## Chapter 4

## Stability

## Topics :

1. Basic Concepts
2. Algebraic Criteria for Linear Systems
3. Lyapunov Theory with Applications to Linear Systems
4. Stability and Control

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Stability was probably the first question in classical dynamical systems which was dealt with in a satisfactory way. Stability questions motivated the introduction of new mathematical concepts (tools) in engineeering, particularly in control engineering. Stability theory has been of interest to mathematicians and astronomers for a long time and has had a stimulating impact on these fields. The specific problem of attempting to prove that the solar system is stable accounted for the introduction of many new methods.

Our treatment of stability will apply to (control) systems described by sets of linear or nonlinear equations. As is to be expected, however, our most explicit results will be obtained for linear systems.

### 4.1 Basic Concepts

Consider the nonlinear dynamical system $\Sigma$ described by

$$
\begin{equation*}
\dot{x}=F(t, x), \quad x \in \mathbb{R}^{m} \tag{4.1}
\end{equation*}
$$

where $x(\cdot)$ is a curve in the state space $\mathbb{R}^{m}$ and $F$ is a vector-valued mapping having components $F_{i}, \quad i=1,2, \ldots, m$.

Note : We shall assume that the components $F_{i}$ are continuous and satisfy standard conditions, such as having continuous first order partial derivatives so that the solution curve of (4.1) exists and is unique for any given initial condition (state).

From a geometric point of view, the right-hand side (rhs) $F$ can be interpreted as a time-dependent vector field on $\mathbb{R}^{m}$. So a (nonlinear) dynamical system is essentially the same as (and thus can be identified with) a vector field on the state space. This point of view is very fruitfull and extremely useful in investigating the properties of the dynamical system (especially when the state space is a manifold).

If the functions $F_{i}$ do not depend explicitly on $t$, then system $\Sigma$ is called autonomous (or time-independent); otherwise, nonautonomous (or time-dependent).

## Equilibrium states

4.1.1 Definition. If $F(t, c)=0$ for all $t$, then $c \in \mathbb{R}^{m}$ is said to be an equilibrium (or critical) state.

It follows at once from (4.1) that (for an equilibrium state $c$ ) if $x\left(t_{0}\right)=c$, then $x(t)=c$ for all $t \geq t_{0}$. Thus solution curves starting at $c$ remain there

Clearly, by introducing new variables $x_{i}^{\prime}=x_{i}-c_{i}$ we can arrange for the equilibrium state to be transferred to the origin (of the state space $\mathbb{R}^{m}$ ); we shall assume that this has been done for any equilibrium state under consideration (there may well be several for a given system $\Sigma$ ), so that we then have

$$
F(t, 0)=0
$$

for all $t \geq t_{0}$.
We shall also assume that there is no other constant solution curve in the neighborhood of the origin, so this is an isolated equilibrium state.
4.1.2 EXAMPLE. The intuitive idea of stability in a dynamical setting is that for "small" perturbations from the equilibrium state at some time $t_{0}$, subsequent motions $t \mapsto x(t), t \geq t_{0}$ should not be too "large". Consider a ball resting in equilibrium on a sheet of metal bent into various shapes with cross-sections as shown.


If frictional forces can be neglected, then small perturbations lead to :

- oscillatory motion about equilibrium (case (i)) ;
- the ball moving away without returning to equilibrium (case (ii));
- oscillatory motion about equilibrium, unless the initial perturbation is so large that the ball is forced to oscillate about a new equilibrium position (case (iii)).

If friction is taken into account then the oscillatory motions steadily decrease until the equilibrium state is returned to.

## Stability

There is no single concept of stability, and many different definitions are possible. We shall consider only the following fundamental statements.
4.1.3 Definition. An equilibrium state $x=0$ is said to be :
(a) stable if for any positive scalar $\varepsilon$ there exists a positive scalar $\delta$ such that $\left\|x\left(t_{0}\right)\right\|<\delta$ implies $\|x(t)\|<\varepsilon$ for all $t \geq t_{0}$.
(b) asymptotically stable if it is stable and if in addition $x(t) \rightarrow 0$ as $t \rightarrow \infty$.
(c) unstable if it is not stable; that is, there exists an $\varepsilon>0$ such that for every $\delta>0$ there exists an $x\left(t_{0}\right)$ with $\left\|x\left(t_{0}\right)\right\|<\delta,\left\|x\left(t_{1}\right)\right\| \geq \varepsilon$ for some $t_{1}>t_{0}$.
(d) completely unstable if there exists an $\varepsilon>0$ such that for every $\delta>0$ and for every $x\left(t_{0}\right)$ with $\left\|x\left(t_{0}\right)\right\|<\delta,\left\|x\left(t_{1}\right)\right\| \geq \varepsilon$ for some $t_{1}>t_{0}$.

Note : The definition (a) is often called "stability in the sense of Lyapunov" (stability i.s.L.) after the Russian mathematician Aleksandr M. Lyapunov (18571918), whose important work features prominently in current control theory.

In Example 4.1.2, case ( $i$ ) represents stability i.s.L. if friction is ignored, and asymptotic stability if friction is taken into account, whereas case (ii) represents instability. If the metal sheet in (i) were thought to extend indefinitely then, if friction is present, the ball would eventually return to equilibrium no matter how large the disturbance. This is an illustration of asymptotic stability in the large, which means that every motion converges to a single equilibrium point (state) as $t \rightarrow \infty$, and clearly does not apply to case (iii). Asymptotic stability in the large implies that all motions are bounded. Generally,
4.1.4 Definition. An equilibrium state $x=0$ is said to be bounded (or Lagrange stable) if there exists a constant $M$, which may depend on $t_{0}$ and $x\left(t_{0}\right)$, such that $\|x(t)\| \leq M$ for all $t \geq t_{0}$.

Some remarks can be made at this stage.

1. Regarded as a function of $t$ in the ( $n$-dimensional) state space, the solution $x(\cdot)$ of (4.1) is called a trajectory (or motion). In two dimensions we can give the definitions a simple geometric interpretation.

- If the origin $O$ is stable, then given the outer circle $\mathcal{C}$, radius $\varepsilon$, there exists an inner circle $\mathcal{C}_{1}$, radius $\delta_{1}$, such that trajectories starting within $\mathcal{C}_{1}$ never leave $\mathcal{C}$.
- If $O$ is asymptotically stable, then there is some circle $\mathcal{C}_{2}$, radius $\delta_{2}$, having the same property as $\mathcal{C}_{1}$ but, in addition, trajectories starting inside $\mathcal{C}_{2}$ tend to $O$ as $t \rightarrow \infty$.

2. We refer to the stability of an equilibrium state of $\Sigma$, not the system itself, as different equilibrium states may have different stability properties.
3. A weakness of the definition of stability i.s.L. for practical purposes is that only the existence of some positive $\delta$ is required, so $\delta$ may be very small compared to $\varepsilon$; in other words, only very small disturbances from equilibrium may be allowable.
4. In engineering applications, asymptotic stability is more desirable than stability since it ensures eventual return to equilibrium, whereas stability allows continuing deviations "not to far" from the equilibrium state.

## Examples

Some further aspects of stability are now illustrated through some examples.
4.1.5 Example. We return again to the environment problem involving rabbits anf foxes, and let the equations have the following numerical form

$$
\left\{\begin{array}{l}
\dot{x}_{1}=2 x_{1}-3 x_{2} \\
\dot{x}_{2}=2 x_{1}-x_{2}
\end{array}\right.
$$

These equations have a single equilibrium point (state) at the origin. With arbitrary initial numbers $x_{1}(0)$ and $x_{2}(0)$ of rabbits and foxes, respectively, the solution is

$$
x_{1}(t)=x_{1}(0) e^{\frac{t}{2}}\left(\cos \frac{\sqrt{15}}{2} t+\frac{3}{\sqrt{15}} \sin \frac{\sqrt{15}}{2} t\right)-\frac{6 x_{2}(0)}{\sqrt{15}} e^{\frac{t}{2}} \sin \frac{\sqrt{15}}{2} t
$$

with a similar expression for $x_{2}(t)$. Clearly, $x_{1}(t)$ tends to infinity as $t \rightarrow \infty$, irrespective of the initial state, so the origin is unstable.
4.1.6 Example. Consider the (initialized) dynamical system on $\mathbb{R}$ described by

$$
\dot{x}=x^{2}, \quad x(0)=x_{0} \in \mathbb{R} .
$$

It is clear that the solution exists and is unique; in fact, by integrating, we easily obtain

$$
-\frac{1}{x}=t-\frac{1}{x_{0}} .
$$

Hence

$$
x(t)=\frac{1}{\frac{1}{x_{0}}-t}
$$

so that if $x_{0}>0, x(t) \rightarrow \infty$ as $t \rightarrow \frac{1}{x_{0}}$. The solution is said to "escape" to infinity in a finite time, or to have a finite escape time. We shall henceforth exclude this situation and assume that (4.1) has a finite solution for all finite $t \geq t_{0}$, for otherwise (4.1) cannot be a mathematical model of a real-life situation.
4.1.7 Example. We demonstrate that the origin is a stable equilibrium state for the (initialized) system described by

$$
\dot{x}=(1-2 t) x, \quad x\left(t_{0}\right)=x_{0} \in \mathbb{R}
$$

by determining explicitly the scalar $\delta$ in the definition. Integrating the equation gives

$$
x(t)=x_{0} e^{t-t^{2}} e^{t_{0}^{2}-t_{0}} .
$$

The condition $|x(t)|<\varepsilon$ leads to

$$
\left|x_{0}\right|<\varepsilon e^{t^{2}-t} e^{t_{0}-t_{0}^{2}}
$$

Since $t \mapsto e^{t^{2}-t}$ has a minimum value of $e^{-\frac{1}{4}}$ when $t=\frac{1}{2}$, it follows that we can take

$$
\delta=\varepsilon e^{-\left(t_{0}-\frac{1}{2}\right)^{2}} .
$$

In general, $\delta$ will depend upon $\varepsilon$, but in this example it is also a function of the initial time. If $\delta$ is independent of $t_{0}$, the stability is called uniform.
4.1.8 Example. Consider the (initialized) dynamical system on $\mathbb{R}$ described by

$$
\dot{x}(t)=\left\{\begin{array}{rc}
x(t)-2 & \text { if } x(t)>2 \\
0 & \text { if } x(t) \leq 2 .
\end{array} \quad, \quad x\left(t_{0}\right)=x_{0} \in \mathbb{R}\right.
$$

The solution is easily found to be

$$
x(t)=\left\{\begin{aligned}
2+\left(x_{0}-2\right) e^{t-t_{0}} & \text { if } x_{0}>2 \\
x_{0} & \text { if } x_{0} \leq 2 .
\end{aligned}\right.
$$

The condition $|x(t)|<\varepsilon$ is implied by $\left|x_{0}\right|<2$ when $\varepsilon \geq 2$, for then $|x(t)|=$ $\left|x_{0}\right|<2<\varepsilon$. When $\varepsilon<2, \quad|x(t)|<\varepsilon$ is implied by $\left|x_{0}\right|<\varepsilon$, for then again $|x(t)|=\left|x_{0}\right|<\varepsilon$. Thus according to the definition, the origin is a stable equilibrium point (state). However, if $x_{0}>2$ then $x(t) \rightarrow \infty$, so for initial perturbations $x_{0}>2$ from equilibrium motions are certainly unstable in a practical sense.
4.1.9 Example. Consider the equation

$$
\dot{x}=f(t) x, \quad x(0)=x_{0} \in \mathbb{R}
$$

where

$$
f(t)=\left\{\begin{array}{cc}
\ln 10 & \text { if } 0 \leq t \leq 10 \\
-1 & \text { if } t>10
\end{array}\right.
$$

The solution is

$$
x(t)=\left\{\begin{aligned}
10^{t} x_{0} & \text { if } 0 \leq t \leq 10 \\
10^{10} x_{0} e^{10-t} & \text { if } t>10 .
\end{aligned}\right.
$$

Clearly, $x(t) \rightarrow 0$ as $t \rightarrow \infty$, and the origin is asymptotically stable. However, if $x_{0}$ changes by a very small amount, say $10^{-5}$, then the corresponding change in $x(t)$ is relatively large - for example, when $t=20$, the change in $x(t)$ is $10^{10} 10^{-5} e^{10-20} \approx 4.5$.

Note: Examples 4.1.8 and 4.1.9 show that an equilibrium state may be stable according to Lyapunov's definitions and yet the system's behaviour may be unsatisfactory from a practical point of view. The converse situation is also possible, and this has led to a definition of "practical stability" being coined for systems which are unstable in Lyapunov's sense but have an acceptable performance in practice, namely that for pre-specified deviations from equilibrium the subsequent motions also lie within specified limits.

### 4.2 Algebraic Criteria for Linear Systems

We return to the general linear (time-invariant) system given by

$$
\begin{equation*}
\dot{x}=A x, \quad x \in \mathbb{R}^{m} \tag{4.2}
\end{equation*}
$$

where $A \in \mathbb{R}^{m \times m}$ and (4.2) may represent the closed or open loop system. Provided the matrix $A$ is nonsingular, the only equilibrium state of (4.2) is the origin, so it is meaningful to refer to the stability of the system (4.2). If the system is stable (at the origin) but not asymptotically stable we shall call it neutrally stable.

One of the basic results on which the development of linear system stability theory relies is now given. The proof will be omitted.
4.2.1 Theorem. (Stability Properties of a Linear System) Consider the linear system (4.2), and for each eigenvalue $\lambda$ of $A$, suppose that $m_{\lambda} d e-$ notes the algebraic multiplicity of $\lambda$ and $d_{\lambda}$ the geometric multiplicity of $\lambda$. Then :
(a) The system is asymptotically stable if and only if $A$ is a stability matrix; that is, every eigenvalue of $A$ has a negative real part.
(b) The system is neutrally stable if and only if

- every eigenvalue of $A$ has a nonpositive real part, and
- at least one eigenvalue has a zero real part, and $d_{\lambda}=m_{\lambda}$ for every eigenvalue $\lambda$ with a zero real part.
(c) The system is unstable if and only if
- some eigenvalue of $A$ has a positive real part, or
- there is an eigenvalue $\lambda$ with a zero real part and $d_{\lambda}<m_{\lambda}$.

Note : (1) Suppose all the eigenvalues of $A$ have nonpositive real parts. One can prove that if all eigenvalues having zero real parts are distinct, then the origin is neutrally stable.
(2) Also, if every eigenvalue of $A$ has a positive real part, then the system is completely unstable.
4.2.2 Example. In Example 4.1 .5 the system matrix is of the form

$$
A=\left[\begin{array}{ll}
a_{1} & -a_{2} \\
a_{3} & -a_{4}
\end{array}\right]
$$

where $a_{1}, a_{2}, a_{3}, a_{4}>0$. It is easy to show that

$$
\operatorname{det}\left(\lambda I_{2}-A\right)=\lambda(\lambda-d)
$$

using the condition

$$
\frac{a_{1}}{a_{3}}=\frac{a_{2}}{a_{4}}
$$

Hence $A$ has a single zero eigenvalues, so the system is neutrally stable provided $d\left(=a_{1}-a_{4}\right)$ is negative. (If $a_{1}=a_{4}$ then the system is unstable.)

Note : The preceding theorem applies if $A$ is real or complex, so the stability determination of (4.2) can be carried out by computing the eigenvalues using one of the powerful standard computer programs now available. However, if $m$ is small (say less than six), or if some of the elements of $A$ are in parametric form, or if access to a digital computer is not possible, then the classical results given below are useful.

Because of its practical importance the linear system stability problem has attracted attention for a considerable time, an early study being by James C. MAXWELL (1831-1879) in connection with the governing of steam engines. The original formulation of the problem was not of course in matrix terms, the system model being

$$
\begin{equation*}
z^{(m)}+k_{1} z^{(m-1)}+\cdots+k_{m} z=u(t) . \tag{4.3}
\end{equation*}
$$

This is equivalent to working with the characteristic polynomial of $A$, which we shall write in this section as

$$
\begin{equation*}
a(\lambda):=\operatorname{det}\left(\lambda I_{m}-A\right)=\lambda^{m}+a_{1} \lambda^{m-1}+\cdots+a_{m-1} \lambda+a_{m} \tag{4.4}
\end{equation*}
$$

The first solutions giving necessary and sufficient conditions for all the roots of $a(\lambda)$ in (4.4) to have negative real parts were given by Augustin L. Cauchy (1789-1857), Jacques C.F. Sturm (1803-1855), and Charles Hermite (1822-1901).

We give here a well-known result due to Adolf Hurwitz (1859-1919) for the case when all the coefficients $a_{i}$ are real. The proof will be omitted.
4.2.3 Theorem. (Hurwitz) The $m \times m$ Hurwitz matrix associated
with the characteristic polynomial $a(\lambda)$ of $A$ in (4.4) is

$$
H:=\left[\begin{array}{ccccc}
a_{1} & a_{3} & a_{5} & \ldots & a_{2 m-1}  \tag{4.5}\\
1 & a_{2} & a_{4} & \ldots & a_{2 m-2} \\
0 & a_{1} & a_{3} & \ldots & a_{2 m-3} \\
0 & 1 & a_{2} & \ldots & a_{2 m-4} \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \ldots & a_{m}
\end{array}\right]
$$

where $a_{r}=0, r>m$. Let $H_{i}$ denote the $i^{t h}$ leading principal minor of $H$. Then all the roots of $a(\lambda)$ have negative real parts $(a(\lambda)$ is a Hurwitz polynomial ) if and only if $H_{i}>0, i=1,2, \ldots, m$.

Note : A disadvantage of Theorem 4.2.3 is the need to evaluate determinants of increasing order, and a convenient way of avoiding this is due to Edward J. Routh (1831-1907). We will give only the test as it applies to polynomials of degree no more than four, although the test can be extended to any polynomial.
4.2.4 Proposition. (The Routh Test) All the roots of the polynomial $a(\lambda)$ (with real coefficients) have negative real parts precisely when the given conditions are met.

- $\lambda^{2}+a_{1} \lambda+a_{2}$ : all the coefficients are positive ;
- $\lambda^{3}+a_{1} \lambda^{2}+a_{2} \lambda+a_{3}$ : all the coefficients are positive and $a_{1} a_{2}>a_{3}$;
- $\lambda^{4}+a_{1} \lambda^{3}+a_{2} \lambda^{2}+a_{3} \lambda+a_{4}$ : all the coefficients are positive, $a_{1} a_{2}>a_{3}$ and $a_{1} a_{2} a_{3}>a_{1}^{2} a_{4}+a_{3}^{2}$.

Note : The Hurwitz and Routh tests can be useful for determining stability of (4.3) and (4.2) in certain cases. However, it should be noted that a practical disadvantage of application to (4.2) is that it is very difficult to calculate accurately the $a_{i}$ in (4.4). This is important because small errors in the $a_{i}$ can lead to large errors in the roots of $a(\lambda)$.
4.2.5 Example. Investigate the stability of the linear system whose characteristic equation is

$$
\lambda^{4}+2 \lambda^{3}+9 \lambda^{2}+4 \lambda+1=0 .
$$

Solution : The polynomial $a(\lambda)=\lambda^{4}+2 \lambda^{3}+9 \lambda^{2}+4 \lambda+1$ has positive coefficients, and since $a_{1}=2, a_{2}=9, a_{3}=4$ and $a_{4}=1$, the polynomial meets the Routh conditions :

$$
18=a_{1} a_{2}>a_{3}=4 \quad \text { and } \quad 72=a_{1} a_{2} a_{3}>a_{1}^{2} a_{4}+a_{3}^{2}=20 .
$$

So all the roots have negative real parts, and hence the linear system is asymptotically stable.

Since we have assumed that the coefficients $a_{i}$ are real it is easy to derive a simple necessary condition for asymptotic stability :
4.2.6 Proposition. If the coefficients $a_{i}$ in (4.4) are real and $a(\lambda)$ corresponds to an asymptotically stable system, then

$$
a_{i}>0, \quad i=1,2, \ldots, m .
$$

Proof: Any complex root of $a(\lambda)$ will occur in conjugate pairs $\alpha \pm i \beta$, the corresponding factor of $a(\lambda)$ being

$$
(\lambda-\alpha-i \beta)(\lambda-\alpha+i \beta)=\lambda^{2}-2 \alpha \lambda+\alpha^{2}+\beta^{2} .
$$

By Theorem 4.2.1, $\alpha<0$, and similarly any real factor of $a(\lambda)$ can be written $(\lambda+\gamma)$ with $\gamma>0$. Thus

$$
a(\lambda)=\prod(\lambda+\gamma) \prod\left(\lambda^{2}-2 \alpha \lambda+\alpha^{2}+\beta^{2}\right)
$$

and since all the coefficients above are positive, $a_{i}$ must also all be positive.
Note : Of course the condition above is not a sufficient condition, but it provides a useful initial check : if any $a_{i}$ are negative or zero, then a $(\lambda)$ cannot be asymptotically stable.

When we turn to linear time-varying systems

$$
\begin{equation*}
\dot{x}=A(t) x, \quad x\left(t_{0}\right)=x_{0} \tag{4.6}
\end{equation*}
$$

the situation is much more complicated. In view of Theorem 4.2.1 it might be thought that if the eigenvalues of $A(t)$ all have negative real parts for all $t \geq t_{0}$, then the origin of (4.6) would be asymptotically stable. Unfortunately, this conjecture is not true (see Exercise 73).

### 4.3 Lyapunov Theory

We shall develop the so-called "direct" method of Lyapunov in relation to the (initialized) nonlinear autonomous dynamical system $\Sigma$ given by

$$
\begin{equation*}
\dot{x}=F(x), \quad x(0)=x_{0} \in \mathbb{R}^{m} ; \quad F(0)=0 . \tag{4.7}
\end{equation*}
$$

Note : Modifications needed to deal with the (nonautonomous) case

$$
\dot{x}=F(t, x), \quad x\left(t_{0}\right)=x_{0}
$$

are straightforward.
The aim is to determine the stability nature of the equilibrium state (at the origin) of system $\Sigma$ without obtaining the solution $x(\cdot)$. This of course has been done algebraically for linear time invariant systems in section $\mathbf{4 . 2}$. The essential idea is to generalize the concept of energy $V$ for a conservative system in mechanics, where a well-known result states that an equilibrium point is stable if the energy is minimum. Thus $V$ is a positive function which has $\dot{V}$ negative in the neighborhood of a stable equilibrium point. More generally,
4.3.1 Definition. We define a Lyapunov function $V: \mathbb{R}^{m} \rightarrow \mathbb{R}$ as follows :

- $V$ and all its partial derivatives $\frac{\partial V}{\partial x_{i}}$ are continuous ;
- $V$ is positive definite; that is, $V(0)=0$ and $V(x)>0$ for $x \neq 0$ in some neighborhood $\{x \mid\|x\| \leq k\}$ of the origin.

Consider now the (directional) derivative of $V$ with respect to (the vector field) $F$, namely

$$
\begin{aligned}
\dot{V} & :=\frac{\partial V}{\partial x} F=\left[\begin{array}{lll}
\frac{\partial V}{\partial x_{1}} & \cdots & \frac{\partial V}{\partial x_{m}}
\end{array}\right]\left[\begin{array}{c}
F_{1} \\
\vdots \\
F_{m}
\end{array}\right] \\
& =\frac{\partial V}{\partial x_{1}} F_{1}+\frac{\partial V}{\partial x_{2}} F_{2}+\cdots+\frac{\partial V}{\partial x_{m}} F_{m} .
\end{aligned}
$$

A Lyapunov function $V$ for the system (4.7) is said to be

- strong if the derivative $\dot{V}$ is negative definite; that is, $\dot{V}(0)=0$ and $\dot{V}(x)<0$ for $x \neq 0$ such that $\|x\| \leq k$.
- weak if the derivative $\dot{V}$ is negative semi-definite; that is, $\dot{V}(0)=0$ and $\dot{V}(x) \leq 0$ for all $x$ such that $\|x\| \leq k$.

NOTE : The definitions for positive or negative definiteness or semi-definiteness are generalizations of those for quadratic forms. Here is the definiteness test for planar quadratic forms: Suppose that $Q=Q(x, y)$ is the quadratic form $a x^{2}+2 b x y+c y^{2}$, where $a, b, c \in \mathbb{R}$. Then $Q$ is :

- positive definite $\Longleftrightarrow a, c>0$ and $b^{2}<a c$;
- positive semi-definite $\Longleftrightarrow a, c \geq 0$ and $b^{2} \leq a c$;
- negative definite $\Longleftrightarrow a, c<0$ and $b^{2}<a c$;
- negative semi-definite $\Longleftrightarrow a, c \leq 0$ and $b^{2} \leq a c$.

Otherwise, $Q$ is indefinite.

## The Lyapunov stability theorems

The statements of the two basic theorems of Lyapunov are remarkably simple. The proofs will be omitted.
4.3.2 Theorem. (Lyapunov's First Theorem) Suppose that there is a strong Lyapunov function $V$ for system $\Sigma$. Then system $\Sigma$ is asymptotically stable.

The conclusion of this theorem is plausible, since the values of the strong Lyapunov function $V(x(t))$ must continually diminish along each orbit $x=$ $x(t)$ as $t$ increases (since $\dot{V}$ is negative definite). This means that the orbit $x=x(t)$ must cut across level sets $V(x)=C$ with ever smaller values of $C$. In fact, $\lim _{t \rightarrow \infty} V(x(t))=0$, which implies that $x(t) \rightarrow 0$ as $t \rightarrow \infty$ since $x(t)$ and $V(x)$ are continuous and $V$ has value zero only at the origin (that's where the positive definiteness of $V$ comes into play).
4.3.3 Theorem. (Lyapunov's Second Theorem) Suppose that there is a weak Lyapunov function $V$ for system $\Sigma$. Then system $\Sigma$ is stable.

The result seems reasonable since the negative semi-definiteness of $\dot{V}$ keeps an orbit that starts near the origin close to the origin as $t$ increases. But orbits don't have to cut across level sets of $V$.

Note : If the conditions on $V$ in Theorem 4.3.2 hold everywhere in state space it does not necessarily follow that the origin is asymptotically stable in the large. For this to be the case $V$ must have the additional property that it is radially unbounded, which means that

$$
V(x) \rightarrow \infty \quad \text { for all } x \text { such that }\|x\| \rightarrow \infty .
$$

For instance,

$$
V=x_{1}^{2}+x_{2}^{2}
$$

is radially unbounded, but

$$
V=\frac{x_{1}^{2}}{1+x_{1}^{2}}+x_{2}^{2}
$$

is not since, for example,

$$
V \rightarrow 1 \quad \text { as } \quad x_{1} \rightarrow \infty, \quad x_{2} \rightarrow 0
$$

A similar line of reasoning shows that if $\Omega$ is the set of points "outside" a bounded region containing the origin, and if throughout $\Omega, V>0, \dot{V} \leq 0$ and $V$ is radially unbounded, then the origin is Lagrange stable.
4.3.4 Example. Consider a unit mass suspended from a fixed support by a spring, $z$ being the displacement from the equilibrium. If first the spring is assumed to obey Hooke's law, then the equation of motion is

$$
\begin{equation*}
\ddot{z}+k z=0 \tag{4.8}
\end{equation*}
$$

where $k$ is the spring constant. Taking $x_{1}:=z, x_{2}:=\dot{z},(4.8)$ becomes

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{2} \\
\dot{x}_{2}=-k x_{1} .
\end{array}\right.
$$

Since the system is conservative, the total energy

$$
E=\frac{1}{2} k x_{1}^{2}+\frac{1}{2} x_{2}^{2}
$$

is a Lyapunov function and it is easy to see that

$$
\dot{E}=k x_{1} x_{2}-k x_{2} x_{1}=0
$$

so by Lyapunov's Second Theorem the origin is stable. (Of course, this is trivial since (4.8) represents simple harmonic motion.)

Suppose now that the force exerted by the spring, instead of being linear is some function $x_{1} k\left(x_{1}\right)$ satisfying

$$
k(0)=0, \quad k\left(x_{1}\right)>0 \text { for } x_{1} \neq 0 .
$$

The total energy is now

$$
E=\frac{1}{2} x_{2}^{2}+\int_{0}^{x_{1}} \tau k(\tau) d \tau
$$

and

$$
\dot{E}=-k x_{1} x_{2}+k x_{1} \dot{x_{1}}=0 .
$$

So again by Lyapunov's Second Theorem the origin is stable for any nonlinear spring satisfying the above conditions.
4.3.5 Example. Consider now the system of the previous example but with a damping force $d \dot{z}$ added, so that the equation of motion is

$$
\begin{equation*}
\ddot{z}+d \dot{z}+k z=0 . \tag{4.9}
\end{equation*}
$$

(Equation (4.9) can also be used to describe an LCR series circuit, motion of a gyroscope, and many other problems.)

Assume first that both $d$ and $k$ are constant, and for simplicity let $d=$ $1, k=2$. The system equations in state space form are

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{2} \\
\dot{x}_{2}=-2 x_{1}-x_{2}
\end{array}\right.
$$

and the total energy is

$$
E=x_{1}^{2}+\frac{1}{2} x_{2}^{2}
$$

so that

$$
\dot{E}=2 x_{1} x_{2}+x_{2}\left(-2 x_{1}-x_{2}\right)=-x_{2}^{2}
$$

which is negative semi-definite, so by Lyapunov's Second Theorem the origin is stable. However, now consider the function

$$
V=7 x_{1}^{2}+2 x_{1} x_{2}+3 x_{2}^{2}
$$

Then

$$
\dot{V}=-4 x_{1}^{2}-4 x_{2}^{2}
$$

Clearly, $\dot{V}$ is negative definite and it is easy to verify that the quadratic form $V$ is positive definite, so by Lyapunov's First Theorem the origin is asymptotically stable (in fact, in the large).

This example illustrates that a suitably-chosen Lyapunov function can provide more information than the energy function. However, when $\dot{V}$ is only negative semi-definite, the following result is often useful.
4.3.6 Proposition. Suppose that there is a (weak) Lyapunov function $V$ such that $\dot{V}$ does not vanish identically on any nontrivial trajectories of $\Sigma$. Then (the origin of) system $\Sigma$ is asymptotically stable.
4.3.7 Example. Consider again the damped mass-spring system described by (4.9), but now suppose that both $d$ and $k$ are not constant. Let $k\left(x_{1}\right)$ be as defined in Example 4.3.4 and let $d\left(x_{2}\right)$ have the property

$$
d\left(x_{2}\right)>0 \quad \text { for } x_{2} \neq 0 ; \quad d(0)=0 .
$$

The state equations are

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{2} \\
\dot{x}_{2}=-x_{1} k\left(x_{1}\right)-x_{2} d\left(x_{2}\right) .
\end{array}\right.
$$

So, if $E$ is

$$
E=\frac{1}{2} x_{2}^{2}+\int_{0}^{x_{1}} \tau k(\tau) d \tau
$$

then

$$
\dot{E}=x_{2}\left(-x_{1} k-d x_{2}\right)+x_{1} k \dot{x}_{1}=-x_{2}^{2} d \leq 0 .
$$

Now $E$ is positive definite and $\dot{E}$ vanishes only when $x_{2}(t) \equiv 0$, which implies $k\left(x_{1}\right) \equiv 0$, which in turn implies $x_{1}(t) \equiv 0$. Thus $\dot{E}$ vanishes only on the trivial solution of (4.9), and so by Proposition 4.3.6 the origin is asymptotically stable.

### 4.3.8 Example. The van der Pol equation

$$
\begin{equation*}
\ddot{z}+\epsilon\left(z^{2}-1\right) \dot{z}+z=0 \tag{4.10}
\end{equation*}
$$

where $\epsilon$ is a constant, arises in a number of engineering problems. (In a control context it can be thought of as application of nonlinear feedback

$$
u=-z+\epsilon\left(1-z^{2}\right) \dot{z}
$$

to the (linear) system described by

$$
\ddot{z}=u .)
$$

We shall assume that $\epsilon<0$. As usual take $x_{1}:=z, x_{2}:=\dot{z}$ to transform (4.10) into

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{2} \\
\dot{x}_{2}=-x_{1}-\epsilon\left(x_{1}^{2}-1\right) x_{2}
\end{array}\right.
$$

(The only equilibrium state of this system is the origin.) Try as a potential Lyapunov function $V=x_{1}^{2}+x_{2}^{2}$ which is obviously positive definite. Then

$$
\begin{aligned}
\dot{V} & =2 x_{1} \dot{x}_{1}+2 x_{2} \dot{x}_{2} \\
& =2 \epsilon x_{2}^{2}\left(1-x_{1}^{2}\right)
\end{aligned}
$$

Thus $\dot{V} \leq 0$ if $x_{1}^{2}<1$, and then by Proposition 4.3 .6 the origin is asymptotically stable. It follows that all trajectories starting inside the region $\Gamma: x_{1}^{2}+x_{2}^{2}<1$ converge to the origin as $t \rightarrow \infty$, and $\Gamma$ is therefore called a region of asymptotic stability.

Note : You may be tempted to think that the infinite strip $S: x_{1}^{2}<1$ is a region of asymptotic stability. This is not in fact true, since a trajectory starting outside $\Gamma$ can move inside the strip whilst continuing in the direction of decreasing $V$ circles, and hence lead to divergence.

In general, if a closed region

$$
R: \quad V(x) \leq \text { constant }
$$

is bounded and has $\dot{V}$ negative throughout, then region $R$ is a region of asymptotic stability.

Suppose that we now take as state variables

$$
x_{1}:=z, \quad x_{3}:=\int_{0}^{t} z(\tau) d \tau
$$

The corresponding state equations are

$$
\left\{\begin{array}{l}
\dot{x}_{1}=-x_{3}-\epsilon\left(\frac{1}{3} x_{1}^{3}-x_{1}\right) \\
\dot{x}_{3}=x_{1} .
\end{array}\right.
$$

Indeed,

$$
\begin{aligned}
\dot{x}_{1} & =\dot{z} \\
& =-\int_{0}^{t} z d \tau-\epsilon \int_{0}^{t}\left(z^{2}-1\right) \dot{z} d \tau \\
& =-x_{3}-\epsilon\left(\frac{1}{3} z^{3}-z\right) \\
& =-x_{3}-\epsilon\left(\frac{1}{3} x_{1}^{3}-x_{1}\right) .
\end{aligned}
$$

Hence, using $V=x_{1}^{2}+x_{3}^{2}$, we have

$$
\begin{aligned}
\dot{V} & =2 x_{1}\left(-x_{3}-\frac{1}{3} \epsilon x_{1}^{3}+\epsilon x_{1}\right)+2 x_{3} x_{1} \\
& =2 \epsilon x_{1}^{2}\left(1-\frac{1}{3} x_{1}^{2}\right) \leq 0 \quad \text { if } x_{1}^{2}<3
\end{aligned}
$$

so the region of asymptotic stability obtained by this different set of state variables is $R: x_{1}^{2}+x_{3}^{2}<3$, larger than before.

Note : In general, if the origin is an asymptotically stable equilibrium point (state), then the total set of initial points (states) from which trajectories converge to the origin as $t \rightarrow \infty$ is called the domain of attraction. Knowledge of this domain is of great value in practical problems since it enables permissible deviations from equilibrium to be determined. However, Example 4.3.8 illustrates the fact that since a particular Lyapunov function gives only sufficient conditions for stability, the region of asymptotic stability obtained can be expected to be only part of the domain of attraction. Different Lyapunov functions or different sets of state variables may well yield different stability regions.

The general problem of finding "optimum" Lyapunov functions, which give best possible estimates for the domain of attraction, is a difficult one.

It may be a waste of effort trying to determine the stability properties of an equilibrium point, since the point may be unstable. The following result is then useful :
4.3.9 Theorem. (Lyapunov's Third Theorem) Let a function $V: \mathbb{R}^{m} \rightarrow$ $\mathbb{R}$ with $V(0)=0$ have continuous first order partial derivatives. If there is some neighborhood containing the origin in which $V$ takes negative values, and if in addition $\dot{V}$ is negative semi-definite, then the origin of (4.7) is not asymptotically stable. If $\dot{V}$ is negative definite, the origin is unstable, and if both $V$ and $\dot{V}$ are negative definite, the origin is completely unstable.

Note : In all three Lyapunov's theorems the terms "positive" and "negative" can be interchanged simply by using $-V$ instead of $V$. It is only the relative signs of the Lyapunov function and its derivative which matter.

## Application to linear systems

We now return to the linear time-invariant system

$$
\begin{equation*}
\dot{x}=A x, \quad x \in \mathbb{R}^{m} \tag{4.11}
\end{equation*}
$$

In section 4.2 we gave algebraic criteria for determining asymptotic stability via the characteristic equation of $A$. We now show how Lyapunov theory can be used to deal directly with (4.11) by taking as a potential Lyapunov function the quadratic form

$$
\begin{equation*}
V=x^{T} P x \tag{4.1.1}
\end{equation*}
$$

where the matrix $P \in \mathbb{R}^{m \times m}$ is symmetric. The (directional) derivative of $V$ with respect to (4.11) (in fact, with respect to the vector field $x \mapsto A x$ ) is

$$
\begin{aligned}
\dot{V} & =\dot{x}^{T} P x+x^{T} P \dot{x} \\
& =x^{T} A^{T} P x+x^{T} P A x \\
& =-x^{T} Q x
\end{aligned}
$$

where

$$
\begin{equation*}
A^{T} P+P A=-Q \tag{4.13}
\end{equation*}
$$

and it is easy to see that $Q$ is also symmetric. If $P$ and $Q$ are both positive definite, then by Lyapunov's First Theorem the (origin of) system (4.11) is asymptotically stable. If $Q$ is positive definite and $P$ is negative definite or indefinite, then in both cases $V$ can take negative values in the neighborhood of the origin so by Lyapunov's Third Theorem, (4.11) is unstable. We have therefore proved :
4.3.10 Proposition. The matrix $A \in \mathbb{R}^{m \times m}$ is a stability matrix if and only if for any given positive definite symmetric matrix $Q$, there exists a positive definite symmetric matrix $P$ that satisfies the Lyapunov matrix equation (4.13).

Moreover, if the matrix $A$ is a stability matrix, then $P$ is the unique solution of the Lyapunov matrix equation (see Exercise 77).

It would be no use choosing $P$ to be positive definite and calculating $Q$ from (4.13). For unless $Q$ turned out to be definite or semi-definite (which is unlikely) nothing could be inferred about asymptotic stability from the Lyapunov theorems.

Note : Equations similar in form to (4.13) also arise in other areas of control theory. However, it must be admitted that since a digital computer will be required to solve (4.13) except for small values of $n$, so far as stability determination of (4.11) is concerned it will be preferable instead to find the eigenvalues of $A$. The true value and importance of Proposition 4.3.10 lies in its use as a theoretical tool.

## Linearization

The usefulness of linear theory can be extended by use of the idea of linearization. Suppose the components of (the vector field) $F$ in

$$
\dot{x}=F(x), \quad x(0)=x_{0} ; \quad F(0)=0
$$

are such that we can apply Taylor's theorem to obtain

$$
\begin{equation*}
F(x)=A x+g(x) \tag{4.14}
\end{equation*}
$$

Here $A=D F(0):=\left.\frac{\partial F}{\partial x}\right|_{x=0} \in \mathbb{R}^{m \times m}, g(0)=0 \in \mathbb{R}^{m}$, and the components of $g$ have power series expansions in $x_{1}, x_{2}, \ldots, x_{m}$ beginning with terms of at least second degree. The linear system

$$
\begin{equation*}
\dot{x}=A x \tag{4.15}
\end{equation*}
$$

is called the linearization (or first approximation) of the given (nonlinear) system (at the origin). We then have:
4.3.11 Theorem. (Lyapunov's Linearization Theorem) If (4.15) is asymptotically stable or unstable, then the origin for $\dot{x}=F(x)$, where $F$ is given by (4.14), has the same stability property.

Proof : Consider the function

$$
V=x^{T} P x
$$

where $P$ satisfies

$$
A^{T} P+P A=-Q
$$

$Q$ being an arbitrary positive definite symmetric matrix. If (4.15) is asymptotically stable, then by Proposition 4.3.10 $P$ is positive definite. The derivative of $V$ with respect to (4.14) is

$$
\dot{V}=-x^{T} Q x+2 g^{T} P x .
$$

Because of the nature of $g$, the term $2 g^{T} P x$ has degree three at least, and so for $x$ sufficiently close to the origin, $\dot{V}<0$.

Exercise 66 Let $a, b>0$ and consider the function

$$
f: \mathbb{R} \rightarrow \mathbb{R}, \quad t \mapsto-a t^{2}+b t^{3} .
$$

Show that, for $t$ sufficiently close to the origin (i.e. for $|t|<\epsilon$ ), we have $f(t)<0$.

Hence, by Lyapunov's First Theorem, the origin of (4.14) is asymptotically stable.

If (4.15) is unstable, $\dot{V}$ remains negative definite but $P$ is indefinite, so $V$ can take negative values and therefore satisfies the conditions of Lyapunov's Third Theorem for instability.

Note : If (4.15) is stable but not asymptotically stable, Lyapunov's Linearization Theorem provides no information about the stability of the origin of (4.14), and other methods must be used.

Furthermore, it is clear that linearization cannot provide any information about regions of asymptotic stability for nonlinear systems, since if the first approximation is asymptotically stable, then it is so in the large. Thus the extent of asymptotic stability for

$$
\dot{x}=F(x), \quad F(0)=0
$$

is determined by the nonlinear terms in (4.14).
4.3.12 Example. Consider the differential equation

$$
\ddot{z}+a \dot{z}+b z+g(z, \dot{z})=0
$$

or

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{2} \\
\dot{x}_{2}=-b x_{1}-a x_{2}-g\left(x_{1}, x_{2}\right) .
\end{array}\right.
$$

The linear part of this system is asymptotically stable if and only if $a>0$ and $b>0$, so if $g$ is any function of $x_{1}$ and $x_{2}$ satisfying the conditions of Theorem 4.3.11, the origin of the system is also asymptotically stable.

### 4.4 Stability and Control

We now consider some stability problems associated explicitly with the control variables.

## Input-output stability

Our definitions in section 4.1 referred to stability with respect to perturbations from an equilibrium state. When a system is subject to inputs it is useful to define a new type of stability.
4.4.1 Definition. The control system (with outputs) $\Sigma$ described by

$$
\left\{\begin{array}{l}
\dot{x}=F(t, x, u), \quad F(t, 0,0)=0 \\
y=h(t, x, u)
\end{array}\right.
$$

is said to be bounded input-bounded output stable (b.i.b.o. stable) if any bounded input produces a bounded output; that is, given

$$
\|u(t)\|<L_{1}, \quad t \geq t_{0}
$$

where $L_{1}$ is any positive constant, then there exists a number $L_{2}>0$ such that

$$
\|y(t)\|<L_{2}, \quad t \geq t_{0}
$$

regardless of initial state $x\left(t_{0}\right)$. The problem of studying b.i.b.o. stability for nonlinear systems is a difficult one, but we can give some results for the usual linear time-invariant system

$$
\left\{\begin{array}{l}
\dot{x}=A x+B u(t)  \tag{4.16}\\
y=C x
\end{array}\right.
$$

Exercise 67 Let $A \in \mathbb{R}^{m \times m}$ be a matrix having $m$ distinct eigenvalues with negative real parts. Show that (for all $t \geq 0$ )

$$
\|\exp (t A)\| \leq K e^{-a t}
$$

for some constants $K, a>0$. (The result remains valid for any stability matrix, but the proof is more difficult.)
4.4.2 Proposition. If the linear system

$$
\dot{x}=A x
$$

is asymptotically stable, then the system described by (4.16) is b.i.b.o. stable.
Proof: Using

$$
x(t)=\exp (t A)\left[x_{0}+\int_{0}^{t} \exp (-\tau A) B u(\tau) d \tau\right]
$$

and properties of norms, we have

$$
\begin{aligned}
\|y(t)\| & \leq\|C\|\|x(t)\| \\
& \leq\|C\|\left\|\exp (t A) x_{0}\right\|+\|C\| \int_{0}^{t}\|\exp (t-\tau) A\|\|B u\| d \tau .
\end{aligned}
$$

If $A$ is a stability matrix, then

$$
\|\exp (t A)\| \leq K e^{-a t} \leq K, \quad t \geq 0
$$

for some positive constants $K$ and $a$. Thus

$$
\begin{aligned}
\|y(t)\| & \leq\|C\|\left[K\left\|x_{0}\right\|+L_{1} K\|B\| \frac{1-e^{-a t}}{a}\right] \\
& \leq\|C\|\left[K\left\|x_{0}\right\|+\frac{L_{1} K\|B\|}{a}\right], \quad t \geq 0
\end{aligned}
$$

showing that the output is bounded, since $\|C\|$ and $\|B\|$ are positive numbers.
The converse of this result holds if $(A, B, C)$ is a minimal realization. In other words,
4.4.3 Proposition. If the control system described by (4.16) is c.c. and c.o. and b.i.b.o. stable, then the linear (dynamical) system

$$
\dot{x}=A x
$$

is asymptotically stable.

We shall not give a proof.
Note : For linear time-varying systems Proposition 4.4.2 is not true, unless for all $t$ the norms of $B(t)$ and $C(t)$ are bounded and the norm of the state transition matrix $\Phi\left(t, t_{0}\right)$ is bounded and tends to zero as $t \rightarrow \infty$ independently of $t_{0}$.

In the definition of complete controllability no restrictions were applied to $u(\cdot)$, but in practical situations there will clearly always be finite bounds on the magnitudes of the control variables and on the duration of their application. It is then intuitively obvious that this will imply that not all states are attainable. As a trivial example, if a finite thrust is applied to a rocket for a finite time, then there will be a limit to the final velocity which can be achieved. We give here one formal result for linear systems.
4.4.4 Proposition. If $A \in \mathbb{R}^{m \times m}$ is a stability matrix, then the linear control system $\Sigma$ given by

$$
\dot{x}=A x+B u(t)
$$

with $u(\cdot)$ bounded is not completely controllable.
Proof : Let $V$ be a quadratic form Lyapunov function (i.e. $V=x^{T} P x$, where $P$ is a symmetric matrix) for the (unforced) system $\dot{x}=A x$. Then (with respect to $\Sigma$ ) we have

$$
\begin{aligned}
\dot{V}=\dot{V}^{T} & =(A x+B u)^{T}(\nabla V) \\
& =x^{T} A^{T}(\nabla V)+u^{T} B^{T}(\nabla V) \\
& =-x^{T} Q x+u^{T} B^{T}(\nabla V)
\end{aligned}
$$

where $P$ and $Q$ satisfy (the Lyapunov matrix equation)

$$
A^{T} P+P A=-Q
$$

and

$$
\nabla V:=\left[\begin{array}{llll}
\frac{\partial V}{\partial x_{1}} & \frac{\partial V}{\partial x_{2}} & \cdots & \frac{\partial V}{\partial x_{m}}
\end{array}\right]^{T}
$$

is the gradient of $V$. The term $u^{T} B^{T}(\nabla V)$ is linear in $x$ and since $u(\cdot)$ is bounded, it follows that for $\|x\|$ sufficiently large, $\dot{V}=\dot{x}^{T}(\nabla V)$ is negative. This shows that $\dot{x}(\cdot)$ points into the interior of the region $R: V(x)=M$ for some $M$ sufficiently large. Hence points outside this region cannot be reached, so by definition the system is not c.c.

## Linear feedback

Consider again the linear system (4.16). If the open loop system is unstable (for instance, by Theorem 4.2.1, if one or more of the eigenvalues of $A$ has a positive real part), then an essential practical objective would be to apply control so as to stabilize the system; that is, to make the closed loop system asymptotically stable.

If (4.16) is c.c., then we saw (Theorem 3.1.3) that stabilization can always be achieved by linear feedback $u=K x$, since there are an infinity of matrices $K$ which will make $A+B K$ a stability matrix.

If the pair $(A, B)$ is not c.c., then we can define the weaker property that $(A, B)$ is stabilizable if (and only if) there exists a constant matrix $K$ such that $A+B K$ is asymptotically stable.
4.4.5 Example. Return to the linear system $\Sigma$ described by

$$
\dot{x}=\left[\begin{array}{rrr}
4 & 3 & 5 \\
1 & -2 & -3 \\
2 & 1 & 8
\end{array}\right] x+\left[\begin{array}{r}
2 \\
1 \\
-1
\end{array}\right] u(t)
$$

in Example 3.4.4. The eigenvalues of the uncontrollable part are the roots of the polynomial

$$
p(\lambda)=\lambda^{2}-7 \lambda-23
$$

which is not asymptotically stable since it has negative coefficients (PropoSItion 4.2.6). The system $\Sigma$ is not stabilizable.

By duality (see Theorem 3.2.4) we define the pair $(A, C)$ to be detectable if (and only if) the pair $\left(A^{T}, C^{T}\right)$ is stabilizable.
4.4.6 Example. Consider the linear control system (with outputs)

$$
\left\{\begin{array}{l}
\dot{x}=A x+b u(t) \\
y=c x
\end{array}\right.
$$

where

$$
A=\left[\begin{array}{cc}
-2 & 2 \\
1 & -1
\end{array}\right], \quad b=\left[\begin{array}{l}
1 \\
1
\end{array}\right], \quad c=\left[\begin{array}{ll}
1 & 0
\end{array}\right] .
$$

This system is both stabilizable and detectable : using the feedback matrix $K$ and the observer matrix $L$ given by

$$
K=\left[\begin{array}{ll}
-2 & 0
\end{array}\right], \quad L=\left[\begin{array}{c}
0 \\
-1
\end{array}\right]
$$

the matrix $A+b K$ then has eigenvalues $-3,-2$ and the matrix $A+L c$ has eigenvalues $-2,-1$. (Other choices for $K$ and $L$ are also possible.)

Stabilizability and detectability are not guaranteed.
4.4.7 Example. Consider the linear control system (with outputs)

$$
\left\{\begin{array}{l}
\dot{x}=A x+b u(t) \\
y=c x
\end{array}\right.
$$

where

$$
A=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right], \quad b=\left[\begin{array}{l}
1 \\
0
\end{array}\right], \quad c=\left[\begin{array}{ll}
0 & 1
\end{array}\right] .
$$

In this case, any $1 \times 2$ feedback matrix $K$ produces a closed loop matrix $A+b K$ with zero as an eigenvalue; therefore, the system is not stabilizable.

Also, any $2 \times 1$ matrix $L$ yields a matrix $A+L c$ with zero as an eigenvalue; therefore, the system is not detectable.

The following simple test for stabilizability holds :
4.4.8 Proposition. Suppose $A \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{m \times \ell}$. The pair $(A, B)$ is stabilizable if and only if

$$
\operatorname{rank}\left[\begin{array}{cc}
s I_{m}-A & B
\end{array}\right]=m
$$

for all $s \in \mathbb{C}_{e+}:=\{s \mid \operatorname{Re}(s) \geq 0\}$.
The proof is not difficult and will be omitted.
4.4.9 Corollary. The pair $(A, B)$ is stabilizable if and only if

$$
\operatorname{rank}\left[\begin{array}{ll}
\lambda I_{m}-A & B
\end{array}\right]=m
$$

for all eigenvalues $\lambda \in \mathbb{C}_{e+}$ of $A$.

Note : It can be proved that the pair $(A, B)$ is c.c. if and only if

$$
\operatorname{rank}\left[\begin{array}{cc}
\lambda I_{m}-A & B
\end{array}\right]=m
$$

for all eigenvalues $\lambda$ of $A$. (The eigenvalues at which the rank drops below $m$ are the so-called uncontrollable modes.) Clearly, if the pair $(A, B)$ is c.c., then it is stabilizable.

By duality, the following (test for detectability) is now immediate.
4.4.10 Proposition. Suppose $A \in \mathbb{R}^{m \times m}$ and $C \in \mathbb{R}^{\ell \times m}$. The pair $(A, C)$ is detectable if and only if

$$
\operatorname{rank}\left[\begin{array}{c}
s I_{m}-A \\
C
\end{array}\right]=m
$$

for all $s \in \mathbb{C}_{e+}:=\{s \mid \operatorname{Re}(s) \geq 0\}$.
4.4.11 Corollary. The pair $(A, C)$ is detectable if and only if

$$
\operatorname{rank}\left[\begin{array}{c}
\lambda I_{m}-A \\
C
\end{array}\right]=m
$$

for all eigenvalues $\lambda \in \mathbb{C}_{e+}$ of $A$.

Note: By duality again, the pair $(A, C)$ is c.o. if and only if

$$
\operatorname{rank}\left[\begin{array}{c}
\lambda I_{m}-A \\
C
\end{array}\right]=m
$$

for all eigenvalues $\lambda$ of $A$. (The eigenvalues at which the rank drops below $m$ are the so-called unobservable modes.) Clearly, if the pair $(A, C)$ is c.o., then it is detectable.

Let $A \in \mathbb{R}^{m \times m}, B \in \mathbb{R}^{m \times \ell}$, and $C \in \mathbb{R}^{n \times m}$. Recall that if $A$ is a stability (or Hurwitz) matrix, then the usual (time-invariant) linear control system with outputs

$$
\begin{aligned}
\dot{x} & =A x+B u \\
y & =C x
\end{aligned}
$$

is b.i.b.o. stable (see Proposition 4.4.2). As we know, the converse is not true, in general. (For example, the usual linear control system with $A=$ $\left[\begin{array}{cc}2 & 0 \\ 0 & -1\end{array}\right], b=\left[\begin{array}{l}1 \\ 1\end{array}\right]$, and $c=\left[\begin{array}{ll}0 & 1\end{array}\right]$ is b.i.b.o. stable but not asymptotically stable.)

The following interesting result is given without a proof.
4.4.12 Proposition. Suppose a usual linear control system $\Sigma$ is stabilizable and detectable. $\Sigma$ is asymptotically stable if and only if it is b.i.b.o. stable.

## Application

It is interesting to consider here a simple application of the Lyapunov methods. First notice that if the linear system

$$
\dot{x}=A x
$$

is asymptotically stable with Lyapunov function $V=x^{T} P x$, where $P$ satisfies

$$
A^{T} P+P A=-Q
$$

then

$$
\frac{\dot{V}}{V}=-\frac{x^{T} Q x}{x^{T} P x} \leq-\sigma
$$

where $\sigma$ is the minimum value of the ratio $\frac{x^{T} Q x}{x^{T} P x}$ (in fact, this is equal to the smallest eigenvalue of $Q P^{-1}$ ). Integrating with respect to $t$ gives

$$
V(x(t)) \leq e^{-\sigma t} V(x(0))
$$

Since $V(x(t)) \rightarrow 0$ as $t \rightarrow \infty$, this can be regarded as a measure of the way in which trajectories approach the origin, so the larger $\sigma$ the "faster" does $x(t) \rightarrow 0$.

Suppose now we apply the control

$$
\begin{equation*}
u=\left(S-Q_{1}\right) B^{T} P x \tag{4.17}
\end{equation*}
$$

to (4.16), where $P$ is the solution of

$$
A^{T} P+P A=-Q
$$

and $S$ and $Q_{1}$ are arbitrary skew-symmetric and positive definite symmetric matrices, respectively. The closed loop system is thus

$$
\begin{equation*}
\dot{x}=\left(A+B\left(S-Q_{1}\right) B^{T} P\right) x \tag{4.18}
\end{equation*}
$$

and it is easy to verify that if $V=x^{T} P x$, then the (directional) derivative with respect to (4.18) is

$$
\dot{V}=-x^{T} Q x-2 x^{T} P B Q_{1} B^{T} P x<-x^{T} Q x
$$

since $P B Q_{1} B^{T} P=(P B) Q_{1}(P B)^{T}$ is positive definite. Hence, by the argument just developed, it follows that (4.18) is "more stable" than the open loop system

$$
\dot{x}=A x
$$

in the sense that trajectories will approach the origin more quickly.

Note : (4.17) is of rather limited practical value because it requires asymptotic stability of the open loop system, but nevertheless the power of Lyapunov theory is apparent by the ease with which the asymptotic stability of (4.18) can be established. This would be impossible using the classical methods requiring the calculation of the characteristic equation of (4.18). Furthermore, the Lyapunov approach often enables extensions to nonlinear problems to be made.

### 4.5 Exercises

Exercise 68 Determine the equilibrium point (other than the origin) of the system described by

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{1}-2 x_{1} x_{2} \\
\dot{x}_{2}=-2 x_{2}+x_{1} x_{2}
\end{array}\right.
$$

Apply a transformation of coordinates which moves this point to the origin, and find the new system equations. (The equations are an example of a predator-pray population model due to Vito Volterra (1860-1940) and used in biology, and are more general than the simple linear rabbit-fox model.)

Exercise 69 Determine whether the following polynomials are asymptotically stable:
(a) $\lambda^{3}+17 \lambda^{2}+2 \lambda+1$.
(b) $\lambda^{4}+\lambda^{3}+4 \lambda^{2}+4 \lambda+3$.
(c) $\lambda^{4}+6 \lambda^{3}+2 \lambda^{2}+\lambda+3$.

Exercise 70 Determine for what range of values of $k$ the polynomial

$$
(3-k) \lambda^{3}+2 \lambda^{2}+(5-2 k) \lambda+2
$$

is asymptotically stable.
Exercise 71 Determine for what range of values of $k \in \mathbb{R}$ the linear (dynamical) system

$$
\dot{x}=\left[\begin{array}{rrr}
0 & 1 & 0 \\
0 & 0 & 1 \\
-k & -1 & -2
\end{array}\right] x
$$

is asymptotically stable. If $k=-1$, and a control term

$$
\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right] u(t)
$$

is added, find a linear feedback control which makes all the eigenvalues of the closed loop system equal to -1 .

Exercise 72 Verify that the eigenvalues of

$$
A(t)=\left[\begin{array}{cc}
-4 & 3 e^{-8 t} \\
-e^{8 t} & 0
\end{array}\right]
$$

are both constant and negative, but the solution of the linear time-varying system

$$
\dot{x}=A(t) x
$$

diverges as $t \rightarrow \infty$.

Exercise 73 Write the system

$$
\ddot{z}+\dot{z}+z^{3}=0
$$

in state space form. Let

$$
V=a x_{1}^{4}+b x_{1}^{2}+c x_{1} x_{2}+d x_{2}^{2}
$$

and choose the constants $a, b, c, d$ such that

$$
\dot{V}=-x_{1}^{4}-x_{2}^{2}
$$

Hence investigate the stability nature of the equilibrium point at the origin.
Exercise 74 Using the function

$$
V=5 x_{1}^{2}+2 x_{1} x_{2}+2 x_{2}^{2}
$$

show that the origin of

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{2} \\
\dot{x}_{2}=-x_{1}-x_{2}+\left(x_{1}+2 x_{2}\right)\left(x_{2}^{2}-1\right)
\end{array}\right.
$$

is asymptotically stable by considering the region $\left|x_{2}\right|<1$. State the region of asymptotic stability thus determined.

Exercise 75 Investigate the stability nature of the origin of the system

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{1}^{2}-x_{2}^{2} \\
\dot{x}_{2}=-2 x_{1} x_{2}
\end{array}\right.
$$

using the function

$$
V=3 x_{1} x_{2}^{2}-x_{1}^{3}
$$

Exercise 76 Given the matrix

$$
A=\left[\begin{array}{rr}
-1 & -1 \\
2 & -4
\end{array}\right]
$$

and taking

$$
P=\left[\begin{array}{ll}
p_{1} & p_{2} \\
p_{2} & p_{3}
\end{array}\right]
$$

solve the equation

$$
A^{T} P+P A=-I_{2}
$$

Hence determine the stability nature of $A$.

Exercise 77 Integrate both sides of the matrix ODE

$$
\dot{W}(t)=A W(t)+W(t) B, \quad W(0)=C
$$

with respect to $t$ from $t=0$ to $t=\infty$. Hence deduce that if $A$ is a stability matrix, the solution of the equation

$$
A^{T} P+P A=-Q
$$

can be written as

$$
P=\int_{0}^{\infty} \exp \left(t A^{T}\right) Q \exp (t A) d t
$$

Exercise 78 Convert the second order ODE

$$
\ddot{z}+a_{1} \dot{z}+a_{2} z=0
$$

into the state space form. Using $V=x^{T} P x$ with $\dot{V}=-x_{2}^{2}$, obtain the necessary and sufficient conditions $a_{1}>0, a_{2}>0$ for asymptotic stability.

Exercise 79 By using the quadratic Lyapunov function $V=V\left(x_{1}, x_{2}\right)$ which has derivative $-2\left(x_{1}^{2}+x_{2}^{2}\right)$, determine the stability of the system

$$
\left\{\begin{array}{l}
\dot{x}_{1}=-k x_{1}-3 x_{2} \\
\dot{x}_{2}=k x_{1}-2 x_{2}
\end{array}\right.
$$

when $k=1$. Using the same function $V$, obtain sufficient conditions on $k$ for the system to be asymptotically stable.

Exercise 80 Investigate the stability nature of the equilibrium state at the origin for the system

$$
\left\{\begin{array}{l}
\dot{x}_{1}=7 x_{1}+2 \sin x_{2}-x_{2}^{4} \\
\dot{x}_{2}=e^{x_{1}}-3 x_{2}-1+5 x_{1}^{2}
\end{array}\right.
$$

Exercise 81 Investigate the stability nature of the origin.
(a)

$$
\left\{\begin{array}{l}
\dot{x}_{1}=-x_{1}+x_{2} \\
\dot{x}_{2}=\left(x_{1}+x_{2}\right) \sin x_{1}-3 x_{2} .
\end{array}\right.
$$

(b)

$$
\left\{\begin{array}{l}
\dot{x}_{1}=-x_{1}^{3}+x_{2} \\
\dot{x}_{2}=-a x_{1}-b x_{2} ; \quad a, b>0
\end{array}\right.
$$

Exercise 82 Show that the system described by

$$
\left\{\begin{array}{l}
\dot{x}=\left[\begin{array}{rrr}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{array}\right] x+\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right] u(t) \\
y=x
\end{array}\right.
$$

is stable i.s.L. and b.i.b.o. stable, but not asymptotically stable. [It is easy to verify that this system is not completely controllable.]

Exercise 83 Consider the system described by the ODE

$$
\dot{x}=-\frac{1}{t+3} x+u(t), \quad x(0)=x_{0} \in \mathbb{R}
$$

Show that if $u(t) \equiv 0, t \geq 0$ then the origin is asymptotically stable, but that if $u(\cdot)$ is the unit step function, then $\lim _{t \rightarrow \infty} x(t)=\infty$.

Exercise 84 A system is described by the equations

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{2}-x_{1} \\
\dot{x}_{2}=-x_{1}-x_{2}-x_{1}^{2}
\end{array}\right.
$$

Determine the equilibrium state which is not at the origin. Transform the system equations so that this point is transferred to the origin, and hence verify that this equilibrium state is unstable.

Exercise 85 Determine for what range of values of the real parameter $k$ the linear system

$$
\dot{x}=\left[\begin{array}{rrc}
0 & 1 & 0 \\
0 & 0 & 1 \\
-5 & -k & k-6
\end{array}\right] x
$$

is asymptotically stable.

Exercise 86 A particle moves in the $x y$-plane so that its position in any time $t$ is given by

$$
\ddot{x}+\dot{y}+3 x=0, \quad \ddot{y}+\lambda \dot{x}+3 y=0 .
$$

Determine the stability nature of the system if $(a) \lambda=-4$ and (b) $\lambda=16$.
Exercise 87 Use the Lyapunov function

$$
V=2 x_{1}^{2}+x_{2}^{2}
$$

to find a region of asymptotic stability for the origin of the system

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{1}^{3}-3 x_{1}+x_{2} \\
\dot{x}_{2}=-2 x_{1}
\end{array}\right.
$$

Exercise 88 Use Lyapunov's Linearization Theorem to show that the origin for the system

$$
\left\{\begin{array}{l}
\dot{x}_{1}=-3 x_{2} \\
\dot{x}_{2}=x_{1}-\alpha\left(2 x_{2}^{3}-x_{2}\right)
\end{array}\right.
$$

is asymptotically stable provided the real parameter $\alpha$ is negative. Use the Lyapunov function

$$
V=\frac{1}{2}\left(x_{1}^{2}+3 x_{2}^{2}\right)
$$

to determine a region of asymptotic stability about the origin.
Exercise 89 A system has state equations

$$
\left\{\begin{array}{l}
\dot{x}_{1}=-x_{1}+x_{2}-x_{3}+\alpha x_{1}^{3} \\
\dot{x}_{2}=-x_{1}-x_{2} \\
\dot{x}_{3}=x_{1}-x_{2}-\beta x_{3}+u(t) .
\end{array}\right.
$$

(a) If $\alpha=0$ show that the system is b.i.b.o. stable for any output which is linear in the state variables, provided $\beta>-1$.
(b) If $\alpha=\beta=1$ and $u(\cdot)=0$, investigate the stability nature of the equilibrium state at the origin by using the Lyapunov function

$$
V=x_{1}^{2}+x_{2}^{2}+x_{3}^{2} .
$$

Exercise 90 Investigate for stabilizability and detectability the following control systems.
(a)

$$
\begin{aligned}
\dot{x} & =\left[\begin{array}{cc}
2 & 0 \\
0 & -1
\end{array}\right] x+\left[\begin{array}{l}
1 \\
1
\end{array}\right] u(t) \\
y & =\left[\begin{array}{ll}
0 & 1
\end{array}\right] x
\end{aligned}
$$

(b)

$$
\begin{aligned}
\dot{x} & =\left[\begin{array}{cc}
0 & 0 \\
0 & -1
\end{array}\right] x+\left[\begin{array}{l}
0 \\
1
\end{array}\right] u(t) \\
y & =\left[\begin{array}{ll}
1 & 1
\end{array}\right] x .
\end{aligned}
$$

