{2} & \ldots & w_{m}
\end{array}\right]
$$

where $w_{i}$ is an eigenvector corresponding to the eigenvalue $\lambda_{i}$. With linear feedback $u=-K x$, suppose that the eigenvalues of $A$ and $A-B K$ are ordered so that those of $A-B K$ are to be

$$
\mu_{1}, \mu_{2}, \ldots, \mu_{r}, \lambda_{r+1}, \ldots, \lambda_{m} \quad(r \leq m)
$$

Then provided the linear system $\Sigma$ is c.c., a suitable matrix is

$$
K=f g \widetilde{W}
$$

where $\widetilde{W}$ consists of the first $r$ rows of $W^{-1}$, and

$$
\begin{aligned}
& g=\left[\begin{array}{llll}
\frac{\alpha_{1}}{\beta_{1}} & \frac{\alpha_{2}}{\beta_{2}} & \cdots & \frac{\alpha_{r}}{\beta_{r}}
\end{array}\right] \\
& \alpha_{i}= \begin{cases}\frac{\prod_{j=1}^{r}\left(\lambda_{i}-\mu_{j}\right)}{\frac{\prod_{j}}{r=1}} \begin{array}{ll}
\prod_{\substack{ \\
j \neq i}}^{r}\left(\lambda_{i}-\lambda_{j}\right) & \\
\lambda_{1}-\mu_{1} & \text { if } r>1 \\
&
\end{array} \text { if=1}\end{cases} \\
& \beta=\left[\begin{array}{llll}
\beta_{1} & \beta_{2} & \ldots & \beta_{r}
\end{array}\right]^{T}=\widetilde{W} B f
\end{aligned}
$$

$f$ being any column $\ell$-vector such that all $\beta_{i} \neq 0$.
3.3.6 Example. Consider the linear system

$$
\dot{x}=\left[\begin{array}{rr}
0 & 1 \\
-2 & -3
\end{array}\right] x+\left[\begin{array}{l}
2 \\
1
\end{array}\right] u(t)
$$

We have

$$
\begin{gathered}
\lambda_{1}=-1, \quad \lambda_{2}=-2 \quad \text { and } \\
W=\left[\begin{array}{rr}
1 & 1 \\
-1 & -2
\end{array}\right], \quad W^{-1}=\left[\begin{array}{rr}
2 & 1 \\
-1 & -1
\end{array}\right]
\end{gathered}
$$

Suppose that

$$
\mu_{1}=-3, \quad \mu_{2}=-4, \quad \text { so } \quad \widetilde{W}=W^{-1}
$$

We have

$$
\alpha_{1}=6, \quad \alpha_{2}=-2
$$

and $\beta=\widetilde{W} B f$ gives

$$
\left[\begin{array}{l}
\beta_{1} \\
\beta_{2}
\end{array}\right]=\left[\begin{array}{rr}
2 & 1 \\
-1 & -1
\end{array}\right]\left[\begin{array}{l}
2 \\
1
\end{array}\right] f=\left[\begin{array}{r}
5 f_{1} \\
-3 f_{1}
\end{array}\right]
$$

Hence we can take $f_{1}=1$, which results in

$$
g=\left[\begin{array}{ll}
\frac{6}{5} & \frac{2}{3}
\end{array}\right] .
$$

Finally, the desired feedback matrix is

$$
K=1 \cdot\left[\begin{array}{ll}
\frac{6}{5} & \frac{2}{3}
\end{array}\right] W^{-1}=\left[\begin{array}{cc}
\frac{26}{15} & \frac{8}{15}
\end{array}\right] .
$$

3.3.7 EXAMPLE. Consider now the linear control system

$$
\dot{x}=\left[\begin{array}{rr}
0 & 1 \\
-2 & -3
\end{array}\right] x+\left[\begin{array}{ll}
2 & 1 \\
1 & 0
\end{array}\right] u(t) \text {. }
$$

We now obtain

$$
\left[\begin{array}{l}
\beta_{1} \\
\beta_{2}
\end{array}\right]=\left[\begin{array}{c}
5 f_{1}+2 f_{2} \\
-3 f_{1}-f_{2}
\end{array}\right]
$$

so that $f_{1}=1, \quad f_{2}=0$ gives

$$
K=\left[\begin{array}{cc}
\frac{26}{15} & \frac{8}{15} \\
0 & 0
\end{array}\right] .
$$

However $f_{1}=1, \quad f_{2}=-1$ gives

$$
\beta_{1}=3, \quad \beta_{2}=-2 \quad \text { so that } g=\left[\begin{array}{ll}
2 & 1
\end{array}\right]
$$

and from $K=f g \widetilde{W}$ we now have

$$
K=\left[\begin{array}{r}
1 \\
-1
\end{array}\right]\left[\begin{array}{ll}
2 & 1
\end{array}\right] W^{-1}=\left[\begin{array}{rr}
3 & 1 \\
-3 & -1
\end{array}\right] .
$$

### 3.4 Realization Theory

The realization problem may be viewed as "guessing the equations of motion (i.e. state equations) of a control system from its input/output behaviour" or, if one prefers, "setting up a physical model which explains the experimental data".

Consider the linear control system (with outputs) $\Sigma$ described by

$$
\left\{\begin{array}{l}
\dot{x}=A x+B u(t)  \tag{3.11}\\
y=C x
\end{array}\right.
$$

where $A \in \mathbb{R}^{m \times m}, B \in \mathbb{R}^{m \times \ell}$ and $C \in \mathbb{R}^{n \times m}$.
Taking Laplace transforms of (3.11) and assuming zero initial conditions gives

$$
s \bar{x}(s)=A \bar{x}(s)+B \bar{u}(s)
$$

and after rearrangement

$$
\bar{x}(s)=\left(s I_{m}-A\right)^{-1} B \bar{u}(s) .
$$

The Laplace transform of the output is

$$
\bar{y}(s)=C \bar{x}(s)
$$

and thus

$$
\bar{y}(s)=C\left(s I_{m}-A\right)^{-1} B \bar{u}(s)=G(s) \bar{u}(s)
$$

where the $n \times \ell$ matrix

$$
\begin{equation*}
G(s):=C\left(s I_{m}-A\right)^{-1} B \tag{3.12}
\end{equation*}
$$

is called the transfer function matrix since it relates the Laplace transform of the output vector to that of the input vector.

Exercise 41 Evaluate (the Laplace transform of the exponential)

$$
\mathcal{L}\left[e^{a t}\right](s):=\int_{0}^{\infty} e^{-s t} e^{a t} d t
$$

and then show that (for $A \in \mathbb{R}^{m \times m}$ ):

$$
\mathcal{L}[\exp (t A)](s)=\left(s I_{m}-A\right)^{-1} .
$$

Using relation

$$
\begin{equation*}
\left(s I_{m}-A\right)^{-1}=\frac{s^{m-1} I_{m}+s^{m-2} B_{1}+s^{m-3} B_{2}+\cdots+B_{m-1}}{\operatorname{char}_{A}(s)} \tag{3.13}
\end{equation*}
$$

where the $k_{i}$ and $B_{i}$ are determined successively by

$$
\begin{aligned}
& B_{1}=A+k_{1} I_{m}, \quad B_{i}=A B_{i-1}+k_{i} I_{m} ; \quad i=2,3, \ldots, m-1 \\
& k_{1}=-\operatorname{tr}(A), \quad k_{i}=-\frac{1}{i} \operatorname{tr}\left(A B_{i-1}\right) ; \quad i=2,3, \ldots, m
\end{aligned}
$$

the expression (3.12) becomes

$$
G(s)=\frac{s^{m-1} G_{0}+s^{m-2} G_{1}+\cdots+G_{m-1}}{\chi(s)}=\frac{H(s)}{\chi(s)}
$$

where $\chi(s)=\operatorname{char}_{A}(s)$ and $G_{k}=\left[g_{i j}^{(k)}\right] \in \mathbb{R}^{n \times \ell}, \quad k=0,1,2, \ldots, m-1$. The $n \times \ell$ matrix $H(s)$ is called a polynomial matrix, since each of its entries is itself a polynomial; that is,

$$
h_{i j}=s^{m-1} g_{i j}^{(0)}+s^{m-2} g_{i j}^{(1)}+\cdots+g_{i j}^{(m-1)} .
$$

NOTE : The formulas above, used mainly for theoretical rather than computational purposes, constitute Leverrier's algorithm.
3.4.1 Example. Consider the electrically-heated oven described in section 1.3, and suppose that the values of the constants are such that the state equations are

$$
\dot{x}=\left[\begin{array}{rr}
-2 & 2 \\
1 & -1
\end{array}\right] x+\left[\begin{array}{l}
1 \\
0
\end{array}\right] u(t) .
$$

Suppose that the output is provided by a thermocouple in the jacket measuring the jacket (excess) temperature, i.e.

$$
y=\left[\begin{array}{ll}
1 & 0
\end{array}\right] x .
$$

The expression (3.12) gives

$$
G(s)=\left[\begin{array}{ll}
1 & 0
\end{array}\right]\left[\begin{array}{rr}
s+2 & -2 \\
-1 & s+1
\end{array}\right]^{-1}\left[\begin{array}{l}
1 \\
0
\end{array}\right]=\frac{s+1}{s^{2}+3 s}
$$

using

$$
\left(s I_{2}-A\right)^{-1}=\frac{1}{\operatorname{char}_{A}(s)} \operatorname{adj}\left(s I_{2}-A\right) .
$$

## Realizations

In practice it often happens that the mathematical description of a (linear time-invariant) control system - in terms of differential equations - is not known, but $G(s)$ can be determined from experimental measurements or other considerations. It is then useful to find a system - in our usual state space form - to which $G(\cdot)$ corresponds.

In formal terms, given an $n \times \ell$ matrix $G(s)$, whose elements are rational functions of $s$, we wish to find (constant) matrices $A, B, C$ having dimensions $m \times m, m \times \ell$ and $n \times m$, respectively, such that

$$
G(s)=C\left(s I_{m}-A\right)^{-1} B
$$

and the system equations will then be

$$
\left\{\begin{array}{l}
\dot{x}=A x+B u(t) \\
y=C x
\end{array}\right.
$$

The triple $(A, B, C)$ is termed a realization of $G(\cdot)$ of order $m$, and is not, of course, unique. Amongst all such realizations some will include matrices $A$ having least dimensions - these are called minimal realizations, since the corresponding systems involve the smallest possible number of state variables.

Note : Since each element in

$$
\left(s I_{m}-A\right)^{-1}=\frac{\operatorname{adj}\left(s I_{m}-A\right)}{\operatorname{det}\left(s I_{m}-A\right)}
$$

has the degree of the numerator less than that of the denominator, it follows that

$$
\lim _{s \rightarrow \infty} C\left(s I_{m}-A\right)^{-1} B=0
$$

and we shall assume that any given $G(s)$ also has this property, $G(\cdot)$ then being termed strictly proper.
3.4.2 EXAMPLE. Consider the scalar transfer function

$$
g(s)=\frac{2 s+7}{s^{2}-5 s+6}
$$

It is easy to verify that one realization of $g(\cdot)$ is

$$
A=\left[\begin{array}{cc}
0 & 1 \\
-6 & 5
\end{array}\right], \quad b=\left[\begin{array}{l}
0 \\
1
\end{array}\right], \quad c=\left[\begin{array}{ll}
7 & 2
\end{array}\right] .
$$

It is also easy to verify that a quite different triple is

$$
A=\left[\begin{array}{ll}
2 & 0 \\
0 & 3
\end{array}\right], \quad b=\left[\begin{array}{l}
1 \\
1
\end{array}\right], \quad c=\left[\begin{array}{ll}
-11 & 13
\end{array}\right] .
$$

Note : Both these realizations are minimal, and there is in consequence a simple relationship between them, as we shall see later.

## Algebraic equivalence and realizations

It is now appropriate to return to the idea of algebraic equivalence of linear control systems (defined in section 3.1), and discuss its implications for the realization problem. The (linear) transformation

$$
\tilde{x}=P x
$$

produces a linear control system with matrices

$$
\begin{equation*}
\widetilde{A}=P A P^{-1}, \quad \widetilde{B}=P B, \quad \widetilde{C}=C P^{-1} \tag{3.14}
\end{equation*}
$$

Exercise 42 Show that if $(A, B, C)$ represents a c.c. (or c.o.) linear control system, then so does $(\widetilde{A}, \widetilde{B}, \widetilde{C})$.

Exercise 43 Show that if two linear control systems are algebraically equivalent, then their transfer function matrices are identical (i.e.

$$
\left.C\left(s I_{m}-A\right)^{-1} B=\widetilde{C}\left(s I_{m}-\widetilde{A}\right)^{-1} \widetilde{B}\right)
$$

## Characterization of minimal realizations

We can now state and prove the central result of this section, which links together the three basic concepts of controllability, observability, and realization.
3.4.3 Theorem. A realization $(A, B, C)$ of a given transfer function matrix $G(\cdot)$ is minimal if and only if the pair $(A, B)$ is c.c. and the pair $(A, C)$ is c.o.

Proof : $\quad(\Leftarrow)$ Sufficiency. Let $\mathcal{C}$ and $\mathcal{O}$ be the controllability and observability matrices, respectively; that is,
$\mathcal{C}=\mathcal{C}(A, B)=\left[\begin{array}{lllll}B & A B & A^{2} B & \ldots & A^{m-1} B\end{array}\right] \quad$ and $\quad \mathcal{O}=\mathcal{O}(A, C)=\left[\begin{array}{c}C \\ C A \\ C A^{2} \\ \vdots \\ C A^{m-1}\end{array}\right]$.

We wish to show that if these both have rank $m$, then (the realization of) $G(\cdot)$ has least order $m$. Suppose that there exists a realization $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ of $G(\cdot)$ with $\widetilde{A}$ having order $\widetilde{m}$. Since

$$
C\left(s I_{m}-A\right)^{-1} B=\widetilde{C}\left(s I_{m}-\widetilde{A}\right)^{-1} \widetilde{B}
$$

it follows that

$$
C \exp (t A) B=\widetilde{C} \exp (t \widetilde{A}) \widetilde{B}
$$

which implies that

$$
C A^{i} B=\widetilde{C} \widetilde{A}^{i} \widetilde{B}, \quad i=0,1,2, \ldots
$$

Consider the product

$$
\begin{aligned}
\mathcal{O C} & =\left[\begin{array}{c}
C \\
C A \\
\vdots \\
C A^{m-1}
\end{array}\right]\left[\begin{array}{llll}
B & A B & \ldots & A^{n-1} B
\end{array}\right] \\
& =\left[\begin{array}{cccc}
C B & C A B & \ldots & C A^{m-1} B \\
C A B & C A^{2} B & \ldots & C A^{m} B \\
\vdots & \vdots & & \vdots \\
C A^{m-1} B & C A^{m} B & \ldots & C A^{2 m-2} B
\end{array}\right] \\
& =\left[\begin{array}{c}
\widetilde{C} \\
\widetilde{C} \widetilde{A} \\
\vdots \\
\widetilde{C} \widetilde{A}^{m-1}
\end{array}\right]\left[\begin{array}{llll}
\widetilde{B} & \widetilde{A} \widetilde{B} & \ldots & \widetilde{A}^{m-1} \widetilde{B}
\end{array}\right] \\
& =\widetilde{\mathcal{O} \widetilde{C} .}
\end{aligned}
$$

Exercise 44 Given $A \in \mathbb{R}^{\ell \times m}$ and $B \in \mathbb{R}^{m \times n}$, show that if $\operatorname{rank}(A)=\operatorname{rank}(B)=$ $m$, then $\operatorname{rank}(A B)=m$. [Hint: Use the results of Exercise 3.]

The matrix $\mathcal{O C}$ has rank $m$, so the matrix $\widetilde{\mathcal{O}} \widetilde{\mathcal{C}}$ also has rank $m$. However, the rank of $\widetilde{\mathcal{O}} \widetilde{C}$ cannot be greater than $\widetilde{m}$. That is, $m \leq \widetilde{m}$, so there can be no realization of $G(\cdot)$ having order less than $m$.
$(\Rightarrow)$ Necessity. We show that if the pair $(A, B)$ is not completely controllable, then there exists a realization of $G(\cdot)$ having order less than $m$. The corresponding part of the proof involving observability follows from duality.

Let the rank of $\mathcal{C}$ be $m_{1}<m$ and let $u_{1}, u_{2}, \ldots, u_{m_{1}}$ be any set of $m_{1}$ linearly independent columns of $\mathcal{C}$. Consider the (linear) transformation

$$
\widetilde{x}=P x
$$

with the $m \times m$ matrix $P$ defined by

$$
P^{-1}=\left[\begin{array}{lllllll}
u_{1} & u_{2} & \ldots & u_{m_{1}} & u_{m_{1}+1} & \ldots & u_{m} \tag{3.15}
\end{array}\right]
$$

where the columns $u_{m_{1}+1}, \ldots, u_{m}$ are any vectors which make the matrix $P^{-1}$ nonsingular. Since $\mathcal{C}$ has rank $m_{1}$ it follows that all its columns can be expressed as a linear combination of the basis $u_{1}, u_{2}, \ldots, u_{m_{1}}$. The matrix

$$
A \mathcal{C}=\left[\begin{array}{llll}
A B & A^{2} B & \ldots & A^{m} B
\end{array}\right]
$$

contains all but the first $\ell$ columns of $\mathcal{C}$, so in particular it follows that the vectors $A u_{i}, \quad i=1,2, \ldots, m_{1}$ can be expressed in terms of the same basis. Multiplying both sides of (3.15) on the left by $P$ shows that $P u_{i}$ is equal to the $i^{\text {th }}$ column of $I_{m}$. Combining these facts together we obtain

$$
\begin{aligned}
\widetilde{A} & =P A P^{-1} \\
& =P\left[\begin{array}{lllll}
A u_{1} & \ldots & A u_{m_{1}} & \ldots & A u_{m}
\end{array}\right] \\
& =\left[\begin{array}{cc}
A_{1} & A_{2} \\
0 & A_{3}
\end{array}\right]
\end{aligned}
$$

where $A_{1}$ is $m_{1} \times m_{1}$. Similarly, since $u_{1}, u_{2}, \ldots, u_{m_{1}}$ also forms a basis for the columns of $B$ we have from (3.14) and (3.15)

$$
\widetilde{B}=P B=\left[\begin{array}{c}
B_{1} \\
0
\end{array}\right]
$$

where $B_{1}$ is $m_{1} \times \ell$. Writing

$$
\widetilde{C}=C P^{-1}=\left[\begin{array}{ll}
C_{1} & C_{2}
\end{array}\right]
$$

we have (see Exercise 43 and also Exercise 60)

$$
\begin{aligned}
G(s) & =\widetilde{C}\left(s I_{m}-\widetilde{A}\right)^{-1} \widetilde{B} \\
& =\left[\begin{array}{ll}
C_{1} & C_{2}
\end{array}\right]\left[\begin{array}{cc}
s I_{m_{1}}-A_{1} & -A_{2} \\
0 & s I_{m-m_{1}}-A_{3}
\end{array}\right]^{-1}\left[\begin{array}{c}
B_{1} \\
0
\end{array}\right] \\
& =\left[\begin{array}{ll}
C_{1} & C_{2}
\end{array}\right]\left[\begin{array}{cc}
\left(s I_{m_{1}}-A_{1}\right)^{-1} & \left(s I_{m_{1}}-A_{1}\right)^{-1} A_{2}\left(s I_{m-m_{1}}-A_{3}\right)^{-1} \\
0 & \left(s I_{m-m_{1}}-A_{3}\right)^{-1}
\end{array}\right]\left[\begin{array}{c}
B_{1} \\
0
\end{array}\right] \\
& =C_{1}\left(s I_{m_{1}}-A_{1}\right)^{-1} B_{1}
\end{aligned}
$$

showing that $\left(A_{1}, B_{1}, C_{1}\right)$ is a realization of $G(\cdot)$ having order $m_{1}<m$. This contradicts the assumption that $(A, B, C)$ is minimal, hence the pair $(A, B)$ must be c.c.
3.4.4 EXAMPLE. We apply the procedure introduced in the second part of the proof of THEOREM 3.4 .5 to split up the linear control system

$$
\dot{x}=\left[\begin{array}{rrr}
4 & 3 & 5  \tag{3.16}\\
1 & -2 & -3 \\
2 & 1 & 8
\end{array}\right] x+\left[\begin{array}{r}
2 \\
1 \\
-1
\end{array}\right] u(t)
$$

into its controllable and uncontrollable parts, as displayed below :

$$
\left[\begin{array}{c}
\dot{x}_{(1)} \\
\dot{x}_{(2)}
\end{array}\right]=\left[\begin{array}{rr}
A_{1} & A_{2} \\
0 & A_{3}
\end{array}\right]\left[\begin{array}{l}
x_{(1)} \\
x_{(2)}
\end{array}\right]+\left[\begin{array}{c}
B_{1} \\
0
\end{array}\right] u(t)
$$

where $x_{(1)}, x_{(2)}$ have orders $m_{1}$ and $m-m_{1}$, respectively, and $\left(A_{1}, B_{1}\right)$ is c.c.

The controllability matrix for (3.16) is

$$
\left[\begin{array}{lll}
B & A B & A^{2} B
\end{array}\right]=\left[\begin{array}{rrr}
2 & 6 & 18 \\
1 & 3 & 9 \\
-1 & -3 & -9
\end{array}\right]
$$

which clearly has rank $m_{1}=1$. For the transformation $\widetilde{x}=P x$ we follow (3.15) and set

$$
P^{-1}=\left[\begin{array}{rrr}
2 & 1 & 0 \\
1 & 0 & 1 \\
-1 & 0 & 0
\end{array}\right]
$$

where the column in (3.16) has been selected, and the remaining columns are simply arbitrary choices to produce a nonsingular matrix. It is then easy to
compute the inverse of the matrix above, and from (3.14)

$$
\begin{aligned}
& \widetilde{A}=P A P^{-1}=\left[\begin{array}{rrr}
3 & -2 & -1 \\
0 & 8 & 5 \\
0 & 3 & -1
\end{array}\right]=\left[\begin{array}{rr}
A_{1} & A_{2} \\
0 & A_{3}
\end{array}\right] \\
& \widetilde{B}=P B=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]=\left[\begin{array}{c}
B_{1} \\
0
\end{array}\right] .
\end{aligned}
$$

Notice that the transformation matrix is not unique. However, all possible matrices $\widetilde{A}$ will be similar to $3 \times 3$ matrix above. In particular, the eigenvalues of the uncontrollable part are those of $A_{3}$, namely the roots of

$$
0=\operatorname{det}\left(\lambda I_{2}-A_{3}\right)=\left|\begin{array}{cc}
\lambda-8 & -5 \\
-3 & \lambda+1
\end{array}\right|=\lambda^{2}-7 \lambda-23
$$

and these roots cannot be altered by applying linear feedback to (3.16).
Note : For any given transfer function matrix $G(\cdot)$ there are an infinite number of minimal realizations satisfying the conditions of Theorem 3.4.3. However, one can show that the relationship between any two minimal realizations is just that of algebraic equivalence: If $\mathcal{R}=(A, B, C)$ is a minimal realization of $G(\cdot)$, then $\widetilde{\mathcal{R}}=(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is also a minimal realization if and only if the following holds :

$$
\widetilde{A}=P A P^{-1}, \quad \widetilde{B} P B, \quad \widetilde{C}=C P^{-1} .
$$

## Algorithm for constructing a minimal realization

We do not have room to discuss the general problem of efficient construction of minimal realizations. We will give here one simple but nevertheless useful result.
3.4.5 Proposition. Let the denominators of the elements $g_{i j}(s)$ of $G(s) \in$ $\mathbb{R}^{n \times \ell}$ have simple roots $s_{1}, s_{2}, \ldots, s_{q}$. Define

$$
K_{i}:=\lim _{s \rightarrow s_{i}}\left(s-s_{i}\right) G(s), \quad i=1,2, \ldots, q
$$

and let

$$
r_{i}:=\operatorname{rank}\left(K_{i}\right), \quad i=1,2, \ldots, q .
$$

If $L_{i}$ and $M_{i}$ are $n \times r_{i}$ and $r_{i} \times \ell$ matrices, respectively, each having rank $r_{i}$ such that

$$
K_{i}=L_{i} M_{i}
$$

then a minimal realization of $G(\cdot)$ is

$$
A=\left[\begin{array}{cccc}
s_{1} I_{r_{1}} & & & O \\
& s_{2} I_{r_{2}} & & \\
& & \ddots & \\
O & & & s_{q} I_{r_{q}}
\end{array}\right], \quad B=\left[\begin{array}{c}
M_{1} \\
M_{2} \\
\vdots \\
M_{q}
\end{array}\right], \quad C=\left[\begin{array}{llll}
L_{1} & L_{2} & \ldots & L_{q}
\end{array}\right] .
$$

(To verify that $(A, B, C)$ is a realization of $G(\cdot)$ is straightforward. Indeed,

$$
\begin{aligned}
C\left(s I_{m}-A\right)^{-1} B & =\left[\begin{array}{lll}
L_{1} & \cdots & L_{q}
\end{array}\right]\left[\begin{array}{ccc}
\frac{1}{s-s_{1}} I_{r_{1}} & \cdots & 0 \\
& \ddots & \\
0 & \cdots & \frac{1}{s-s_{q}} I_{r_{q}}
\end{array}\right]\left[\begin{array}{c}
M_{1} \\
\vdots \\
M_{q}
\end{array}\right] \\
& =\frac{L_{1} M_{1}}{s-s_{1}}+\cdots+\frac{L_{q} M_{q}}{s-s_{q}}=\frac{K_{1}}{s-s_{1}}+\cdots+\frac{K_{q}}{s-s_{q}}=G(s) .
\end{aligned}
$$

Since $\operatorname{rank} \mathcal{C}(A, B)=\operatorname{rank} \mathcal{O}(A, C)=m$, this realization is minimal.)
3.4.6 Example. Consider the scalar transfer function

$$
g(s)=\frac{2 s+7}{s^{2}-5 s+6} .
$$

We have

$$
\begin{aligned}
& K_{1}=\lim _{s \rightarrow 2} \frac{(s-2)(2 s+7)}{(s-2)(s-3)}=-11, \quad r_{1}=1 \\
& K_{2}=\lim \frac{(s-3)(2 s+7)}{(s-2)(s-3)}=13, \quad r_{2}=1 .
\end{aligned}
$$

Taking

$$
L_{1}=K_{1}, \quad M_{1}=1, \quad L_{2}=K_{2}, \quad M_{2}=1
$$

produces a minimal realization

$$
A=\left[\begin{array}{ll}
2 & 0 \\
0 & 3
\end{array}\right], \quad b=\left[\begin{array}{l}
1 \\
1
\end{array}\right], \quad c=\left[\begin{array}{cc}
-11 & 13
\end{array}\right] .
$$

However,

$$
b=\left[\begin{array}{c}
m_{1} \\
m_{2}
\end{array}\right] \quad \text { and } \quad c=\left[\begin{array}{ll}
-\frac{11}{m_{1}} & \frac{13}{m_{2}}
\end{array}\right]
$$

can be used instead, still giving a minimal realization for arbitrary nonzero values of $m_{1}$ and $m_{2}$.

### 3.5 Exercises

Exercise 45 Verify that the control system described by

$$
\left[\begin{array}{l}
\dot{x}_{1} \\
\dot{x}_{2}
\end{array}\right]=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]+\left[\begin{array}{rr}
0 & 0 \\
1 & -1
\end{array}\right]\left[\begin{array}{l}
u_{1}(t) \\
u_{2}(t)
\end{array}\right]
$$

is c.c.

Exercise 46 Given the control system described by

$$
\dot{x}=\left[\begin{array}{rr}
-1 & -1 \\
2 & -4
\end{array}\right] x+b u(t)
$$

find for what vector $b$ the system is not c.c.

Exercise 47 For the (initialized) control system

$$
\dot{x}=\left[\begin{array}{rr}
-4 & 2 \\
4 & -6
\end{array}\right] x+\left[\begin{array}{l}
1 \\
2
\end{array}\right] u(t), \quad x(0)=\left[\begin{array}{l}
2 \\
3
\end{array}\right]
$$

apply a control in the form

$$
u(t)=c_{1}+c_{2} e^{-2 t}
$$

so as to bring the system to the origin at time $t=1$. Obtain, but do not solve, the equations which determine the constants $c_{1}$ and $c_{2}$.

Exercise 48 For each of the following cases, determine for what values of the real parameter $\alpha$ the control system is not c.c.

$$
\begin{aligned}
& \text { (1) } \dot{x}=\left[\begin{array}{rrr}
-1 & 1 & -1 \\
0 & -1 & \alpha \\
0 & 1 & 3
\end{array}\right] x+\left[\begin{array}{l}
0 \\
2 \\
1
\end{array}\right] u(t) \\
& \text { (2) } \dot{x}=\left[\begin{array}{rr}
2 & \alpha-3 \\
0 & 2
\end{array}\right] x+\left[\begin{array}{rr}
1 & 1 \\
0 & \alpha^{2}-\alpha
\end{array}\right] u(t) .
\end{aligned}
$$

In part (2), if the first control variable $u_{1}(\cdot)$ ceases to operate, for what additional values (if any) of $\alpha$ the system is not c.c. under the remaining scalar control $u_{2}(\cdot)$ ?

Exercise 49 Consider the control system defined by

$$
\left[\begin{array}{c}
\dot{x}_{1} \\
\dot{x}_{2}
\end{array}\right]=\left[\begin{array}{rr}
-1 & 0 \\
0 & -2
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2}
\end{array}\right]+\left[\begin{array}{l}
1 \\
3
\end{array}\right] u(t) .
$$

Such a system could be thought of as a simple representation of a vehicle suspension system: in this interpretation, $x_{1}$ and $x_{2}$ are the displacements of the end points of the platform from equilibrium. Verify that the system is c.c. If the ends of the platform are each given an initial displacement of 10 units, find using

$$
u^{*}(t)=-B^{T}(t) \Phi^{T}\left(t_{0}, t\right) W_{c}^{-1}\left(t_{0}, t_{1}\right)\left[x_{0}-\Phi\left(t_{0}, t_{1}\right) x_{f}\right]
$$

a control function which returns the system to equilibrium at $t=1$.

Exercise 50 Prove that $U\left(t_{0}, t_{1}\right)$ defined by

$$
U\left(t_{0}, t_{1}\right)=\int_{t_{0}}^{t_{1}} \Phi\left(t_{0}, \tau\right) B(\tau) B^{T}(\tau) \Phi^{T}\left(t_{0}, \tau\right) d \tau
$$

satisfies the matrix differential equation

$$
\dot{U}\left(t, t_{1}\right)=A(t) U\left(t, t_{1}\right)+U\left(t, t_{1}\right) A^{T}(t)-B(t) B^{T}(t), \quad U\left(t_{1}, t_{1}\right)=0 .
$$

Exercise 51 In the preceding exercise let $A$ and $B$ be time-invariant, and put

$$
W\left(t, t_{1}\right)=U\left(t, t_{1}\right)-U_{0}
$$

where the constant matrix $U_{0}$ satisfies

$$
A U_{0}+U_{0} A^{T}=B B^{T}
$$

Write down the solution of the resulting differential equation for $W$ using the result in Exercise 32 and hence show that

$$
U\left(t, t_{1}\right)=U_{0}-\exp \left(\left(t-t_{1}\right) A\right) U_{0} \exp \left(\left(t-t_{1}\right) A^{T}\right)
$$

Exercise 52 Consider again the rabbit-fox environment problem described in section 1.3 (see also, Exercise 29). If it is possible to count only the total number of animals, can the individual numbers of rabbits and foxes be determined ?

Exercise 53 For the system (with outputs)

$$
\dot{x}=\left[\begin{array}{rr}
-1 & -1 \\
2 & -4
\end{array}\right] x, \quad y=\left[\begin{array}{ll}
1 & 2
\end{array}\right] x
$$

find $x(0)$ if $y(t)=-20 e^{-3 t}+21 e^{-2 t}$.

Exercise 54 Show that the control system described by

$$
\dot{x}_{1}=x_{2}, \quad \dot{x}_{2}=-2 x_{1}-3 x_{2}+u, \quad y=x_{1}+x_{2}
$$

is not c.o. Determine initial states $x(0)$ such that if $u(t)=0$ for $t \geq 0$, then the output $y(t)$ is identically zero for $t \geq 0$.

Exercise 55 Prove that $V\left(t_{0}, t_{1}\right)$ defined by

$$
V\left(t_{0}, t_{1}\right)=\int_{t_{0}}^{t_{1}} \Phi^{T}\left(\tau, t_{0}\right) C^{T}(\tau) C(\tau) \Phi\left(\tau, t_{0}\right) d \tau
$$

satisfies the matrix differential equation

$$
\dot{V}\left(t, t_{1}\right)=-A^{T}(t) V\left(t, t_{1}\right)-V\left(t, t_{1}\right) A(t)-C^{T}(t) C(t), \quad V\left(t_{1}, t_{1}\right)=0
$$

Exercise 56 Consider the time-invariant control system

$$
\dot{x}=\left[\begin{array}{rr}
-1 & -1 \\
2 & -4
\end{array}\right] x+\left[\begin{array}{l}
1 \\
3
\end{array}\right] u(t) .
$$

Find a $1 \times 2$ matrix $K$ such that the closed loop system has eigenvalues -4 and -5 .

Exercise 57 For the time-invariant control system

$$
\dot{x}=\left[\begin{array}{rrr}
1 & 0 & -1 \\
1 & 2 & 1 \\
2 & 2 & 3
\end{array}\right] x+\left[\begin{array}{l}
1 \\
0 \\
1
\end{array}\right] u(t)
$$

find a suitable matrix $K$ so as to make the closed loop eigenvalues $-1,-1 \pm 2 i$.
Exercise 58 Given the time-invariant control system

$$
\dot{x}=A x+B u(t)
$$

where

$$
A=\left[\begin{array}{rrr}
0 & 1 & 0 \\
0 & 0 & 1 \\
6 & -11 & 6
\end{array}\right], \quad B=\left[\begin{array}{ll}
1 & 0 \\
0 & 1 \\
1 & 1
\end{array}\right]
$$

find a suitable matrix $K$ which makes the eigenvalues of $A-B K$ equal to $1,1,3$.

Exercise 59 Determine whether the system described by

$$
\dot{x}=\left[\begin{array}{rrr}
-1 & 0 & 3 \\
0 & -3 & 0 \\
1 & 0 & -3
\end{array}\right] x+\left[\begin{array}{c}
1 \\
1 \\
-1
\end{array}\right] u(t)
$$

is c.c. Show that under (linear) feedback of the form $u=\alpha x_{1}+\beta x_{3}$, the closed loop system has two fixed eigenvalues, one of which is equal to -3 . Determine the second fixed eigenvalue, and also values of $\alpha$ and $\beta$ such that the third closed loop eigenvalue is equal to -4 .

Exercise 60 Show that if $X=\left[\begin{array}{ll}A & B \\ 0 & C\end{array}\right]$ is a block matrix with $A$ and $C$ invertible, then $X$ is invertible and

$$
X^{-1}=\left[\begin{array}{cc}
A^{-1} & -A^{-1} B C^{-1} \\
0 & C^{-1}
\end{array}\right]
$$

Exercise 61 Use Proposition 3.4.5 to obtain a minimal realization of

$$
G(s)=\frac{1}{g(s)}\left[\begin{array}{cc}
\left(s^{2}+6\right) & \left(s^{2}+s+4\right) \\
\left(2 s^{2}-7 s-2\right) & \left(s^{2}-5 s-2\right)
\end{array}\right]
$$

where $g(s)=s^{3}+2 s^{2}-s-2$.

Exercise 62 Show that the order of a minimal realization of

$$
G(s)=\frac{1}{s^{2}+3 s+2}\left[\begin{array}{cc}
(s+2) & 2(s+2) \\
-1 & (s+1)
\end{array}\right]
$$

is three. (Notice the fallacy of assuming that the order is equal to the degree of the common denominator.)

Exercise 63 If $\left(A_{1}, B_{1}, C_{1}\right)$ and $\left(A_{2}, B_{2}, C_{2}\right)$ are realizations of $G_{1}(\cdot)$ and $G_{2}(\cdot)$, respectively, show that

$$
A=\left[\begin{array}{cc}
A_{1} & B_{1} C_{2} \\
0 & A_{2}
\end{array}\right], \quad B=\left[\begin{array}{c}
0 \\
B_{2}
\end{array}\right], \quad C=\left[\begin{array}{ll}
C_{1} & 0
\end{array}\right]
$$

is a realization of $G_{1}(\cdot) G_{2}(\cdot)$, assuming that this product exists.

Exercise 64 Verify that algebraic equivalence

$$
\widetilde{A}=P A P^{-1}, \quad \widetilde{B}=P B, \quad \widetilde{C}=C P^{-1}
$$

can be written as the transformation

$$
\left[\begin{array}{rr}
P & 0 \\
0 & I_{n}
\end{array}\right]\left[\begin{array}{rr}
s I_{n}-A & B \\
-C & 0
\end{array}\right]\left[\begin{array}{rr}
P^{-1} & 0 \\
0 & I_{n}
\end{array}\right]=\left[\begin{array}{rr}
s I_{n}-\widetilde{A} & \widetilde{B} \\
-\widetilde{C} & 0
\end{array}\right] .
$$

Exercise 65 Determine values of $b_{1}, b_{2}, c_{1}$ and $c_{2}$ such that

$$
\mathcal{R}=\left(\left[\begin{array}{rr}
-2 & 0 \\
0 & -3
\end{array}\right], \quad\left[\begin{array}{l}
b_{1} \\
b_{2}
\end{array}\right], \quad\left[\begin{array}{ll}
1 & 1
\end{array}\right]\right)
$$

and

$$
\widetilde{\mathcal{R}}=\left(\left[\begin{array}{rr}
0 & 1 \\
-6 & -5
\end{array}\right], \quad\left[\begin{array}{l}
0 \\
1
\end{array}\right], \quad\left[\begin{array}{ll}
c_{1} & c_{2}
\end{array}\right]\right)
$$

are realizations of the transfer function

$$
g(s)=\frac{s+4}{s^{2}+5 s+6} .
$$

Determine a matrix $P$ such that the algebraic equivalence relationship holds between the two realizations.

