2. Actions of Groups on Sets

Group actions • Orbits and stabilizers • Particular G-sets • Examples of group actions.

2.1. Group actions. A valuable technique in studying a group is to represent it in terms of something familiar and concrete: if the elements happen to be permutations or matrices, however, we may be able to obtain results by using this extra information.

NOTE: We began with (examples of) transformation groups, i.e., subgroups of the symmetric group $\mathfrak{S}_{\mathsf{M}}$ on a set M . This approach is consistent both with the historical path along which group theory developed and with the importance of transformation groups in other areas of mathematics. The so-called abstract theory of groups, which arose in a later era – the first half of the 20th century – has gone far beyond transformation groups, but many of the concepts of this theory bear the imprint of earlier times. In fact, the most common source of these concepts is the idea of a *realization* (or *representation*) of a given group G in $\mathfrak{S}_{\mathsf{M}}$, where M is some suitably chosen set.

Let G be a group and M a (non-empty) set.

DEFINITION 18. A (left) **action** of G on M is a function $\theta : \mathsf{G} \times \mathsf{M} \to \mathsf{M}$ such that (LA1)' $\theta(g_2, \theta(g_1, x)) = \theta(g_2g_1, x)$ for all $g_1, g_2 \in \mathsf{G}$ and $x \in \mathsf{M}$; (LA2)' $\theta(\mathbf{1}, x) = x$ for all $x \in \mathsf{M}$.

We write $g \cdot x$ in place of the pedantic notation $\theta(g, x)$. We can now write the above conditions (axioms) as follows:

(LA1) $g_2 \cdot (g_1 \cdot x) = (g_2 g_1) \cdot x$ for all $g_1, g_2 \in \mathsf{G}$ and $x \in \mathsf{M}$;

(LA2) $\mathbf{1} \cdot x = x$ for all $x \in \mathsf{M}$.

The set M is called a (left) G-set. One also says that the group G acts on the set M.

NOTE: In this definition, the elements of G act from the left. There is a "right" version of G-sets that is sometimes convenient. Define a **right action** of G on M to be a function $\tau : \mathsf{M} \times \mathsf{G} \to \mathsf{M}$, $(x, g) \mapsto x \cdot g$ such that

(RA1) $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$ for all $g_1, g_2 \in \mathsf{G}$ and $x \in \mathsf{M}$;

(RA2) $x \cdot \mathbf{1} = x$ for all $x \in \mathsf{M}$.

It is easy to see that every right action $\tau : \mathsf{M} \times \mathsf{G} \to \mathsf{M}$ gives rise to a left action $\theta : \mathsf{G} \times \mathsf{M} \to \mathsf{M}$ if one defines $\theta(g, x) := x \cdot g^{-1}$.

Any action of G on M induces an action of G on (the Cartesian product) $M^k = M \times \cdots \times M$ (k factors) by the obvious rule:

$$g \cdot (x_1, \ldots, x_k) := (g \cdot x_1, \ldots, g \cdot x_k).$$

 \diamond Exercise 19. Let G be a group and let $\theta_1 : G \times M_1 \to M_1$ and $\theta_2 : G \times M_2 \to M_2$ be actions of G on the sets M_1 and M_2 , respectively. Define

$$\mathsf{M}_1 + \mathsf{M}_2 := (\mathsf{M}_1 \times \{1\}) \cup (\mathsf{M}_2 \times \{2\})$$

and $\theta^{\vee}: \mathsf{G} \times (\mathsf{M}_1 + \mathsf{M}_2) \to \mathsf{M}_1 + \mathsf{M}_2$ by

$$(g, (x, i)) \mapsto (g \cdot x, i)$$

for $i = 1, 2, x \in M_i$ and $g \in G$. Show that θ^{\vee} is an action (of G on the sum $M_1 + M_2$).

 $\diamond \ \mathbf{Exercise} \ \mathbf{20.} \ \ \mathrm{Let} \ \mathsf{G} \ \mathrm{be} \ \mathrm{a} \ \mathrm{group} \ \mathrm{and} \ \mathrm{let} \ \theta_1: \mathsf{G} \times \mathsf{M}_1 \to \mathsf{M}_1 \ \mathrm{and} \ \theta_2: \mathsf{G} \times \mathsf{M}_2 \to \mathsf{M}_2 \ \mathrm{be} \ \mathrm{actions} \ \mathrm{of} \ \mathsf{G} \ \mathrm{on} \ \mathrm{the} \ \mathrm{sets} \ \mathsf{M}_1 \ \mathrm{and} \ \mathsf{M}_2, \ \mathrm{respectively}. \ \mathrm{Define} \ \theta^{\wedge}: \mathsf{G} \times (\mathsf{M}_1 \times \mathsf{M}_2) \to \mathsf{M}_1 \times \mathsf{M}_2 \ \mathrm{by}$

$$(g, (x_1, x_2)) \mapsto (g \cdot x_1, g \cdot x_2)$$

for $x_1 \in M_1, x_2 \in M_2$ and $g \in G$. Show that θ^{\wedge} is an action (of G on the *product* $M_1 \times M_2$).

There is also an induced action of G on the set of all subsets $\mathcal{P}(\mathsf{M})$ of M. We set $g \cdot \emptyset := \emptyset$ and, if S is a non-empty subset of M, then we set $g \cdot \mathsf{S} := \{g \cdot x : x \in \mathsf{S}\}.$

 \diamond Exercise 21. Let M and N be sets, and let θ be an action of the group G on the set M. Consider the set N^M of all N-valued functions defined on M. Show that the correspondence

$$(g,F)\mapsto g\cdot F:=F\circ\theta_{q^{-1}}$$

for $g \in G$ and $F : M \to N$, defines an action (of G on the function set N^{M}). (For $\mathsf{M} = \mathbb{R}$ this gives the induced action on the function set $\mathfrak{F}_{\mathsf{M}} = \mathbb{R}^{\mathsf{M}}$; for $\mathsf{N} = \{0, 1\}$ this gives the induced action on the the power set $\mathcal{P}(\mathsf{M}) = \{0, 1\}^{\mathsf{M}}$.)

 \diamond Exercise 22. Let L and M be sets, and let τ be an action of the group G on the set M. Consider the set M^L of all M-valued functions defined on L. Show that the correspondence

$$(g,C) \mapsto g \cdot C := \tau_g \circ C$$

for $g \in \mathsf{G}$ and $C : \mathsf{L} \to \mathsf{M}$, defines an action (of G on the function set M^{L}). (For $\mathsf{L} = \mathbb{R}$ this gives the induced action on the set $\mathfrak{C}_{\mathsf{M}} = \mathbb{M}^{\mathbb{R}}$ of M -valued parametrised curves.)

EXAMPLE 19. Let G be a subgroup of the symmetric group \mathfrak{S}_M on M: $G \leq \mathfrak{S}_M$ (G is a transformation group). Then the function

$$\mathsf{G} \times \mathsf{M} \ni (\alpha, x) \mapsto \alpha(x) \in \mathsf{M}$$

is an action of G on M; this is the most frequent case. For example, G can be defined as a subgroup of \mathfrak{S}_M satisfying certain conditions.

EXAMPLE 20. (The regular representation) Given a group G, we can make G into a G-set (i.e., take M to be G) by defining $g \cdot x$ to be the group product: the function

$$\mathsf{G} \times \mathsf{G} \ni (g, x) \mapsto gx \in \mathsf{G}$$

is an action of G on itself. The map $L_a : G \to G$, $g \mapsto ag$ is the left translation by a. Our action (by left translations) induces an action of G on the set of subsets of G. In particular, let $H \leq G$. It is clear that the function (denoted λ^{H})

$$\mathsf{G} \times \mathsf{G}/\mathsf{H} \ni (g, a \mathsf{H}) \mapsto g(a \mathsf{H}) := (ga) \mathsf{H}$$

is an action of G on the orbit set G/H. The corresponding homomorphism

$$\Phi^{\mathsf{H}}:\mathsf{G}\to\mathfrak{S}_{\mathsf{G}/\mathsf{H}},\quad g\mapsto\lambda_{a}^{\mathsf{H}}\;(:\mathsf{G}/\mathsf{H}\to\mathsf{G}/\mathsf{H})$$

is the so-called (left) **regular representation** of G. (Here λ_g^{H} takes the left coset $a \mathsf{H}$ to $(ga) \mathsf{H}$.)

 $\diamond \text{ Exercise 23.} \quad \text{Show that the map } \Phi^{\mathsf{H}}:\mathsf{G}\to\mathfrak{S}_{\mathsf{G}/\mathsf{H}}, \quad g\mapsto\lambda_g^{\mathsf{H}} \text{ is a homomorphism.}$

The regular representation of the group G by permutations of cosets of the subgroup H in G is much more efficient than the one obtained using CAYLEY'S THEOREM.

EXAMPLE 21. (The conjugation action) Another way to make G into a G-set is to use *conjugation*: the function

$$\mathsf{G} \times \mathsf{G} \ni (g, x) \mapsto g x g^{-1} \in \mathsf{G}$$

is also an action of G on itself. Clearly, $\mathbf{1}x = x$ and

$$g_2(g_1x) = g_2\left(g_1xg_1^{-1}\right)g_2^{-1} = (g_2g_1)x(g_2g_1)^{-1} = (g_2g_1)x(g_2g_1)^{-1}$$

for all $x \in G$. The conjugation action carries over to subsets and subgroups of G. Two subsets $S, T \subseteq G$ are **conjugate** if $T = g S g^{-1}$ for some $g \in G$. Let $H \leq G$. It is customary to call (the group) $N(H) := \{g \in G : g H g^{-1} = H\}$ the **normalizer** of H in G. (The subgroup H is normal in G precisely when N(H) = H.)

NOTE: The first mathematicians who studied group-theoretic problems (e.g., J.L. LA-GRANGE) were concerned with the question: What happens to the polynomial $f(X_1, \ldots, X_m)$ if one permutes the variables ? More precisely, if $\pi \in \mathfrak{S}_m$, define

$$\pi \cdot f(X_1,\ldots,X_m) := f(X_{\pi(1)},\ldots,X_{\pi(m)});$$

given $f \in \mathbb{R}[X_1, \ldots, X_m]$, how many distinct polynomials $\pi \cdot f$ are there? (Here $\mathbb{R}[X_1, \ldots, X_m]$ denotes the set – in fact, ring – of polynomials in m variables X_1, \ldots, X_m with real coefficients.)

If $\pi \cdot f = f$ for all $\pi \in \mathfrak{S}_m$, then (the polynomial) f is called a **symmetric function**. If a polynomial $f(X) = \sum_{i=0}^{m} a_i X^i \in \mathbb{R}[X]$ has roots r_1, \ldots, r_m , then each of the coefficients a_i of $f(X) = a_m \prod_{i=0}^{m} (X - r_i)$ is a symmetric function of r_1, \ldots, r_m . Other interesting functions of the roots may not be symmetric. For example, the **discriminant** of f(X) is defined to be

the number d^2 , where $d := \prod_{i < j} (r_i - r_j)$. If $D(X_1, \ldots, X_m) = \prod_{i < j} (X_i - X_j)$, then it is easy to see that $\pi \cdot D = \pm D$ for every $\pi \in \mathfrak{S}_m$. Indeed, D is an **alternating function** of the roots: $\pi \cdot D = D$ if and only if $\pi \in \mathfrak{A}_m$. This suggests a slight change in viewpoint. Given $f(X_1, \ldots, X_m)$, find $\mathcal{S}(f) := \{\pi \in \mathfrak{S}_m : \pi \cdot f = f\}$; this is precisely what LAGRANGE did. It is easy to see that $\mathcal{S}(f) \leq \mathfrak{S}_m$; moreover, f is symmetric if and only if $\mathcal{S}(f) = \mathfrak{S}_m$, while $\mathcal{S}(D) = \mathfrak{A}_m$.

Modern mathematicians are concerned with the same type of problem. If M is a G-set, then the set of all $\alpha : M \to M$ such that $\alpha(g \cdot x) = \alpha(x)$ for all $x \in M$ and all $g \in G$ is usually valuable in analyzing M.

EXAMPLE 22. Let k be a *field* (think of either \mathbb{R} or \mathbb{C}). The symmetric group \mathfrak{S}_m acts on the set $\mathsf{M} = \Bbbk[X_1, \ldots, X_m]$ by

$$\mathfrak{S}_m \times \mathsf{M} \ni (\pi, f) \mapsto \pi \cdot f$$

where $\pi \cdot f(X_1, ..., X_m) = f(X_{\pi(1)}, ..., X_{\pi(m)}).$

 \diamond Exercise 24. For any permutation $\pi \in \mathfrak{S}_m$ and $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, define

$$\pi \cdot x := (x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(m)}).$$

Show that $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$ for $\alpha, \beta \in \mathfrak{S}_m$ and $x \in \mathbb{R}^m$. (This means that the map $(\pi, x) \mapsto \pi \cdot x$ is an action of the symmetric group \mathfrak{S}_m on the set \mathbb{R}^m . What is the induced action on the function set $\mathfrak{F}_{\mathbb{R}^m}$? But on the ring of polynomial functions $\mathbb{R}[X_1, \ldots, X_m]$?)

Any homomorphism $\Phi: \mathsf{G} \to \mathfrak{S}_{\mathsf{M}}$ gives rise to an action θ of G on M defined by

$$\theta(g, x) = g \cdot x := \Phi(g)(x)$$

for all $g \in G$ and all $x \in M$. This really is an action because

$$g_2 \cdot (g_1 \cdot x) = \Phi(g_2) \left(\Phi(g_1)(x) \right) = \left(\Phi(g_2) \Phi(g_1) \right) (x) = \Phi(g_2 g_1)(x) = (g_2 g_1) \cdot x$$

for all $g_1, g_2 \in \mathsf{G}$ and all $x \in \mathsf{M}$, and

$$\mathbf{1} \cdot x = \Phi(\mathbf{1})(x) = \mathbf{1}_{\mathsf{M}}(x) = x$$

for all $x \in M$.

Conversely, suppose that θ is an action of G on M. For a fixed element $g \in G$ consider the mapping

$$\theta(g, \cdot) := \theta_g : \mathsf{M} \to \mathsf{M}, \quad x \mapsto g \cdot x,$$

This is invertible: it has an inverse, namely $\theta_{g^{-1}}$ because (for all $x \in \mathsf{M}$)

$$\theta_g \theta_{g^{-1}}(x) = \theta_g \left(\theta_{g^{-1}}(x) \right) = g \cdot (g^{-1} \cdot x) = (gg^{-1}) \cdot x = \mathbf{1} \cdot x = x$$

and similarly $\theta_{g^{-1}}\theta_g(x) = x$, which shows that

$$\theta_g \theta_{g^{-1}} = \theta_{g^{-1}} \theta_g = \mathbf{1}_{\mathsf{M}}$$

In this way each element of G acts as a permutation of M. Furthermore, the map

$$\Phi: \mathsf{G} \to \mathfrak{S}_\mathsf{M}, \quad q \mapsto \theta_q$$

is a homomorphism. Indeed, for all $x \in M$, we have

$$\Phi(g_2g_1)(x) = (g_2g_1) \cdot x = g_2 \cdot (g_1 \cdot x) = \Phi(g_2) \left(\Phi(g_1)(x)\right) = \left(\Phi(g_2)\Phi(g_1)\right)(x)$$

and so

$$\Phi(g_2g_1) = \Phi(g_2)\Phi(g_1).$$

We call a homomorphism $\Phi: G \to \mathfrak{S}_M$ a **permutation representation** of G on M or a *representation* of G as a group of transformations (permutations) of M. What we have just shown is that every such representation gives rise to an action of G on M and that, conversely, every action gives rise to a permutation representation.

To summarize, we have

PROPOSITION 15. There is a one-to-one correspondence between actions of the group G on the set M and the representations of G by permutations of M.

In view of this result we shall use the language of group actions and of permutation representations interchangeably.

NOTE: All this can be done with right actions, but a little care must be exercised. If $(x,g) \mapsto x \cdot g$ is the right action of G on M, the corresponding permutation representation of G on M is given by $g \mapsto \tau(\cdot, g^{-1})$. (Without this inverse we would not obtain a homomorphism from G to \mathfrak{S}_M but an "anti-homomorphism" or, if one prefers, a homomorphism from the *opposite* group G^{op} to \mathfrak{S}_{M} .)

One can use *right translations* $R_a : \mathsf{G} \to \mathsf{G}$, $g \mapsto ga$ to define natural actions of G on itself and on the set of *right* cosets (denoted by $\mathsf{G}\backslash\mathsf{H}$). The action ρ^{H} of G on the set $\mathsf{G}\backslash\mathsf{H}$ has a corresponding *right regular representation*

$$\mathsf{G} \to \mathfrak{S}_{\mathsf{G} \setminus \mathsf{H}}, \quad g \mapsto \rho_{q^{-1}}^{\mathsf{H}}.$$

A major theme of mathematical endeavour is to understand groups in terms of their actions. An interesting (and important) case is when the set M on which the group acts carries some extra structure—which will generally have a "geometric" flavour—and we will require the group action to respect this structure.

Consider the case when $\mathsf{M} = \mathbb{R}^m$, which is a vector space, and require G to act on \mathbb{R}^m by *linear* transformations. That is, we replace the symmetric group $\mathfrak{S}_{\mathbb{R}^m}$ with $\mathsf{GL}(\mathbb{R}^m)$, the group of invertible linear maps $\mathbb{R}^m \to \mathbb{R}^m$; this group is (isomorphic to) the general linear group $\mathsf{GL}(m,\mathbb{R})$. A homomorphism $\rho: \mathsf{G} \to \mathsf{GL}(m,\mathbb{R})$ is called a **linear representation** of G . To put it another way, a linear representation of G is a concrete realization of the group G as a collection of invertible matrices.

A faithful linear representation of a group G is an *embedding* of G into a matrix group; that is, the homomorphism $\rho : G \to GL(m, \mathbb{R})$ is injective: distinct elements of the group correspond to distinct matrices. In this case we refer to G as a linear group.

EXAMPLE 23. For every $m \in \mathbb{N}$, the symmetric group \mathfrak{S}_m can be embedded into $\mathsf{GL}(m,\mathbb{R})$. In order to embed \mathfrak{S}_m into (the general linear group) $\mathsf{GL}(m,\mathbb{R})$, we must find an injective homomorphism $\rho : \mathfrak{S}_m \to \mathsf{GL}(m,\mathbb{R})$; this involves assigning to each permutation $\pi \in \mathfrak{S}_m$ an invertible linear map $\rho_\pi : \mathbb{R}^m \to \mathbb{R}^m$.

Given an ordered basis $(e_i)_{1 \leq i \leq m}$ of \mathbb{R}^m , let $\rho_{\pi} : \mathbb{R}^m \to \mathbb{R}^m$ be the <u>unique</u> linear map that permutes these elements according to the permutation $\pi \in \mathfrak{S}_m$. That is, each π corresponds to a bijection $\pi : [m] \to [m]$; define now the action of ρ_{π} on the vectors e_i by

$$\rho_{\pi}(e_i) := e_{\pi(i)}, \quad i = 1, \dots, m.$$

Since the vectors e_i are linearly independent, ρ_{π} extends <u>uniquely</u> to a linear map on their span, which is \mathbb{R}^m : it sends the vector $x = x_1e_1 + \cdots + x_me_m \in \mathbb{R}^m$ to the vector

$$\rho_{\pi}(x) = x_{1}\rho_{\pi}(e_{1}) + \dots + x_{m}\rho_{\pi}(e_{m})$$

= $x_{1}e_{\pi(1)} + \dots + x_{m}e_{\pi(m)}$
= $x_{\pi^{-1}(1)}e_{1} + \dots + x_{\pi^{-1}(m)}e_{m} \in \mathbb{R}^{n}$

(The action $(\pi, x) \mapsto \pi \cdot x := \rho_{\pi}(x)$ of \mathfrak{S}_m on \mathbb{R}^m is given by $\pi \cdot (x_1, \ldots, x_m) = (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(m)})$.) The map $\rho : \pi \mapsto \rho_{\pi}$ is a homomorphism since

$$\rho_{\pi}\rho_{\sigma}(e_i) = \rho_{\pi}\left(e_{\sigma(i)}\right) = e_{\pi(\sigma(i))} = \rho_{\pi\sigma}(e_i)$$

(It is easy to see that this homomorphism is injective.)

Choosing the standard basis for the vector space \mathbb{R}^m , the linear transformations ρ_{π} are represented by *permutation matrices* $P = \Phi(\pi) \in \mathsf{GL}(m, \mathbb{R})$ (see EXAMPLE 15).

 \diamond Exercise 25. Verify that the natural homomorphism $\rho : \mathfrak{S}_m \to \mathsf{GL}(m,\mathbb{R})$ is injective.

One wants to think of two G-sets M and \overline{M} as being "essentially the same" if M can be identified with \overline{M} in such a way that the actions θ and $\overline{\theta}$ become the same. Formally, we say that M and \overline{M} are **equivalent** if there exist an automorphism $\phi \in Aut(G)$ and a bijection $\beta : M \to \overline{M}$ such that for all $g \in G$ the following diagram commutes:

$$\begin{array}{ccc} \mathsf{M} & \stackrel{\beta}{\longrightarrow} & \overline{\mathsf{M}} \\ \\ {}^{\theta_g} \! \downarrow & & \downarrow_{\overline{\theta}_{\phi(g)}} \\ \\ \mathsf{M} & \stackrel{\beta}{\longrightarrow} & \overline{\mathsf{M}} \end{array}$$

In other words, $\beta(g \cdot x) = \phi(g) \cdot \beta(x)$ for all $x \in M$ and all $g \in G$. (We say that β is ϕ -equivariant.)

♦ Exercise 26. Show that two G-sets M and \overline{M} are equivalent if and only if there exist an automorphism $\phi \in Aut(G)$ and a bijection $\beta : M \to \overline{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{G} \times \mathsf{M} & \stackrel{\phi \times \beta}{\longrightarrow} & \mathsf{G} \times \overline{\mathsf{M}} \\ \\ \theta \\ \downarrow & & & \downarrow \overline{\theta} \\ \mathsf{M} & \stackrel{\beta}{\longrightarrow} & \overline{\mathsf{M}} \end{array}$$

2.2. Orbits and stabilizers. There are two fundamental aspects of G-sets: orbits and stabilizers. Let M be a G-set (with respect to $\Phi : G \to \mathfrak{S}_M$). Two points $x, y \in M$ are said to be G-equivalent if $y = g \cdot x$ for some $g \in G$.

 \diamond Exercise 27. Verify that the G-equivalence relation is a genuine equivalence relation on M (which divides M into disjoint equivalence classes).

Each equivalence class is called a G-orbit. (Usually, we will say simply *orbit* instead of G-orbit.) The orbit containing $x \in M$ is denoted Orb(x) or Gx; thus

$$\mathsf{Orb}\,(x) = \mathsf{G}x := \{g \cdot x \, : \, g \in \mathsf{G}\}$$

NOTE: The notion of an orbit arose from geometry. For example, if G = SO(2) is the group of rotations of the (Euclidean) plane about the origin, then the orbit of a point P is the circle centered at the origin passing through P, and the set $M = \mathbb{R}^2$ is the union of all the concentric circles, including the one with zero radius (consisting of a single point, the origin).

Let $x \in M$. Consider the set

$$\mathsf{St}(x) = \mathsf{G}_x := \{g \in \mathsf{G} : g \cdot x = x\}.$$

It is called the **stabilizer** (or the **isotropy group**) of x.

♦ Exercise 28. Show that for any G-set M and any element $x \in M$, the stabilizer St (x) is a subgroup of G.

EXAMPLE 24. If G acts on itself by left translations and $x \in G$, then Orb(x) = G (there is only one orbit) and St(x) is the trivial group $\{1\}$.

EXAMPLE 25. If G acts on itself by conjugation and $x \in G$, then Orb(x) is the conjugacy class C(x) of x (i.e., the set of all group elements of the form gxg^{-1} as g varies over G), and

$$\mathsf{St}(x) = \left\{ g \in \mathsf{G} \, : \, gxg^{-1} = x \right\} = \left\{ g \in \mathsf{G} \, : \, gx = xg \right\}$$

(the *centralizer* Z(x) of x).

EXAMPLE 26. If G acts by conjugation on the set of all its subgroups and $H \leq G$, then $Orb(H) = \{g H g^{-1} : g \in G\}$ (all the conjugates of H) and St(H) is the normalizer N(H) of H.

EXAMPLE 27. Let $\mathsf{M} = \Bbbk[X_1, \ldots, X_m]$ and $\mathsf{G} = \mathfrak{S}_m$. If $f \in \mathsf{M}$, then $\mathsf{Orb}(f)$ is the set of distinct polynomials of the form $\pi \cdot f, \pi \in \mathfrak{S}_m$, and

$$\mathsf{St}(f) = \mathcal{S}(f) = \{\pi \in \mathfrak{S}_m : \pi \cdot f = f\}.$$

♦ Exercise 29. Show that $St(g \cdot x) = g St(x) g^{-1}$ for all $x \in M$ and all $g \in G$. (This means that points in the same orbit have conjugate stabilizers.)

THEOREM 16. (ORBIT-STABILIZER THEOREM) For each $x \in M$, the correspondence $g \cdot x \mapsto g \operatorname{St}(x)$ is a bijection between the orbit $\operatorname{Orb}(x)$ and the set $\operatorname{G/St}(x)$ of left cosets of the stabilizer $\operatorname{St}(x)$ in G .

Proof. The correspondence is clearly surjective. It is injective because if $g \operatorname{St}(x) = g' \operatorname{St}(x)$, then g = g'h for some element $h \in \operatorname{St}(x)$, and therefore

$$g \cdot x = (g'h) \cdot x = g' \cdot (h \cdot x) = g' \cdot x.$$

COROLLARY 17. If G is finite, the size of each orbit is a divisor of the order of G.

Proof. By the ORBIT-STABILIZER THEOREM, the size of the orbit Orb(x) is |G/St(x)| = |G|/|St(x)|, therefore

$$|\mathsf{Orb}(x)| \cdot |\mathsf{St}(x)| = |\mathsf{G}|.$$

♦ Exercise 30. Let M and M be two G-sets such that there exists a bijection $\beta : M \to \overline{M}$ such that $\beta(g \cdot x) = g \cdot \beta(x)$ for all $g \in G$ and all $x \in M$. (This means that M and M are equivalent.) Let $x \in M$ and $\overline{x} \in \overline{M}$ such that $\overline{x} = \beta(x)$. Show that $St(x) = St(\overline{x})$.

2.3. **Particular G-sets.** Let M be a G-set (with respect to the homomorphism $\Phi : G \to \mathfrak{S}_M$). We define some particular properties the action $\theta : G \times M \to M$ can have.

DEFINITION 28. θ is an effective action if Φ is injective (i.e., Ker $\Phi = \{1\}$).

This always happens when $G \leq \mathfrak{S}_M$. We observe that $\operatorname{Ker} \Phi = \bigcap_{x \in M} \operatorname{St}(x)$, an element of $\operatorname{Ker} \Phi$ being exactly an element of G contained in every isotropy group. If θ is not effective, then there exists a factorization $\widehat{\Phi}$ through $G/\operatorname{Ker} \Phi$



and $G/\text{Ker}\Phi$ acts effectively on M.

EXAMPLE 29. The action of G on itself by conjugation (inner automorphisms) has the center Z(G) as kernel.

In case of an effective action, we think of the group G as being identified with its image under (the associated homomorphism) Φ , a subgroup of the symmetric group $\mathfrak{S}_{\mathsf{M}}$ and we are essentially back with the important special case of permutation (transformation) groups.

DEFINITION 30. θ is a **free action** if $g \cdot x = x$ for some $x \in M$ implies g = 1.

This means that the transformation $\theta_g : x \mapsto g \cdot x$ for $g \neq 1$ has no fix points (free means "free of fix points"). The isotropy group is reduced to trivial subgroup: St $(x) = \{1\}$ for every $x \in M$. Clearly, every free action is effective. A G-set with a free action is also called a **principal G**-set.

EXAMPLE 31. The action of G on itself by left translations is free.

DEFINITION 32. θ is a **transitive action** if for $x_1, x_2 \in M$ there exists a $g \in G$ such that $g \cdot x_1 = x_2$, and **simply transitive** if, moreover, the element g is unique.

A simply transitive action is free. Conversely, a free action is simply transitive on each orbit. Indeed, let $x = g_1 \cdot x_0 = g_2 \cdot x_0 \in \operatorname{Orb}(x_0)$. Then

$$x_0 = g_2^{-1} \cdot x = g_2^{-1} \cdot (g_1 \cdot x_0) = (g_2^{-1}g_1) \cdot x_0$$

and therefore $g_2^{-1}g_1 \in \mathsf{St}(x_0) = \{\mathbf{1}\}$, hence $g_1 = g_2$.

NOTE: Stabilizers, in some sense, tell us how far a group is from acting simply transitively: just notice that $g \cdot x = h \cdot x \iff h^{-1}g \in St(x)$.

PROPOSITION 18. If G is an Abelian group, any effective and transitive action is simply transitive.

Proof. Let M be a G-set and let $x, y \in M$. Since our action is transitive, there is at least some element $g \in G$ such that $g \cdot x = y$. Assume that we have $g_1, g_2 \in G$ with $g_1 \cdot x = g_2 \cdot x = y$. We shall prove that, actually, $g_1 \cdot z = g_2 \cdot z$ for all $z \in M$. As our action is effective, we must have $g_1 = g_2$, and this proves our statement.

Let $z \in M$. There is some $g' \in G$ such that $z = g' \cdot x$. Then we have

$$g_{1} \cdot z = g_{1} \cdot (g' \cdot x)$$

$$= (g_{1}g') \cdot x$$

$$= (g'g_{1}) \cdot x \quad (\text{since } \mathsf{G} \text{ is Abelian})$$

$$= g' \cdot (g_{1} \cdot x)$$

$$= g' \cdot (g_{2} \cdot x)$$

$$= (g'g_{2}) \cdot x$$

$$= (g_{2}g') \cdot x \quad (\text{since } \mathsf{G} \text{ is Abelian})$$

$$= g_{2} \cdot (g' \cdot x)$$

$$= g_{2} \cdot z.$$

Therefore, $g_1 \cdot z = g_2 \cdot z$ for all $z \in M$, as claimed.

DEFINITION 33. A G-set M is called homogeneous if G acts transitively on M.

EXAMPLE 34. The action of G on itself by left translations is transitive. For, if $x, y \in M = G$, and if we take $g := yx^{-1}$, then gx = y.

EXAMPLE 35. The action of the general linear group $\mathsf{GL}(n, \Bbbk)$ on $\Bbbk^n \setminus \{0\}$ is transitive. For, given any non-zero vector x in \Bbbk^n , there certainly exists an invertible $n \times n$ matrix A over \Bbbk , whose first column is x and then

$$A\begin{bmatrix}1\\\vdots\\0\end{bmatrix} = x.$$

The transitivity of the action follows. Why?

EXAMPLE 36. The orthogonal group O(n) acts transitively on the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. More generally, an action of G on M induces a transitive action on each orbit.

It is not difficult to see that every G-set is expressible in a unique way as a disjoint union of orbits (this is PROBLEM 7). So many questions about actions of groups (on sets) can be reduced to the study of homogeneous G-sets.

There is a simple method for constructing a homogeneous set (this is EXAMPLE 20). Let G be a group and consider a subgroup $H \leq G$. Then we can define an action of G on the orbit set G/H. The left translation $L_g : G \to G$ satisfies $L_g (a H) = (ga) H$ and therefore defines a transformation (of G/H)

$$\lambda_q^{\mathsf{H}} : \mathsf{G}/\mathsf{H} \to \mathsf{G}/\mathsf{H}, \quad a \,\mathsf{H} \mapsto (ga) \,\mathsf{H}.$$

This makes G/H a G-set, which is *homogeneous*.

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 \diamond Exercise 31. Show that the action (of G on G/H)

 $\mathsf{G} \times \mathsf{G}/\mathsf{H} \ni (g, a \mathsf{H}) \mapsto (ga) \mathsf{H} \in \mathsf{G}/\mathsf{H}$

is transitive.

The method of constructing a homogeneous set (as described above) is, in a certain sense, universal: every homogeneous G-set is equivalent to a (homogeneous) G-set of the form G/H for a suitable $H \leq G$. (This orbit set provides a sort of "canonical form" for homogeneous G-sets, under equivalence). Indeed, let M be an arbitrary homogeneous G-set and let $x_0 \in M$. Put $H = St(x_0) \leq G$. Then the map

$$\beta : \mathsf{M} \ni x = g \cdot x_0 \mapsto g \mathsf{H} \in \mathsf{G}/\mathsf{H}$$

is an equivalence of G. (β is well defined, bijective and equivariant.)

NOTE: If M is a homogeneous G-set, then we have (for each $x \in M$)

 $M = Orb(x) \approx G/St(x)$ (one-to-one correspondence).

This equation is invaluable, for it reduces the study of a set M (usually endowed with some "structure") to an algebraic problem, namely the study of the pair (G, St(x)).

In the context of group actions (on sets), the homogeneous G-sets play a role somewhat similar to that played by the vector spaces \mathbb{k}^n in the context of linear algebra. The result, stated above, regarding the homogeneous G-sets corresponds to the classical result which states that every finite-dimensional vector space V over \mathbb{k} is isomorphic to some (vector space) \mathbb{k}^n . In the same way an isomorphism between V and \mathbb{k}^n assumes the choice of a basis for V, an equivalence (G-isomorphism) between a homogeneous G-set M and G/H assumes the choice of a point in G. Also, in the same way a vector space \mathbb{k}^n admits a preferred basis, a homogeneous set admits a preferred point. Finally, the statement that two vector spaces \mathbb{k}^m and \mathbb{k}^n are isomorphic if and only if m = n, corresponds to the statement that *two homogeneous* G-sets G/H₁ and G/H₂ are equivalent if and only if H₁ and H₂ are conjugate in G (this is PROBLEM 10).

2.4. Examples of group actions. We now give some further examples of group actions on sets.

EXAMPLE 37. Let M = G be a group. Then G acts on itself in several important ways:

(a) $\theta_g(x) = gx$	(left translation);
(b) $\theta_g(x) = xg^{-1}$	(right translation);
(c) $\theta_g(x) = gxg^{-1}$	(inner automorphism)

EXAMPLE 38. If M = V is a vector space (over the field k), its linear group

 $\mathsf{G} = \mathsf{GL}\left(\mathsf{V}\right) := \left\{ \alpha \in \mathsf{V}^{\mathsf{V}} : \alpha \text{ is linear and bijective} \right\} \le \mathfrak{S}_{\mathsf{V}}$

acts on M. When $V = \mathbb{k}^n$, the group $GL(\mathbb{k}^n)$ is (isomorphic to) the general linear group $GL(n,\mathbb{k})$. So the group $GL(n,\mathbb{k})$ acts on \mathbb{k}^n by left multiplication:

$$\mathsf{GL}(n,\Bbbk) \times \Bbbk^n \ni (A,x) \mapsto Ax \in \Bbbk^n$$

(here the elements of \mathbb{k}^n are viewed as $n \times 1$ matrices over \mathbb{k}).

EXAMPLE 39. Let M = E be an *Euclidean vector space*, and put

$$\mathsf{G} = \mathsf{O}(\mathsf{E}) := \{ \alpha \in \mathsf{GL}(\mathsf{E}) : \alpha \text{ is an isometry} \}.$$

Then there is a natural action of G on M. When $\mathsf{E} = \mathbb{R}^n$, the group $\mathsf{O}(\mathbb{R}^n)$ is the orthogonal group $\mathsf{O}(n) \leq \mathsf{GL}(n,\mathbb{R})$. In particular, the rotation group $\mathsf{SO}(n)$ acts naturally on \mathbb{R}^n . For $n \geq 1$, let

$$\mathbb{S}^{n-1} := \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : x_1^2 + \dots + x_n^2 = 1 \right\} \subset \mathbb{R}^n$$

be the *unit sphere*. In particular, \mathbb{S}^2 is the usual sphere in \mathbb{R}^3 . Thus, we have an action

$$\mathsf{SO}(3) \times \mathbb{S}^2 \to \mathbb{S}^2, \quad (R, x) \mapsto Rx.$$

This action is *transitive*. This is so because, for any two points $x, y \in \mathbb{S}^2$, there is a rotation whose axis is perpendicular to the plane containing x, y, and the center of the sphere (this plane is not unique when x and y are antipodal, i.e., on a diameter). Similarly, for $n \ge 1$, we get an action of SO(n) on \mathbb{S}^{n-1} .

NOTE: An Euclidean vector space E is a finite-dimensional vector space over \mathbb{R} , together with a positive definite symmetric bilinear form ϕ (i.e., $\phi : \mathsf{E} \times \mathsf{E} \to \mathbb{R}$ is symmetric and bilinear, and $\phi(x,x) > 0$ for all $x \neq 0$). We write $\phi(x,y) = (x \mid y)$ and call this number the scalar product of x and y. The norm of x is $||x|| := \sqrt{\phi(x,x)} = \sqrt{(x \mid x)}$. If $(x \mid y) = 0$, we say that x and y are orthogonal.

The standard example of an Euclidean vector space is $\mathsf{E} = \mathbb{R}^n$, with

$$\phi((x_1, \dots, x_n), (y_1, \dots, y_n)) = x_1 y_1 + \dots + x_n y_n$$

An orthogonal set of non-zero vectors is linearly independent. If $(e_i)_{1 \le i \le n}$ is an *orthonormal* basis for E, the coefficients of the decomposition $x = x_1e_1 + \cdots + x_ne_n$ are given by $x_i = (x | e_i)$. Moreover,

 $(x_1e_1 + \dots + x_ne_n | y_1e_1 + \dots + y_ne_n) = x_1y_1 + \dots + x_ny_n.$

Let $\mathsf{E}, \overline{\mathsf{E}}$ be two Euclidean vector spaces of the same dimension, and let $\alpha : \mathsf{E} \to \overline{\mathsf{E}}$ be a map. The following conditions are logically equivalent:

- (a) α is linear, and $||\alpha(x)|| = ||x||$ for all $x \in \mathsf{E}$;
- (b) $(\alpha(x) | \alpha(y)) = (x | y)$ for all $x, y \in \mathsf{E}$.

Such a map is necessarily bijective and is called an **isometry**. The set of all such isometries is denoted by $O(E;\overline{E})$. Every n-dimensional Euclidean vector space is isometric to \mathbb{R}^n .

The group O(E) := O(E; E) is called the **orthogonal group** of E; we write $O(n) = O(\mathbb{R}^n)$. The condition $\alpha \in O(E)$ is equivalent to $A^{\top}A = \mathbf{1}$, where A is the matrix of α in some (or any) orthonormal basis and **1** is the identity matrix. In particular, det $\alpha = \pm 1$. We set $SO(E) := \{\alpha \in O(E) : \det \alpha = 1\}$. The elements of (the group) SO(E) are called *rotations*.

EXAMPLE 40. Let k be a field (think of \mathbb{R} or \mathbb{C}). Given $A \in \mathsf{GL}(n, \Bbbk)$ and $b \in \Bbbk^n$, we define a map

$$\tau_{A,b}: \mathbb{k}^n \to \mathbb{k}^n, \quad x \mapsto Ax + b.$$

These maps are known as affine transformations of \mathbb{k}^n ; they constitute the affine group

Aff
$$(n, \mathbb{k}) := \{ \tau_{A, b} : A \in \mathsf{GL}(n, \mathbb{k}), b \in \mathbb{k}^n \} \leq \mathfrak{S}_{\mathbb{k}^n}.$$

The group Aff (n, \mathbb{k}) acts naturally on \mathbb{k}^n : this makes \mathbb{k}^n an Aff (n, \mathbb{k}) -set (known as the *n*-dimensional *affine space* over \mathbb{k}). Notice that $\mathsf{GL}(n, \mathbb{k})$ is a subgroup of Aff (n, \mathbb{k}) ; that is, invertible linear transformations are some of the affine transformations of \mathbb{k}^n . Notice also that the group of translations $x \mapsto x + b$ (which is a normal subgroup of Aff (n, \mathbb{k})) acts on the affine space \mathbb{k}^n regularly (in the sense of EXAMPLE 20).

NOTE: Let M be a (non-empty) set and let V be vector space over the field \Bbbk (think again of \mathbb{R} or \mathbb{C}) considered with its additive group structure. An **affine space** over \Bbbk is a structure $(\mathsf{M},\mathsf{V},\theta)$, where $\theta:\mathsf{V}\times\mathsf{M}\to\mathsf{M}$ is an effective and transitive action on M. The vector space V is said to underlie the affine space M. We put

$$\theta(v, x) = x + v.$$

The map $\theta_v = \theta(v, \cdot)$ is called the *translation* of M by the vector v. The action θ is simply transitive (see PROPOSITION 18), so there exists a function $\Theta : \mathsf{M} \times \mathsf{M} \to \mathsf{V}$ such that $y = \theta(\Theta(x, y), x)$ for all $x, y \in \mathsf{M}$. We set $\overrightarrow{xy} := \Theta(x, y)$, and sometimes say that \overrightarrow{xy} is the *free vector* associated with the pair (x, y). We also write $\overrightarrow{xy} = y - x$. The fact that θ is a V-action can be translated as follows:

$$(x + v) + w = x + (v + w).$$

In particular, Θ satisfies the following conditions:

- (AS1) $\Theta_x : \mathsf{M} \ni y \mapsto \Theta(x, y) \in \mathsf{V}$ is a bijection for all $x \in \mathsf{M}$;
- (AS2) $\Theta(x,y) + \Theta(y,z) = \Theta(x,z)$ for all $x, y, z \in M$

since we have $\Theta_x^{-1}(v) = x + v$. (The identity $\Theta(x, y) + \Theta(y, z) = \Theta(x, z)$ is known as CHASLES'S RELATION.)

<u>Alternative definition</u>. Given a (non-empty) set M and a vector space V over the field k, assume that $\Theta : \mathsf{M} \times \mathsf{M} \to \mathsf{V}$ is a function satisfying conditions (AS1) and (AS2). Then M is an affine space under the action $\theta(v, x) = \Theta_x^{-1}(v)$. This indeed is an equivalent definition, for we have $\Theta(x, x) = 0$, $\Theta(y, x) = -\Theta(x, y)$, $\theta(-v) \circ \theta(v) = \mathbf{1}_{\mathsf{M}}$, and thus

$$\theta(v) \circ \theta(w) = \theta(v+w).$$

Affine maps (morphisms) can be defined between two affine spaces (over the same field \mathbb{k}). Heuristically, such a map consists of a translation and a linear transformation. (If $M = \overline{M} = \mathbb{R}$,

we recover the well-known maps $x \mapsto ax + b$ for $a, b \in \mathbb{R}, a \neq 0$.) The set

$$AGL(M) := \{ \alpha \in \mathfrak{S}_{M} : \alpha \text{ is an affine map} \}$$

is a group, called the **affine group** of M. It turns out that *(the affine space)* M *is a homogeneous* AGL(M)-*set.* The vector space \mathbb{k}^n has a natural affine structure: when $M = V = \mathbb{k}^n$, the function

$$\Bbbk^n \times \Bbbk^n \ni (x, y) \mapsto y - x \in \Bbbk^n$$

induces an action of \mathbb{k}^n on itself. We write $\mathsf{AGL}(\mathbb{k}^n) = \mathsf{Aff}(n,\mathbb{k}) \leq \mathfrak{S}_{\mathbb{k}^n}$.

EXAMPLE 41. Let $G = \mathbb{R}$ and $M = \mathbb{S}^3 := \{x \in \mathbb{R}^4 : ||x|| = 1\} \subset \mathbb{R}^4$. Identifying \mathbb{R}^4 with \mathbb{C}^2 , we can define the action

$$\mathbb{R} \times \mathbb{S}^3 \to \mathbb{S}^3$$
, $(t, z, z') \mapsto (e^{it}z, e^{it}z')$.

This example is of great importance in geometry.

PROBLEMS
$$(6-10)$$

(6) (a) The **upper half-plane** \mathbb{H}^2 is the (open) subset of \mathbb{R}^2 consisting of all points $(x, y) \in \mathbb{R}^2$ with y > 0. It is convenient to identify \mathbb{R}^2 with the set of complex numbers. So

$$\mathbb{H}^2 := \{ z = x + iy \in \mathbb{C} : y > 0 \}$$

Define the map

$$\theta : \mathsf{SL}(2,\mathbb{R}) \times \mathbb{H}^2 \to \mathbb{H}^2, \quad \left(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, z \right) \mapsto \frac{az+b}{cz+d}.$$

Show that θ is an action of the special linear group $\mathsf{SL}(2,\mathbb{R})$ on the upper half-plane \mathbb{H}^2 . Is this action transitive ?

- (b) Consider the set of all Möbius transformations $\mu_{abcd} : \mathbb{C}_{\infty} \to \mathbb{C}_{\infty}$ corresponding to the case $a, b, c, d \in \mathbb{R}$ with ad bc = 1. This set is denoted by $\mathsf{M\"ob}_{\mathbb{R}}^+$. Show that $\mathsf{M\"ob}_{\mathbb{R}}^+$ is a subgroup of the Möbius group $\mathsf{M\"ob}$.
- (c) Define the function

$$\Phi: \mathsf{SL}(2,\mathbb{R}) \to \mathsf{M\ddot{o}b}_{\mathbb{R}}^+, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \mu_{abcd}.$$

Show that Φ is a surjective homomorphism (epimorphism) whose kernel is Ker $(\Phi) = \{\mathbf{1}, -\mathbf{1}\}$. Hence deduce that the group $\mathsf{M\ddot{o}b}_{\mathbb{R}}^+$ is isomorphic to the quotient group $\mathsf{SL}(2,\mathbb{R})/\{\mathbf{1},-\mathbf{1}\}$, denoted by $\mathsf{PSL}(2,\mathbb{R})$. (This latter group turns out to be the group of *projective transformations* of the real projective line \mathbb{RP}^1 .)

- (7) Show that every G-set can be expressed in just one way as the disjoint union of a family of orbits.
- (8) For which n is the special linear group $\mathsf{SL}(n,\mathbb{R})$ acting transitively on $\mathbb{R}^n \setminus \{0\}$?
- (9) Show that two homogeneous G-spaces M and \overline{M} are equivalent if and only if there exist a automorphism $\phi \in Aut(G)$ and elements $x_0 \in M$ and $\overline{x}_0 \in \overline{M}$ such that

$$\phi\left(\mathsf{St}\left(x_{0}\right)\right)=\mathsf{St}\left(\overline{x}_{0}\right).$$

(10) Show that two homogeneous G-sets $M = G/H_1$ and $\overline{M} = G/H_2$ are equivalent if and only if the subgroups H_1 and H_2 are conjugate in G.