Chapter 5

Optimal Control

Topics :

1. PERFORMANCE INDICES

2. ELEMENTS OF CALCULUS OF VARIATIONS

3. PONTRYAGIN'S PRINCIPLE

4. LINEAR REGULATORS WITH QUADRATIC COSTS

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This section deals with the problem of compelling a system to behave in some "best possible" way. Of course, the precise control strategy will depend upon the criterion used to decide what is meant by "best", and we first discuss some choices for measures of system performance. This is followed by a description of some mathematical techniques for determining optimal control policies, including the special case of linear systems with quadratic performance index when a complete analytical solution is possible.

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5.1   Performance Indices

Consider a (nonlinear) *control system* Σ described by

$$\dot{x} = F(t, x, u), \quad x(t_0) = x_0 \in \mathbb{R}^m. \tag{5.1}$$

Here $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix}$ is the *state vector*, $u(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_\ell(t) \end{bmatrix}$ is the *control vector*, and F is a vector-valued mapping having components

$$F_i : t \mapsto F_i(t, x_1(t), x_2(t), \ldots, x_m(t), u_1(t), \ldots, u_\ell(t)), \quad i = 1, 2, \ldots, m.$$

NOTE :   We shall assume that the F_i are continuous and satisfy standard conditions, such as having continuous first order partial derivatives (so that the solution exists and is unique for the given initial condition). We say that F is *continuously differentiable* (or of *class C^1*).

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The optimal control problem

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The general *optimal control problem* (OCP) concerns the *minimization* of some function (functional) $\mathcal{J} = \mathcal{J}[u]$, the **performance index** (or **cost functional**); or, one may want to maximize instead a "utility" functional $\mathcal{J}$, but this amounts to minimizing the cost $-\mathcal{J}$. The performance index $\mathcal{J}$ provides a *measure* by which the performance of the system is judged. We give several examples of performance indices.

(1)   Minimum-time problems.

Here $u(\cdot)$ is to be chosen so as to transfer the system from an initial state x_0 to a specified state in the *shortest possible time*. This is equivalent to *minimizing* the performance index

$$\mathcal{J} := t_1 - t_0 = \int_{t_0}^{t_1} dt \tag{5.2}$$

where t_1 is the first instant of time at which the desired state is reached.

5.1.1 EXAMPLE. An aircraft pursues a ballistic missile and wishes to intercept it as quickly as possible. For simplicity neglect gravitational and aerodynamic forces and suppose that the trajectories are horizontal. At $t = 0$ the aircraft is at a distance a from the missile, whose motion is known to be described by $x(t) = a + bt^2$, where b is a positive constant. The motion of the aircraft is given by $\ddot{x} = u$, where the thrust $u(\cdot)$ is subject to $|u| \leq 1$, with suitably chosen units. Clearly the optimal strategy for the aircraft is to accelerate with maximum thrust $u(t) = 1$. After a time t the aircraft has then travelled a distance $ct + \frac{1}{2}t^2$, where $\dot{x}(0) = c$, so interception will occur at time T where

$$cT + \frac{1}{2}T^2 = a + bT^2.$$

This equation may not have any real positive solution; in other words, *this minimum-time problem may have no solution for certain initial conditions.*

(2) Terminal control.

In this case the final state $x_f = x(t_1)$ is to be brought as near as possible to some desired state $\overline{x}(t_1)$. A suitable performance measure to be *minimized* is

$$\mathcal{J} := e^T(t_1)Me(t_1) \tag{5.3}$$

where $e(t) := x(t) - \overline{x}(t)$ and M is a positive definite symmetric matrix $(M^T = M > 0)$.

A special case is when M is the unit matrix and then

$$\mathcal{J} = \|x_f - \overline{x}(t_1)\|^2.$$

NOTE : More generally, if $M = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$, then the entries λ_i are chosen so as to weight the relative importance of the deviations $(x_i(t_1) - \overline{x}_i(t_1))$. If some of the $\overline{x}_i(t_1)$ are not specified, then the corresponding elements of M will be zero and M will be only positive semi-definite $(M^T = M \geq 0)$.

(3) Minimum effort.

The desired final state is now to be attained with *minimum total expenditure* of control effort. Suitable performance indices to be *minimized* are

$$\mathcal{J} := \int_{t_0}^{t_1} \sum_{i=1}^{\ell} \beta_i |u_i| \, dt \tag{5.4}$$

or

$$\mathcal{J} := \int_{t_0}^{t_1} u^T R u \, dt \tag{5.5}$$

where $R = \left[r_{ij} \right]$ is a positive definite symmetric matrix $(R^T = R > 0)$ and the β_i and r_{ij} are *weighting factors*.

(4)   Tracking problems.

The aim here is to follow or "track" as closely as possible some desired state $\overline{x}(\cdot)$ throughout the interval $[t_0, t_1]$. A suitable performance index is

$$\mathcal{J} := \int_{t_0}^{t_1} e^T Q e \, dt \tag{5.6}$$

where Q is a positive semi-definite symmetric matrix $(Q^T = Q \geq 0)$.

NOTE :   Such systems are called **servomechanisms**; the special case when $\overline{x}(\cdot)$ is constant or zero is called a **regulator**. If the $u_i(\cdot)$ are unbounded, then the minimization problem can lead to a control vector having infinite components. This is unacceptable for real-life problems, so to restrict the total control effort, the following index can be used

$$\mathcal{J} := \int_{t_0}^{t_1} \left(e^T Q e + u^T R u \right) \, dt. \tag{5.7}$$

Expressions (costs) of the form (5.5), (5.6) and (5.7) are termed **quadratic performance indices** (or **quadratic costs**).

5.1.2 EXAMPLE.   A landing vehicle separates from a spacecraft at time $t_0 = 0$ at an altitude h from the surface of a planet, with initial (downward) velocity $\vec{v}$. For simplicity, assume that gravitational forces are neglected and that the mass of the vehicle is constant. Consider vertical motion only, with

upwards regarded as the positive direction. Let x_1 denote *altitude*, x_2 *velocity* and $u(\cdot)$ the *thrust* exerted by the rocket motor, subject to $|u(t)| \leq 1$ with suitable scaling. The *equations of motion* are

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

and the *initial conditions* are

$$x_1(0) = h, \quad x_2(0) = -v.$$

For a "soft landing" at some time T we require

$$x_1(T) = 0, \quad x_2(T) = 0.$$

A suitable *performance index* might be

$$\mathcal{J} := \int_0^T \left(|u| + k \right) \, dt.$$

This expression represents a sum of the *total fuel consumption* and *time to landing*, k being a factor which weights the relative importance of these two quantities.

Simple application

Before dealing with problems of determining optimal controls, we return to the linear time-invariant system

$$\dot{x} = Ax, \quad x(0) = x_0 \tag{5.8}$$

and show *how to evaluate associated quadratic indices (costs)*

$$\mathcal{J}_r := \int_0^\infty t^r x^T Q x \, dt, \quad r = 0, 1, 2, \ldots \tag{5.9}$$

where Q is a positive definite symmetric matrix ($Q^T = Q > 0$).

NOTE : If (5.8) represents a *regulator*, with $x(\cdot)$ being the *deviation* from some desired constant state, then minimizing $\mathcal{J}_r$ with respect to system parameters is

equivalent to making the system approach its desired state in an "optimal" way. Increasing the value of r in (5.9) corresponds to penalizing large values of t in this process.

To evaluate $\mathcal{J}_0$ we use the techniques of Lyapunov theory (cf. section **4.3**). It was shown that

$$\frac{d}{dt}\left(x^T P x\right) = -x^T Q x \tag{5.10}$$

where P and Q satisfy the *Lyapunov matrix equation*

$$A^T P + P A = -Q. \tag{5.11}$$

Integrating both sides of (5.10) with respect to t gives

$$\mathcal{J}_0 = \int_0^\infty x^T Q x\, dt = -\left(x^T(t) P x(t)\right)\Big|_0^\infty = x_0^T P x_0$$

provided A is a stability matrix, since in this case $x(t) \to 0$ as $t \to \infty$ (cf. THEOREM 4.2.1).

NOTE :   The matrix P is positive definite and so $\mathcal{J}_0 > 0$ for all $x_0 \neq 0$.

A repetition of the argument leads to a similar expression for $\mathcal{J}_r$, $r \geq 1$. For example,

$$\frac{d}{dt}\left(t x^T P x\right) = x^T P x - t x^T Q x$$

and integrating we have

$$\mathcal{J}_1 = \int_0^\infty t x^T Q x\, dt = x_0^T P_1 x_0$$

where

$$A^T P_1 + P_1 A = -P.$$

Exercise 91 Show that

$$\mathcal{J}_r := \int_0^\infty t^r x^T Q x\, dt = r!\, x_0^T P_r x_0 \tag{5.12}$$

where

$$A^T P_{r+1} + P_{r+1} A = -P_r, \quad r = 0, 1, 2, \dots\,; \quad P_0 = P. \tag{5.13}$$

Thus *evaluation of (5.9) involves merely successive solution of the linear matrix equations (5.13)*; there is no need to calculate the solution $x(\cdot)$ of (5.8).

5.1.3 EXAMPLE.   A general second-order linear system (the *harmonic oscillator* in one dimension) can be written as

$$\ddot{z} + 2\omega k \dot{z} + \omega^2 z = 0$$

where ω is the natural *frecquency* of the undamped system and k is a *damping coefficient*. With the usual choice of state variables $x_1 := z$, $x_2 := \dot{z}$, and taking $Q = \mathrm{diag}\,(1, q)$ in (5.11), it is easy to obtain the corresponding solution $P = \left[p_{ij}\right]$ with elements

$$p_{11} = \frac{k}{\omega} + \frac{1 + q\omega^2}{4k\omega}, \quad p_{12} = p_{21} = \frac{1}{2\omega^2}, \quad p_{22} = \frac{1 + q\omega^2}{4k\omega^3}.$$

Exercise 92 Work out the preceding computation.

In particular, if $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\mathcal{J}_0 = p_{11}$. Regarding k as a parameter, optimal damping could be defined as that which minimizes $\mathcal{J}_0$. By setting $\frac{d}{dk}\mathcal{J}_0 = 0$, this gives

$$k^2 = \frac{1 + q\omega^2}{4}.$$

For example, if $q = \frac{1}{\omega^2}$ then the "optimal" value of k is $\frac{1}{\sqrt{2}}$.

NOTE :   In fact by determining $x(t)$ it can be deduced that *this value does indeed give the desirable system transient behaviour*. However, there is no *a priori* way of deciding on a suitable value for the factor q, which weights the relative importance of reducing $z(\cdot)$ and $\dot{z}(\cdot)$ to zero. This illustrates a disadvantage of the performance index approach, although in some applications it is possible to use physical arguments to choose values for weighting factors.

5.2   Elements of Calculus of Variations

The *calculus of variations* is the name given to the theory of the optimization of integrals. The name itself dates from the mid-eighteenth century and describes the method used to derive the theory. We have room for only a very brief treatment (in particular, we shall not mention the well-known *Euler-Lagrange equation* approach).

We consider the problem of *minimizing* the functional

$$\mathcal{J}[u] := \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} L(t, x, u)\, dt \qquad (5.14)$$

subject to

$$\dot{x} = F(t, x, u), \quad x(t_0) = x_0 \in \mathbb{R}^m.$$

We assume that

- there are *no constraints* on the control functions $u_i(\cdot)$, $\quad i = 1, 2, \ldots, \ell$ (that is, the control set U is $\mathbb{R}^\ell$);

- $\mathcal{J} = \mathcal{J}[u]$ is *differentiable* (that is, if u and $u + \delta u$ are two controls for which $\mathcal{J}$ is defined, then

$$\Delta \mathcal{J} := \mathcal{J}[u + \delta u] - \mathcal{J}[u] = \delta \mathcal{J}[u, \delta u] + j(u, \delta u) \cdot \|\delta u\|$$

where $\delta \mathcal{J}$ is *linear* in δu and $j(u, \delta u) \to 0$ as $\|\delta u\| \to 0$).

NOTE :   (1)   The cost functional $\mathcal{J}$ is in fact a function on the *function space* $\mathcal{U}$ (of all admissible controls) :

$$\mathcal{J} : u \in \mathcal{U} \mapsto \mathcal{J}[u] \in \mathbb{R}.$$

(2) $\delta \mathcal{J}$ is called the (first) **variation** of $\mathcal{J}$ corresponding to the variation δu in u.

The control u^* is an **extremal**, and $\mathcal{J}$ has a (relative) *minimum*, provided there exists an $\varepsilon > 0$ such that for all functions u satisfying $\|u - u^*\| < \varepsilon$,

$$\mathcal{J}[u] - \mathcal{J}[u^*] \geq 0.$$

A fundamental result (given without proof) is the following :

5.2.1 PROPOSITION. *A necessary condition for u^* to be an extremal is that*

$$\delta \mathcal{J}[u^*, \delta u] = 0 \quad \text{for all } \delta u.$$

We now apply PROPOSITION 5.2.1. Introduce a covector function of *Lagrange multipliers* $p(t) = \begin{bmatrix} p_1(t) & p_2(t) \dots & p_m(t) \end{bmatrix} \in \mathbb{R}^{1 \times m}$ so as to form an *augmented functional* incorporating the constraints :

$$\mathcal{J}_a := \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} (L(t, x, u) + p(F(t, x, u) - \dot{x})) \, dt.$$

Integrating the last term on the rhs by parts gives

$$\begin{aligned}
\mathcal{J}_a &= \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} (L + pF + \dot{p}x) \, dt - px \Big|_{t_0}^{t_1} \\
&= \varphi(x(t_1), t_1) - px \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} (H + \dot{p}x) \, dt
\end{aligned}$$

where the (control) **Hamiltonian function** is defined by

$$\boxed{H(t, p, x, u) := L(t, x, u) + pF(t, x, u).} \tag{5.15}$$

Assume that u is differentiable on $[t_0, t_1]$ and that t_0 and t_1 are fixed. The variation in $\mathcal{J}_a$ corresponding to a variation δu in u is

$$\delta \mathcal{J}_a = \left[\left(\frac{\partial \varphi}{\partial x} - p \right) \delta x \right]_{t=t_1} + \int_{t_0}^{t_1} \left(\frac{\partial H}{\partial x} \delta x + \frac{\partial H}{\partial u} \delta u + \dot{p} \, \delta x \right) dt$$

where δx is the variation in x in the differential equation

$$\dot{x} = F(t, x, u)$$

due to δu. (We have used the notation

$$\frac{\partial H}{\partial x} := \begin{bmatrix} \dfrac{\partial H}{\partial x_1} & \dfrac{\partial H}{\partial x_2} & \cdots & \dfrac{\partial H}{\partial x_m} \end{bmatrix}$$

and similarly for $\dfrac{\partial \varphi}{\partial x}$ and $\dfrac{\partial H}{\partial u}$.)

NOTE :   Since $x(t_0)$ is specified, $\delta x|_{t=t_0} = 0$.

It is convenient to remove the term (in the expression δJ_a) involving δx by suitably choosing p, i.e. by taking

$$\dot{p} = -\frac{\partial H}{\partial x} \quad \text{and} \quad p(t_1) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1} . \tag{5.16}$$

It follows that

$$\delta J_a = \int_{t_0}^{t_1} \left(\frac{\partial H}{\partial u}\, \delta u\right) dt \,.$$

Thus *a necessary condition for u^* to be an extremal is that*

$$\boxed{\left.\frac{\partial H}{\partial u}\right|_{u=u^*} = 0\,, \qquad t_0 \le t \le t_1.} \tag{5.17}$$

We have therefore "established"

5.2.2 THEOREM.   *Necessary conditions for u^* to be an extremal for*

$$J[u] = \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} L(t, x, u)\, dt$$

subject to

$$\dot{x} = F(t, x, u), \quad x(t_0) = x_0$$

are the following :

$$\dot{p} \;\; = \;\; -\frac{\partial H}{\partial x}$$

$$p(t_1) \;\; = \;\; \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1}$$

$$\left.\frac{\partial H}{\partial u}\right|_{u=u^*} \;\; = \;\; 0\,, \qquad t_0 \le t \le t_1.$$

NOTE :   The (vector) *state equation*

$$\dot{x} = F(t, x, u)$$

and the (vector) *co-state equation* (or *adjoint equation*)

$$\dot{p} = -\frac{\partial H}{\partial x}$$

give a total of $2m$ linear *or* nonlinear ODEs with (mixed) *boundary conditions* $x(t_0)$ and $p(t_1)$. In general, *analytical solution is not possible and numerical techniques have to be used.*

5.2.3 EXAMPLE. Choose $u(\cdot)$ so as to minimize

$$\mathcal{J} = \int_0^T \left(x^2 + u^2 \right) dt$$

subject to

$$\dot{x} = -ax + u, \quad x(0) = x_0 \in \mathbb{R}$$

where $a, T > 0$. We have

$$H = L + pF = x^2 + u^2 + p(-ax + u).$$

Also,

$$\dot{p}^* = -\frac{\partial H}{\partial x} = -2x^* + ap^*$$

and

$$\frac{\partial H}{\partial u}\bigg|_{u=u^*} := 2u^* + p^* = 0$$

where x^* and p^* denote the *state* and *adjoint* variables for an optimal solution.

Substitution produces

$$\dot{x}^* = -ax^* - \frac{1}{2}p^*$$

and since $\varphi \equiv 0$, the *boundary condition* is just

$$p(T) = 0.$$

The linear system

$$\begin{bmatrix} \dot{x}^* \\ \dot{p}^* \end{bmatrix} = \begin{bmatrix} -a & -\frac{1}{2} \\ -2 & a \end{bmatrix} \begin{bmatrix} x^* \\ p^* \end{bmatrix}$$

can be solved using the methods of CHAPTER 2. (It is easy to verify that x^* and p^* take the form $c_1 e^{\lambda t} + c_2 e^{-\lambda t}$, where $\lambda = \sqrt{1 + a^2}$ and the constants c_1 and c_2 are found using the conditions at $t = 0$ and $t = T$.)

It follows that the *optimal control* is

$$u^*(t) = -\frac{1}{2}p^*(t).$$

NOTE :   We have only found *necessary conditions* for optimality; further discussion of this point goes far beyond the scope of this course.

If the functions L and F do not *explicitly* depend upon t, then from

$$H(p, x, u) = L(x, u) + pF(x, u)$$

we get

$$
\begin{aligned}
\dot{H} = \frac{d\,H}{dt} &= \frac{\partial L}{\partial u}\dot{u} + \frac{\partial L}{\partial x}\dot{x} + p\left(\frac{\partial F}{\partial u}\dot{u} + \frac{\partial F}{\partial x}\dot{x}\right) + \dot{p}F \\
&= \left(\frac{\partial L}{\partial u} + p\frac{\partial F}{\partial u}\right)\dot{u} + \left(\frac{\partial L}{\partial x} + p\frac{\partial F}{\partial x}\right)\dot{x} + \dot{p}F \\
&= \frac{\partial H}{\partial u}\dot{u} + \frac{\partial H}{\partial x}\dot{x} + \dot{p}F \\
&= \frac{\partial H}{\partial u}\dot{u} + \left(\frac{\partial H}{\partial x} + \dot{p}\right)F.
\end{aligned}
$$

Since on an optimal trajectory

$$\dot{p} = -\frac{\partial H}{\partial x} \quad \text{and} \quad \left.\frac{\partial H}{\partial u}\right|_{u=u^*} = 0$$

it follows that $\dot{H} = 0$ when $u = u^*$, so that

$$H_{u=u^*} = \text{constant}, \qquad t_0 \le t \le t_1.$$

Discussion

We have so far assumed that t_1 is *fixed* and $x(t_1)$ is *free*. If this is not necessary the case, then we obtain

$$\delta \mathcal{J}_a = \left[\left(\frac{\partial \varphi}{\partial x} - p\right)\delta x + \left(H + \frac{\partial \varphi}{\partial t}\right)\delta t\right]_{\substack{u=u^* \\ t=t_1}} + \int_{t_0}^{t_1}\left(\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u + \dot{p}\,\delta x\right)dt.$$

The expression outside the integral must be zero (by virtue of PROPOSITION 5.2.1), making the integral zero. The implications of this for some important special cases are now listed. The initial condition $x(t_0) = x_0$ holds throughout.

A | Final time t_1 specified.

(i) $x(t_1)$ free

We have $\delta t|_{t=t_1} = 0$ but $\delta x|_{t=t_1}$ is arbitrary, so the condition

$$p(t_1) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1}$$

must hold (with $H_{u=u^*} = \text{constant}$, $t_0 \le t \le t_1$ when appropriate), as before.

(ii) $x(t_1)$ specified

In this case $\delta t|_{t=t_1} = 0$ and $\delta x|_{t=t_1} = 0$ so

$$\left[\left(\frac{\partial \varphi}{\partial x} - p\right)\delta x + \left(H + \frac{\partial \varphi}{\partial t}\right)\delta t\right]_{\substack{u=u^* \\ t=t_1}}$$

is automatically zero. The condition is thus

$$x^*(t_1) = x_f$$

(and this replaces $p(t_1) = \left.\dfrac{\partial \varphi}{\partial x}\right|_{t=t_1}$).

B | Final time t_1 free.

(iii) $x(t_1)$ free

Both $\delta t|_{t=t_1}$ and $\delta x|_{t=t_1}$ are now arbitrary so for the expression

$$\left[\left(\frac{\partial \varphi}{\partial x} - p\right)\delta x + \left(H + \frac{\partial \varphi}{\partial t}\right)\delta t\right]_{\substack{u=u^* \\ t=t_1}}$$

to vanish, the conditions

$$p(t_1) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1} \quad \text{and} \quad \left.\left(H + \frac{\partial \varphi}{\partial t}\right)\right|_{\substack{u=u^* \\ t=t_1}} = 0$$

must hold.

NOTE :   In particular, if φ, L, and F do not explicitly depend upon t, then

$$H_{u=u^*} = 0, \qquad t_0 \leq t \leq t_1.$$

(iv) $x(t_1)$ specified

Only $\left.\delta t\right|_{t=t_1}$ is now arbitrary, so the conditions are

$$x^*(t_1) = x_f \quad \text{and} \quad \left.\left(H + \frac{\partial \varphi}{\partial t}\right)\right|_{\substack{u=u^* \\ t=t_1}} = 0.$$

5.2.4 EXAMPLE.   A particle of unit mass moves along the x-axis subject to a force $u(\cdot)$. It is required to determine the control which transfers the particle from rest at the origin to rest at $x = 1$ in unit time, so as to minimize the effort involved, measured by

$$\mathcal{J} := \int_0^1 u^2 \, dt.$$

SOLUTION :   The equation of motion is

$$\ddot{x} = u$$

and taking $x_1 := x$ and $x_2 := \dot{x}$ we obtain the *state equations*

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u.$$

We have

$$H = L + pF = p_1 x_2 + p_2 u + u^2.$$

From

$$\left.\frac{\partial H}{\partial u}\right|_{u=u^*} = 0$$

the *optimal control* is given by

$$2u^* + p_2^* = 0$$

and the *adjoint equations* are

$$\dot{p}_1^* = 0, \quad \dot{p}_2^* = -p_1^*.$$

Integration gives

$$p_2^* = C_1 t + C_2$$

and thus

$$\dot{x}_2^* = -\frac{1}{2}(C_1 t + C_2)$$

which on integrating, and using the given conditions $x_2(0) = 0 = x_2(1)$, produces

$$x_2^*(t) = \frac{1}{2}C_2\left(t^2 - t\right), \quad C_1 = -2C_2.$$

Finally, integrating the equation $\dot{x}_1 = x_2$ and using $x_1(0) = 0, \quad x_1(1) = 1$ gives

$$x_1^*(t) = \frac{1}{2}t^2(3 - 2t), \quad C_2 = -12.$$

Hence the *optimal control* is

$$u^*(t) = 6(1 - 2t).$$

An interesting case

If the state at final time t_1 (assumed fixed) is to lie on a "surface" S (more precisely, an $(m - k)$-*submanifold* of $\mathbb{R}^m$) defined by

$$
\begin{aligned}
g_1(x_1, x_2 \ldots, x_m) &= 0 \\
g_2(x_1, x_2, \ldots, x_m) &= 0 \\
&\vdots \\
g_k(x_1, x_2, \ldots, x_m) &= 0
\end{aligned}
$$

(i.e. $S = g^{-1}(0) \subset \mathbb{R}^m$, where $g = (g_1, \ldots, g_k) : \mathbb{R}^m \to \mathbb{R}^k$, $m \geq k$ is such that $\operatorname{rank} \frac{\partial g}{\partial x} = k$), then (it can be shown that) in addition to the k conditions

$$g_1\left(x^*(t_1)\right) = 0, \ \ldots, \ g_k(x^*(t_1)) = 0 \tag{5.18}$$

there are a further m conditions which can be written as

$$\frac{\partial \varphi}{\partial x} - p = d_1 \frac{\partial g_1}{\partial x} + d_2 \frac{\partial g_2}{\partial x} + \cdots + d_k \frac{\partial g_k}{\partial x} \tag{5.19}$$

both sides being evaluated at $t = t_1$, $u = u^*$, $x = x^*$, $p = p^*$. The d_i are constants to be determined. Together with the $2m$ constants of integration there are thus $2m + k$ unknowns and $2m + k$ conditions (5.18), (5.19), and $x(t_0) = x_0$. If t_1 is free, then in addition

$$\left(H + \frac{\partial \varphi}{\partial t} \right)\Bigg|_{\substack{u=u^* \\ t=t_1}} = 0$$

holds.

5.2.5 EXAMPLE.   A system is described by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + u$$

is to be transformed (steered) from $x(0) = 0$ to the line $\mathcal{L}$ with equation

$$ax_1 + bx_2 = c$$

at time T so as to minimize

$$\int_0^T u^2 \, dt.$$

The values of $a, b, c,$ and T are given.

From

$$H = u^2 + p_1 x_2 - p_2 x_2 + p_2 u$$

we get

$$u^* = -\frac{1}{2} p_2^*. \tag{5.20}$$

The *adjoint equations* are

$$\dot{p}_1^* = 0, \quad \dot{p}_2^* = -p_1^* + p_2^*$$

so that

$$p_1^* = c_1, \quad p_2^* = c_2 e^t + c_1 \tag{5.21}$$

where c_1 and c_2 are constants. We obtain

$$x_1^* = c_3 e^{-t} - \frac{1}{4}c_2 e^t - \frac{1}{2}c_1 t + c_4, \quad x_2^* = -c_3 e^{-t} - \frac{1}{4}c_2 e^t - \frac{1}{2}c_1$$

and the conditions

$$x_1^*(0) = 0, \quad x_2^*(0) = 0, \quad ax_1^*(T) + bx_2^*(T) = c \tag{5.22}$$

must hold.

It is easy to verify that (5.19) produces

$$\frac{p_1^*(T)}{p_2^*(T)} = \frac{a}{b} \tag{5.23}$$

and (5.22) and (5.23) give four equations for the four unknown constants c_i. The optimal control $u^*(\cdot)$ is then obtained from (5.20) and (5.21).

NOTE :   In some problems the restriction on the total amount of control effort which can be expended to carry out a required task may be expressed in the form

$$\int_{t_0}^{t_1} L_0(t, x, u)\, dt = c \tag{5.24}$$

where c is a given constant, such a constraint being termed **isoperimetric**. A convenient way of dealing with (5.24) is to define a new variable

$$x_{m+1}(t) := \int_{t_0}^t L_0(t, x, u)\, d\tau$$

so that

$$\dot{x}_{m+1} = L_0(t, x, u).$$

This ODE is simply added to the original one (5.1) together with the conditions

$$x_{m+1}(t_0) = 0, \quad x_{m+1}(t_1) = c$$

and the previous procedure continues as before, ignoring (5.24).

5.3   Pontryagin's Principle

In real-life problems the control variables are usually subject to constraints on their magnitudes, typically of the form

$$|u_i(t)| \leq K_i, \qquad i = 1, 2, \ldots, \ell.$$

This implies that *the set of final states which can be achieved is restricted.*

Our aim here is to derive the necessary conditions for optimality corresponding to THEOREM 5.2.2 for the unbounded case.

An **admissible control** is one which satisfies the constraints, and we consider variations such that

- $u^* + \delta u$ is admissible

- $\|\delta u\|$ is sufficiently small so that the sign of

$$\Delta \mathcal{J} = \mathcal{J}\left[u^* + \delta u\right] - \mathcal{J}\left[u^*\right]$$

where

$$\mathcal{J}\left[u\right] = \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} L(t, x, u)\, dt$$

is determined by $\delta \mathcal{J}$ in

$$\mathcal{J}\left[u + \delta u\right] - \mathcal{J}\left[u\right] = \delta \mathcal{J}\left[u, \delta u\right] + j(u, \delta u) \cdot \|\delta u\|.$$

Because of the restriction on δu, PROPOSITION 5.2.1 no longer applies, and instead a necessary condition for u^* to minimize $\mathcal{J}$ is

$$\delta \mathcal{J}\left[u^*, \delta u\right] \geq 0.$$

The development then proceeds as in the previous section; Lagrange multipliers $p = \begin{bmatrix} p_1 & p_2 & \ldots & p_m \end{bmatrix}$ are introduced to define $\mathcal{J}_a$ and are chosen so as to satisfy

$$\dot{p} = -\frac{\partial H}{\partial x} \quad \text{and} \quad p(t_1) = \left.\frac{\partial \varphi}{\partial x}\right|_{t=t_1}.$$

The only difference is that the expression for $\delta \mathcal{J}_a$ becomes

$$\delta \mathcal{J}_a \left[u, \delta u \right] = \int_{t_0}^{t_1} \left(H(t, p, x, u + \delta u) - H(t, p, x, u) \right) \, dt.$$

It therefore follows that a necessary condition for $u = u^*$ to be a minimizing control is that

$$\delta \mathcal{J}_a \left[u^*, \delta u \right] \geq 0$$

for all admissible δu. This in turn implies that

$$\boxed{H(t, p^*, x^*, u^* + \delta u) \geq H(t, p^*, x^*, u^*)} \qquad (5.25)$$

for all admissible δu and all t in $[t_0, t_1]$. This states that u^* *minimizes* H, so we have "established"

5.3.1 THEOREM.   (PONTRYAGIN'S MINIMUM PRINCIPLE) *Necessary conditions for* u^* *to minimize*

$$\mathcal{J} \left[u \right] = \varphi(x(t_1), t_1) + \int_{t_0}^{t_1} L(t, x, u) \, dt$$

are the following :

$$\dot{p} \;\; = \;\; -\frac{\partial H}{\partial x}$$

$$p(t_1) \;\; = \;\; \left. \frac{\partial \varphi}{\partial x} \right|_{t=t_1}$$

$$H(t, p^*, x^*, u^* + \delta u) \;\; \geq \;\; H(t, p^*, x^*, u^*) \;\; \textit{for all admissible } \delta u, \; t_0 \leq t \leq t_1.$$

NOTE :   (1)   With a slighty different definition of H, the principle becomes one of maximizing J, and is then referred to as the **Pontryagin's Maximum Principle**.

(2) $u^*(\cdot)$ is now allowed to be *piecewise continuous*. (A rigorous proof is beyond the scope of this course.)

(3)   Our derivation assumed that t_1 was fixed and $x(t_1)$ free; the boundary conditions for other situations are *precisely the same* as those given in the preceding section.

5.3.2 EXAMPLE.   Consider again the *"soft landing"* problem (cf. EXAMPLE 5.1.2), where the performance index

$$\mathcal{J} = \int_0^T \left(|u| + k \right) dt$$

is to be *minimized* subject to

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u.$$

The *Hamiltonian* is

$$H = |u| + k + p_1 x_2 + p_2 u.$$

Since the admissible range of control is $-1 \leq u(t) \leq 1$, it follows that H *will be minimized* by the following :

$$u^*(t) = \begin{cases} -1 & \text{if } 1 < p_2^*(t) \\[2mm] 0 & \text{if } -1 < p_2^*(t) < 1 \\[2mm] +1 & \text{if } p_2^* < -1. \end{cases} \tag{5.26}$$

NOTE :   (1)   Such a control is referred to by the graphic term **bang-zero-bang**, since only maximum thrust is applied in a forward or reverse direction; no intermediate nonzero values are used. If there is no period in which u^* is zero, the control is called **bang-bang**. For example, a racing-car driver approximates to bang-bang operation, since he tends to use either full throttle or maximum braking when attempting to circuit a track as quickly as possible.

(2)   In (5.26) $u^*(\cdot)$ *switches* in value according to the value of $p_2^*(\cdot)$, which is therefore termed (in this example) the **switching function**.

The *adjoint equations* are

$$\dot{p}_1^* = 0, \quad \dot{p}_2^* = -p_1^*$$

and integrating these gives

$$p_1^*(t) = c_1, \quad p_2^*(t) = -c_1 t + c_2$$

where c_1 and c_2 are constants. Since p_2^* is linear in t, it follows that it can take each of the values $+1$ and -1 at most once in $[0, T]$, so $u^*(\cdot)$ can switch at most twice. *We must however use physical considerations to determine an actual optimal control.*

Since the landing vehicle begins with a downwards velocity at altitude h, logical sequences of control would seem to either

$$u^* = 0, \quad \text{followed by} \quad u^* = +1$$

(upwards is regarded as positive), or

$$u^* = -1, \text{then} \quad u^* = 0, \text{then} \quad u^* = +1.$$

Consider the first possibility and suppose that u^* switches from 0 to $+1$ in time t_1. By virtue of (5.26) this sequence of control is possible if p_2^* decreases with time. It is easy to verify (exercise !) that the solution of

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u$$

subject to the initial conditions

$$x_1(0) = h, \quad x_2(0) = -v$$

is

$$x_1^* = \begin{cases} h - vt & \text{if } 0 \leq t \leq t_1 \\ \\ h - vt + \frac{1}{2}(t - t_1)^2 & \text{if } t_1 \leq t \leq T \end{cases} \tag{5.27}$$

$$x_2^* = \begin{cases} -v & \text{if } 0 \leq t \leq t_1 \\ \\ -v + (t - t_1) & \text{if } t_1 \leq t \leq T. \end{cases} \tag{5.28}$$

Substituting the soft landing requirements

$$x_1(T) = 0, \quad x_2(T) = 0$$

into (27) and (28) gives

$$T = \frac{h}{v} + \frac{1}{2}v, \quad t_1 = \frac{h}{v} - \frac{1}{2}v.$$

Because the final time is not specified and because of the form of H equation $H_{u=u^*} = 0$ holds, so in particular $H_{u=u^*} = 0$ at $t = 0$; that is,

$$k + p_1^*(0)x_2^*(0) = 0$$

or

$$p_1^*(0) = \frac{k}{v}.$$

Hence we have

$$p_1^*(t) = \frac{k}{v}, \quad t \geq 0$$

and

$$p_2^*(t) = -\frac{kt}{v} - 1 + \frac{kt_1}{v}$$

using the assumption that $p_2^*(t_1) = -1$. Thus the assumed optimal control will be valid if $t_1 > 0$ and $p_2^*(0) < 1$ (the latter conditions being necessary since $u^* = 0$), and these conditions imply that

$$h > \frac{1}{2}v^2, \quad k < \frac{2v^2}{h - \frac{1}{2}v^2}. \tag{5.29}$$

NOTE :   If these inequalities do not hold, then some different control strategy (such as $u^* = -1$, then $u^* = 0$, then $u^* = +1$), becomes optimal. For example, if k is increased so that the second inequality in (5.29) is violated, then this means that more emphasis is placed on the time to landing in the performance index. It is therefore reasonable to expect this time would be reduced by first accelerating downwards with $u^* = -1$ before coasting with $u^* = 0$.

A general regulator problem

We can now discuss a general *linear regulator problem* in the usual form

$$\dot{x} = Ax + Bu \tag{5.30}$$

where $x(\cdot)$ is the *deviation* from the desired constant state. The aim is to transfer the system from some initial state to the origin in minimum time, subject to

$$|u_i(t)| \leq K_i, \qquad i = 1, 2, \ldots, \ell.$$

The Hamiltonian is

$$
\begin{aligned}
H &= 1 + p\,(Ax + Bu) \\
&= 1 + pAx + \begin{bmatrix} pb_1 & pb_2 & \ldots & pb_\ell \end{bmatrix} u \\
&= 1 + pAx + \sum_{i=1}^{\ell} (p\,b_i) u_i
\end{aligned}
$$

where the b_i are the columns of B. Application of (PMP) (cf. THEOREM 5.3.1) gives the necessary conditions for optimality that

$$u_i^*(t) = -K_i \operatorname{sgn}(s_i(t)), \qquad i = 1, 2, \ldots, \ell$$

where

$$s_i(t) := p^*(t) b_i \tag{5.31}$$

is the *switching function* for the ith variable. The *adjoint* equation is

$$\dot{p}^* = -\frac{\partial}{\partial x}\,(p^* Ax)$$

or

$$\dot{p}^* = -p^* A.$$

The solution of this ODE can be written in the form

$$p^*(t) = p(0) \exp(-tA)$$

so the switching function becomes

$$s_i(t) = p(0) \exp(-tA) b_i.$$

If $s_i(t) \equiv 0$ in some time interval, then $u_i^*(t)$ is *indeterminate* in this interval. We now therefore investigate whether the expression in (5.31) can vanish.

Firstly, we can assume that $b_i \neq 0$. Next, since the final time is free, the condition $H_{u=u^*} = 0$ holds, which gives (for all t)

$$1 + p^* \left(Ax^* + Bu^* \right) = 0$$

so clearly $p^*(t)$ cannot be zero for any value of t. Finally, if the product $p^* b_i$ is zero, then $s_i = 0$ implies that

$$\dot{s}_i(t) = -p^*(t) A b_i = 0$$

and similarly for higher derivatives of s_i. This leads to

$$p^*(t) \begin{bmatrix} b_i & Ab_i & A^2 b_i & \dots & A^{m-1} b_i \end{bmatrix} = 0. \tag{5.32}$$

If the system (5.30) is c.c. by the ith input acting alone (i.e. $u_j \equiv 0$, $j \neq i$), then by THEOREM 3.1.3 the matrix in (5.32) is nonsingular, and equation (5.32) then has only the trivial solution $p^* = 0$. However, we have already ruled out this possibility, so s_i cannot be zero. Thus *provided the controllability condition holds, there is no time interval in which u_i^* is indeterminate.*

The *optimal control* for the ith variable then has the *bang-bang form*

$$u_i^* = \pm K_i.$$

5.4   Linear Regulators with Quadratic Costs

A general closed form solution of the optimal control problem is possible for a linear regulator with quadratic performance index. Specifically, consider the time-varying system

$$\dot{x} = A(t)x + B(t)u \tag{5.33}$$

with a criterion (obtained by combining together (5.3) and (5.7)) :

$$\mathcal{J} := \frac{1}{2} x^T(t_1) M x(t_1) + \frac{1}{2} \int_0^{t_1} \left(x^T Q(t)x + u^T R(t)u \right) dt \tag{5.34}$$

with $R(t)$ positive definite and M and $Q(t)$ positive semi-definite symmetric matrices for $t \geq 0$ (the factors $\frac{1}{2}$ enter only for convenience).

NOTE :   The quadratic term in u in (5.34) ensures that the total amount of control effort is restricted, so that *the control variables can be assumed unbounded.*

The Hamiltonian is

$$H = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u + p(Ax + Bu)$$

and the necessary condition (5.17) for optimality gives

$$\frac{\partial}{\partial u}\left(\frac{1}{2}(u^*)^T R u^* + p^* B u^*\right) = (Ru^*)^T + p^* B = 0$$

so that

$$u^* = -R^{-1}B^T (p^*)^T \qquad (5.35)$$

$R(t)$ being nonsingular (since it is positive definite). The *adjoint equation* is

$$(\dot{p}^*)^T = -Qx^* - A^T (p^*)^T. \qquad (5.36)$$

Substituting (5.35) into (5.33) gives

$$\dot{x}^* = Ax^* - BR^{-1}B^T (p^*)^T$$

and combining this equation with (5.36) produces the system of $2m$ linear ODEs

$$\frac{d}{dt}\begin{bmatrix} x^* \\ (p^*)^T \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & -A^T(t) \end{bmatrix}\begin{bmatrix} x^* \\ (p^*)^T \end{bmatrix}. \qquad (5.37)$$

Since $x(t_1)$ is not specified, the *boundary condition* is

$$(p^*)^T(t_1) = Mx^*(t_1). \qquad (5.38)$$

It is convenient to express the solution of (5.37) as follows :

$$\begin{bmatrix} x^* \\ (p^*)^T \end{bmatrix} = \Phi(t, t_1)\begin{bmatrix} x^*(t_1) \\ (p^*)^T(t_1) \end{bmatrix}$$

$$= \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{bmatrix}\begin{bmatrix} x^*(t_1) \\ (p^*)^T(t_1) \end{bmatrix}$$

where Φ is the transition matrix for (5.37). Hence

$$
\begin{aligned}
x^* &= \phi_1 x^*(t_1) + \phi_2 (p^*)^T(t_1) \\
&= (\phi_1 + \phi_2 M) x^*(t_1).
\end{aligned}
$$

Also we get

$$
\begin{aligned}
(p^*)^T &= (\phi_3 + \phi_4 M) x^*(t_1) \\
&= (\phi_3 + \phi_4 M)(\phi_1 + \phi_2 M)^{-1} x^*(t) \\
&= P(t) x^*(t).
\end{aligned}
$$

(It can be shown that $\phi_1 + \phi_2 M$ is nonsingular for all $t \geq 0$). It now follows that the *optimal control is of linear feedback form*

$$
u^*(t) = -R^{-1}(t) B^T(t) P(t) x^*(t). \tag{5.39}
$$

To determine the matrix $P(t)$, differentiating $(p^*)^T = P x^*$ gives

$$
\dot{P} x^* + P \dot{x}^* - (\dot{p}^*)^T = 0
$$

and substituting for $\dot{x}^*$, $(\dot{p}^*)^T$ (from (5.37)) and $(p^*)^T$, produces

$$
\left(\dot{P} + PA - PBR^{-1}B^T P + Q + A^T P \right) x^*(t) = 0.
$$

Since this must hold throughout $0 \leq t \leq t_1$ it follows that $P(t)$ satisfies

$$
\boxed{\dot{P} = PBR^{-1}B^T P - A^T P - PA - Q} \tag{5.40}
$$

with boundary condition

$$
P(t_1) = M.
$$

Equation (5.40) is often referred to as a **matrix Riccati differential equation**.

NOTE : (1) Since the matrix M is symmetric, it follows that $P(t)$ is symetric for all t, so the (vector) ODE (5.40) represents $\frac{m(m+1)}{2}$ scalar first order (quadratic) ODEs, which can be integrated numerically.

(2) Even when the matrices $A, B, Q,$ and R are all time-invariant the solution $P(t)$ of (5.40), and hence the feedback matrix in (5.39), will in general still be time-varying.

However, of particular interest is the case when in addition the *final time* t_1 tends to infinity. Then there is no need to include the terminal expression in the performance index since the aim is to make $x(t_1) \to 0$ as $t_1 \to \infty$, so we set $M = 0$. Let Q_1 be a matrix having the same rank as Q and such that $Q = Q_1^T Q_1$. It can be shown that *the solution $P(t)$ of (5.40) does become a constant matrix P, and we have :*

5.4.1 PROPOSITION. *If the linear time-invariant control system*

$$\dot{x} = Ax + Bu(t)$$

is c.c. and the pair (A, Q_1) is c.o., then the control which minimizes

$$\int_0^\infty \left(x^T Q x + u^T R u \right) \, dt \tag{5.41}$$

is given by

$$u^*(t) = -R^{-1} B^T P x(t) \tag{5.42}$$

where P is the unique positive definite symmetric matrix which satisfies the so-called **algebraic Riccati equation**

$$\boxed{PBR^{-1}B^T P - A^T P - PA - Q = 0.} \tag{5.43}$$

NOTE : Equation (5.43) represents $\frac{m(m+1)}{2}$ quadratic algebraic equations for the unknown elements (entries) of P, so the solution will not in general be unique. However, it can be shown that *if a positive definite solution of (5.43) exists, then there is only one such solution.*

Interpretation

The matrix Q_1 can be interpreted by defining an *output vector* $y = Q_1 x$ and replacing the quadratic term involving the state in (5.41) by

$$y^T y \left(= x^T Q_1^T Q_1 x \right).$$

The *closed loop system* obtained by substituting (5.42) into (5.33) is

$$\dot{x} = \mathbf{A}x \qquad (5.44)$$

where $\mathbf{A} := A - BR^{-1}B^T P$. It is easy to verify that

$$\mathbf{A}^T P + P\mathbf{A} = A^T P + PA - 2PBR^{-1}B^T P = -PBR^{-1}B^T P - Q \qquad (5.45)$$

using the fact that it is a solution of (5.43). Since R^{-1} is positive definite and Q is positive semi-definite, the matrix on the RHS in (5.45) is negative semi-definite, so PROPOSITION 4.3.10 is *not* directly applicable, unless Q is actually positive definite.

It can be shown that *if the triplet (A, B, Q_1) is neither c.c. nor c.o. but is stabilizable and detectable, then the algebraic Riccati equation (5.43) has a unique solution, and the closed loop system (5.44) is asymptotically stable.*

NOTE :   Thus a solution of the algebraic Riccati equation leads to a *stabilizing linear feedback control* (5.42) irrespective of whether or not the open loop system is stable. (This provides an alternative to the methods of section **3.3**.)

If $x^*(\cdot)$ is the solution of the closed loop system (5.44), then (as in (5.10)) equation (5.45) implies

$$\begin{aligned}
\frac{d}{dt}\left((x^*)^T P x^*\right) &= -(x^*)^T \left(PBR^{-1}B^T P + Q\right) x^* \\
&= -(u^*)^T R u^* - (x^*)^T Q x^*.
\end{aligned}$$

Since $\mathbf{A}$ is a stability matrix, we can integrate both sides of this equality with respect to t (from 0 to ∞) to obtain the *minimum value* of (5.41) :

$$\int_0^\infty \left((x^*)^T Q x^* + (u^*)^T R u^*\right) \, dt = x_0^T P x_0. \qquad (5.46)$$

NOTE :   When $B \equiv 0$, (5.43) and (5.46) reduce simply to

$$A^T P + PA = -Q$$

and

$$\mathcal{J}_0 = \int_0^\infty x^T Q x \, dt = x_0^T P x_0$$

respectively.

5.5   Exercises

Exercise 93 A system is described by

$$\dot{x} = -2x + u$$

and the control $u(\cdot)$ is to be chosen so as to *minimize* the performance index

$$\mathcal{J} = \int_0^1 u^2 \, dt.$$

Show that the optimal control which transfers the system from $x(0) = 1$ to $x(1) = 0$ is

$$u^*(t) = -\frac{4e^{2t}}{e^4 - 1} \cdot$$

Exercise 94 A system is described by

$$\ddot{z} = u(t)$$

where $z(\cdot)$ denotes *displacement.* Starting from some given initial position with given velocity and acceleration it is required to choose $u(\cdot)$ which is constrained by $|u(t)| \le k$, so as to make displacement, velocity, and acceleration equal to zero in the least possible time. Show using (PMP) that the optimal control consists of

$$u^* = \pm \, k$$

with zero, one, or two switchings.

Exercise 95 A linear system is described by

$$\ddot{z} + a\dot{z} + bz = u$$

where $a > 0$ and $a^2 < 4b$. The control variable is subject to $|u(t)| \le k$ and is to be chosen so that the system reaches the state $z(T) = 0$, $\dot{z}(T) = 0$ in *minimum possible time.* Show that the optimal control is

$$u^*(t) = k \, \text{sgn} \, p(t)$$

where $p(\cdot)$ is a periodic function.

Exercise 96 A system is described by

$$\dot{x} = -2x + 2u, \qquad x \in \mathbb{R}.$$

The unconstrained control variable $u(\cdot)$ is to be chosen so as to *minimize* the performance index

$$\mathcal{J} = \int_0^1 \left(3x^2 + u^2\right) dt$$

whilst transferring the system from $x(0) = 0$ to $x(1) = 1$. Show that the optimal control is

$$u^*(t) = \frac{3e^{4t} + e^{-4t}}{e^4 - e^{-4}} \cdot$$

Exercise 97 A system is described by the equations

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_1 - 2x_2 + u$$

and is to be transferred to the origin from some given initial state.

(a) If the control $u(\cdot)$ is unbounded, and is to be chosen so that

$$\mathcal{J} = \int_0^T u^2 \, dt$$

is *minimized*, where T is fixed, show that the optimal control has the form

$$u^*(t) = c_1 e^t \sinh(t\sqrt{2} + c_2)$$

where c_1 and c_2 are certain constants. (**DO NOT** try to determine their values.)

(b) If $u(\cdot)$ is such that $|u(t)| \leq k$, where k is a constant, and the system is to be brought to the origin in the *shortest possible time*, show that the optimal control is *bang-bang*, with at most one switch.

Exercise 98 For the system described by

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 + u$$

determine the control which transfers it from $x(0) = 0$ to the line $\mathcal{L}$ with equation

$$x_1 + 5x_2 = 15$$

and *minimizes* the performance index

$$\mathcal{J} = \frac{1}{2}\left(x_1(2) - 5\right)^2 + \frac{1}{2}\left(x_2(2) - 2\right)^2 + \frac{1}{2}\int_0^2 u^2 \, dt.$$

Exercise 99 Use PROPOSITION 5.4.1 to find the feedback control which *minimizes*

$$\int_0^\infty \left(x_2^2 + \frac{1}{10} u^2 \right) dt$$

subject to

$$\dot{x}_1 = -x_1 + u, \quad \dot{x}_2 = x_1.$$

Exercise 100

(a) Use the *Riccati equation* formulation to determine the feedback control for the system

$$\dot{x} = -x + u, \qquad x \in \mathbb{R}$$

which *minimizes*

$$\mathcal{J} = \frac{1}{2} \int_0^1 \left(3x^2 + u^2 \right) dt.$$

[Hint : In the Riccati equation for the problem put $P(t) = -\dfrac{\dot{w}(t)}{w(t)} \cdot$]

(b) If the system is to be transferred to the origin from an arbitrary initial state with the same performance index, use the calculus of variations to determine the optimal control.