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Lecture Notes CCR

AM3.2 - Linear Control

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The most practical solution is a good theory.

Albert Einstein

Once upon a time, it is said, in the Good Old Days (read nineteenth century), there was only Mathematics, a subject intimately bound up with the ways of Mother Nature. Today this subject has fragmented into two ideological blocs, labelled Pure and Applied. Each views the other with an element of distrust [. . .] Each side has the conviction of the True Believer in its own moral superiority. The Applied Mathematicians accuse the Pure of a lack of contact with reality; the Pure Mathematicians feel that Applied are altogether too slapdash and have never quite grasped the rules of the game [. . .] There is without doubt a great difference in attitudes between those who called themselves pure and those who called themselves applied. Halmos reckons that the main difference is that the applied mathematicians are convinced there is no difference, whereas the pure mathematicians know perfectly well there is. A more salient difference is one of intention. ***The applied mathematician wants an answer; the pure mathematician wants to understand the problem.*** *The pure mathematician observes that sometimes the applied one is so keen to answer that he doesn't worry much whether it's the right answer. The applied mathematician observes that when the pure one can't understand a problem he moves on to another one and tries again. Perhaps the true difference is that applied mathematicians devote a lot of thought to the modelling process – devising an effective mathematical model of a natural phenomenon – whereas this step is largely absent from pure mathematicians.*

Ian Stewart

Scientists use mathematics to build mental universes. They write down mathematical descriptions – models – that capture essential fragments of how they think this world behaves. Then they analyse their consequences. This is called "theory". They test their theories against observations : this is called "experiment". Depending on the result, they may modify the mathematical model and repeat the cycle until theory and experiment agree. Not that it's really that simple; but that's the general gist of it, the essence of the scientific method.

I. Stewart and M. Golubitsky

Differential equations are the particular dialect of the language of mathematics that most effectively describes how nature works.

J.D. Barrow

What is Mathematical Control Theory ?

Mathematical control theory is the area of *application-oriented mathematics* that deals with the basic principles underlying the analysis and design of *control systems*. To *control* an object means to influence its behaviour so as to achieve a desired goal. In order to implement this influence, control engineers build devices that incorporate various mathematical techniques. These devices range from *Watt's steam engine governor*, designed during the English Industrial Revolution, to the sophisticated microprocessor controllers found in items – such as CD players and automobiles – or in industrial robots and airplane autopilots.

Control theory was originally developed to satisfy the design needs of *ser-*

vomechanisms, under the name of **"automatic control theory"**. The classical theory of automatic control mostly deals with *linear feedback* control systems with single input and single output. Mathematical structures of such systems must be, in principle, described in terms of linear ordinary differential equations (ODEs) with constant coefficients. Hence control engineers use *block diagrams* to describe systems, and operational calculus based on *Laplace transforms* to obtain response characteristics. Thus the input/output relation of a system is described in terms of *transfer functions.* One of the remarkable contributions to classical control theory is *Nyquist's criterion* (after its originator, Harry Nyquist (1889-1976)) for stability testing of linear feedback systems. The test consists of plotting the Nyquist diagram of a transfer function in the *frecquency domain* (complex plane), and differs essentially from the *Routh-Hurwitz stability test* for linear ODEs with constant coefficients.

Control theory became recognized as a *mathematical subject* in the 1960's. Around 1960 three remarkable contributions were made concurently; they are

- **dynamic programming** – Richard E. Bellman (1920-1984)
- **Pontryagin's principle** – Lev S. Pontryagin (1908-1988)
- **linear system theory** – Rudolf E. Kalman (1930).

The first two give rise to mathematical tools to solve *optimal control* problems and to design optimal controllers and regulators. In contrast to the classical theory of control, optimal control problems are formulated in terms of systems of linear *or* nonlinear multivariable ODEs with multiinput (control) variables. Linear system theory derives from the concepts of *controllability* and *observability.* These two concepts are concerned with the interaction between (internal) *states* of a system and its *inputs* and *outputs.*

R.E. Kalman challenged the accepted approach to control theory of that period (limited to the use of Laplace transforms and the frecquency domain) by showing that the basic control problems could be studied effectively through the notion of a *state* of the system that evolves in time according to ODEs in which controls appear as parameters. Aside from drawing attention to the mathematical content of control problems, Kalman's work served as a catalyst for further growth of the subject. Liberated from the confines of the frecquency domain and further inspired by the development of computers, automatic control theory became the subject matter of a new science called **systems theory**.

Systems theory grew out of a desire to merge *automata theory*, and *artificial intelligence*, and *discrete* and *continuous control* into a single subject concerned with input/output relations parametrized by the states of the system. The level of generality required to keep these subjects together was well beyond the realm of differential equations, and control theory quickly evolved into *topological dynamical systems*. Systems theory, itself a hybrid of control and automata theory, in its formative period looked to *abstract dynamical systems* and *mathematical logic* for its further growth.

Around 1970 the significance of the *Lie bracket* for problems of control became clear thanks to efforts made by R.M. Hermann, R.W. Brockett, C. Lobry, H.J. Sussmann, V. Jurdjevic, and others. As a result, *differential geometry* entered into an exciting partnership with control theory, marking the birth of **geometric control theory**.

Present day theoretical research in control theory involves a variety of areas of pure mathematics (e.g. *linear and multilinear algebra, Lie semigroups and Lie groups, algebraic geometry, dynamical systems, complex analysis, functional analysis, calculus of variations, topology, differential geometry, probability theory*, etc.). Concepts and results from these areas find applications in control theory; conversely, questions about control systems give rise to new open problems in mathematics.

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