4. Matrix Groups

Matrix algebra • Matrix groups • Linear Lie groups: examples • Complex matrix groups as real matrix groups.

4.1. **Matrix algebra.** Throughout, we shall denote by $\Bbbk$ either the *field* $\mathbb{R}$ of real numbers or the *field* $\mathbb{C}$ of complex numbers. Let $\Bbbk^m$ be the set of all m-tuples of elements of $\Bbbk$. Under the usual addition and scalar multiplication, $\Bbbk^m$ is a *vector space* over $\Bbbk$. The set $\mathsf{Hom}\,(\Bbbk^n, \Bbbk^m)$ of all linear mappings from $\Bbbk^n$ to $\Bbbk^m$ (i.e., mappings $L : \Bbbk^n \to \Bbbk^m$ such that $L(\lambda x + \mu y) = \lambda L(x) + \mu L(y)$ for every $x, y \in \Bbbk^n$ and $\lambda, \mu \in \Bbbk$) is also a *vector space* over $\Bbbk$.

◇ **Exercise 43.** Determine the *dimension* of the vector space $\mathsf{Hom}\,(\Bbbk^n, \Bbbk^m)$.

Let $\Bbbk^{m\times n}$ be the set of all $m \times n$ matrices with elements (entries) from $\Bbbk$. Under the usual matrix addition and multiplication, $\Bbbk^{m\times n}$ is a *vector space* over $\Bbbk$. There is a natural one-to-one correspondence $A \mapsto L_A\,(: x \mapsto Ax)$ between the $m \times n$ matrices with elements from $\Bbbk$ and the linear mappings from $\Bbbk^n$ to $\Bbbk^m$.

◇ **Exercise 44.** Show that the vector spaces $\Bbbk^{m\times n}$ and $\mathsf{Hom}\,(\Bbbk^n, \Bbbk^m)$ are *isomorphic*.

In particular, the (n-dimensional) vector spaces $\Bbbk^{1\times n}$ and $\mathsf{Hom}\,(\Bbbk^n, \Bbbk) = (\Bbbk^n)^*$ (the *dual* of $\Bbbk^n$) are isomorphic. Any matrix $A \in \Bbbk^{m\times n}$ can be *interpreted* as a linear mapping $L_A \in \mathsf{Hom}\,(\Bbbk^n, \Bbbk^m)$, whereas any linear mapping $L \in \mathsf{Hom}\,(\Bbbk^n, \Bbbk^m)$ can be *realized* as a matrix $A \in \Bbbk^{m\times n}$. Henceforth we shall not distinguish notationwise between a matrix A and its corresponding linear mapping $x \mapsto Ax$.

Note : A matrix (or linear mapping, if one prefers) $A \in \Bbbk^{n\times n}$ can be viewed as a **vector field** (on $\Bbbk^n$) : A associates to each point p in $\Bbbk^n$ the tangent vector $A(p) = Ap \in \Bbbk^n$. We may think of a *fluid* in motion, so that the velocity of the fluid particles passing through p is always $A(p)$. The vector field is then the current of the *flow* and the paths of the fluid particles are the trajectories. This kind of flow is, of course, very special : $A(p)$ is independent of time, and depends linearly on p.

The (structured) set $\Bbbk^{n\times n}$ is not just a vector space. It also has a multiplication which is associative and distributes over addition (on either side). In other words, under the usual addition and multiplication, $\Bbbk^{n\times n}$ is a *ring* (in general not commutative), with identity $\mathbf{1}$. Moreover, for all $A, B \in \Bbbk^{n\times n}$ and $\lambda \in \Bbbk$,

$$\lambda(AB) = (\lambda A)B = A(\lambda B).$$

Such a structure is called an (associative) **algebra** over $\Bbbk$.

For $x \in \Bbbk^n$ $(= \Bbbk^{n\times 1})$, let

$$\|x\|_2 := \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

be the 2-norm (or *Euclidean norm*) on $\Bbbk^n$.

NOTE : For $r \geq 1$, the *r-norm* of $x \in \Bbbk^n$ is defined as

$$\|x\|_r := \left(|x_1|^r + |x_2|^r + \cdots + |x_n|^r\right)^{1/r}.$$

The following properties hold (for $x, y \in \Bbbk^n$ and $\lambda \in \Bbbk$) :

$$\|x\|_r \geq 0, \quad \text{and} \quad \|x\|_r = 0 \iff x = 0\ ;$$
$$\|\lambda x\|_r = |\lambda|\,\|x\|_r\ ;$$
$$\|x+y\|_r \leq \|x\|_r + \|y\|_r.$$

In practice, only three of the r-norms are used, and they are :

$$\begin{aligned}
\|x\|_1 &= |x_1| + |x_2| + \cdots + |x_n| \quad \text{(the grid norm)};\\
\|x\|_2 &= \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2} \quad \text{(the Euclidean norm)};\\
\|x\|_\infty = \lim_{r\to\infty} \|x\|_r &= \max\{|x_1|, |x_2|, \ldots, |x_n|\} \quad \text{(the max norm)}.
\end{aligned}$$

For $x \in \Bbbk^n$, we have

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\cdot\|x\|_2 \leq n\cdot\|x\|_\infty$$

and so any two of these norms are *equivalent* (i.e., the associated metric topologies are identical). In fact, *all norms on a finite-dimensional vector space (over* $\Bbbk$ *) are equivalent.*

The metric topology *induced* by (the Euclidean distance) $(x, y) \mapsto \|x - y\|_2$ is the *natural topology* on the set (vector space) $\Bbbk^n$.

$\diamond$ **Exercise 45.** Show that, for $x, y \in \Bbbk^n$,

$$|\,\|x\|_2 - \|y\|_2\,| \leq \|x - y\|_2.$$

Hence deduce that the function $\|\cdot\|_2 : \Bbbk^n \to \mathbb{R}, \quad x \mapsto \|x\|_2$ is *continuous* (with respect to the natural topologies on $\Bbbk^n$ and $\mathbb{R}$).

$\diamond$ **Exercise 46.** Given $A \in \Bbbk^{n\times n}$, show that the linear mapping (on $\Bbbk^n$) $x \mapsto Ax$ is *continuous* (with respect to the natural topology on $\Bbbk^n$).

Let $A \in \Bbbk^{n\times n}$. The 2-norm $\|\cdot\|_2$ on $\Bbbk^{n\times 1}$ induces a (matrix) norm on $\Bbbk^{n\times n}$ by setting

$$\|A\| := \max_{\|x\|_2=1} \|Ax\|_2.$$

The subset $K = \{x \in \Bbbk^n \,:\, \|x\|_2 = 1\} \subset \Bbbk^n$ is closed and bounded, and so is *compact.* [A subset of the metric space $\Bbbk^n$ is compact if and only if it is closed and bounded.] On the other hand, the function $f : K \to \mathbb{R}, \quad x \mapsto \|Ax\|_2$ is *continuous.* [The composition of two continuous maps is a continuous map.] Hence the maximum value $\max_{x\in K}\|Ax\|_2$ *must* exist.

NOTE : The following topological result holds : *If* $K \subset \mathbb{k}^n$ *is a (non-empty) compact set, then any continuous function* $f : K \to \mathbb{R}$ *is bounded; that is, the image set* $f(K) = \{f(x) : x \in K\} \subseteq \mathbb{R}$ *is bounded.* Moreover, f has a global maximum (and a global minimum).

◇ **Exercise 47.** Show that the induced norm $\|\cdot\|$ is *compatible* with its underlying norm $\|\cdot\|_2$; that is (for $A \in \mathbb{k}^{n\times n}$ and $x \in \mathbb{k}^n$),

$$\|Ax\|_2 \leq \|A\| \, \|x\|_2.$$

$\|\cdot\|$ is a *matrix norm* on $\mathbb{k}^{n\times n}$, called the **operator norm**; that is, it has the following four properties (for $A, B \in \mathbb{k}^{n\times n}$ and $\lambda \in \mathbb{k}$) :

(MN1) $\|A\| \geq 0$, and $\|A\| = 0 \iff A = 0$;
(MN2) $\|\lambda A\| = |\lambda| \, \|A\|$;
(MN3) $\|A + B\| \leq \|A\| + \|B\|$;
(MN4) $\|AB\| \leq \|A\| \, \|B\|$.

NOTE : There is a simple procedure (well known in numerical linear algebra) for calculating the operator norm of an $n \times n$ matrix A. This is $\|A\| = \sqrt{\lambda_{\max}}$, where $\lambda_{\max}$ is the largest eigenvalue of the matrix A^*A. Here A^* denotes the *Hermitian conjugate* (i.e., the conjugate transpose) matrix of A; in the case $\mathbb{k} = \mathbb{R}$, $A^* = A^\top$.

We define a *metric* ρ on (the algebra) $\mathbb{k}^{n\times n}$ by

$$\rho(A, B) := \|A - B\|.$$

Associated to this metric is a natural *topology* on $\mathbb{k}^{n\times n}$. Hence fundamental topological concepts, like *open sets, closed sets, compactness, connectedness,* as well as *continuity*, can be introduced. In particular, we can speak of continuous functions $\mathbb{k}^{n\times n} \to \mathbb{k}$.

◇ **Exercise 48.** For $1 \leq i, j \leq n$, show that the coordinate function

$$\mathrm{coord}_{ij} : \mathbb{k}^{n\times n} \to \mathbb{k}, \quad A \mapsto a_{ij}$$

is *continuous.* [HINT : Show first that $|a_{ij}| \leq \|A\|$ and then verify the defining condition for continuity.]

It follows immediately that *if* $f : \mathbb{k}^{n^2} \to \mathbb{k}$ *is continuous, then the associated function*

$$\widetilde{f} = f \circ (\mathrm{coord}_{ij}) : \mathbb{k}^{n\times n} \to \mathbb{k}, \quad A \mapsto f((a_{ij}))$$

is also continuous. Here $(a_{ij}) = (a_{11}, \ldots, a_{n1}, \ldots, a_{1n}, \ldots, a_{nn}) \in \mathbb{k}^{n^2}$.

◇ **Exercise 49.** Show that the *determinant function*

$$\det : \mathbb{k}^{n\times n} \to \mathbb{k}, \quad A \mapsto \det A := \sum_{\sigma \in \mathfrak{S}_n} (-1)^{|\sigma|} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

and the *trace function*

$$\operatorname{tr} : \mathbb{k}^{n\times n} \to \mathbb{k}, \quad A \mapsto \operatorname{tr} A := \sum_{i=1}^{n} a_{ii}$$

are *continuous.*

The metric space $(\mathbb{k}^{n\times n}, \rho)$ is *complete.* This means that *every Cauchy sequence* $(A_r)_{r\geq 0}$ *in* $\mathbb{k}^{n\times n}$ *has a unique limit* $\lim\limits_{r\to\infty} A_r$. Furthermore,

$$\left(\lim_{r\to\infty} A_r\right)_{ij} = \lim_{r\to\infty} (A_r)_{ij}.$$

Indeed, the limit on the RHS exists, so it is sufficient to check that the required matrix limit is the matrix A with $a_{ij} = \lim\limits_{r\to\infty} (A_r)_{ij}$. The sequence $(A_r - A)_{r\geq 0}$ satisfies

$$\|A_r - A\| \leq \sum_{i,j=1}^{n} |(A_r)_{ij} - a_{ij}| \to 0 \quad \text{as} \quad r \to \infty$$

and so $A_r \to A$.

4.2. **Matrix groups.** Let $\mathsf{GL}\,(n, \mathbb{k})$ be the set of all invertible $n \times n$ matrices over $\mathbb{k}$. So

$$\mathsf{GL}\,(n, \mathbb{k}) := \{A \in \mathbb{k}^{n\times n} \, : \, \det A \neq 0\}.$$

$\diamond$ **Exercise 50.** Verify that the set $\mathsf{GL}\,(n, \mathbb{k})$ is a *group* under matrix multiplication.

$\mathsf{GL}\,(n, \mathbb{k})$ is called the **general linear group** over $\mathbb{k}$. We will refer to $\mathsf{GL}\,(n, \mathbb{R})$ and $\mathsf{GL}\,(n, \mathbb{C})$ as the *real* and *complex* general linear group, respectively. A 1×1 matrix over $\mathbb{k}$ is just an element of $\mathbb{k}$ and matrix multiplication of two such elements is just multiplication in $\mathbb{k}$. So we see that

$$\mathsf{GL}\,(1, \mathbb{k}) = \mathbb{k}^{\times} \quad \text{(the multiplicative group of } \mathbb{k} \setminus \{0\}\text{)}\,.$$

Any subgroup of $\mathsf{GL}\,(n, \mathbb{k})$ is customarily referred to as a **linear group** or sometimes as a matrix group.

Proposition 31. *$\mathsf{GL}\,(n, \mathbb{k})$ is an open subset of $\mathbb{k}^{n\times n}$.*

Proof. We have seen that the function $\det : \mathbb{k}^{n\times n} \to \mathbb{k}$ is continuous (see **Exercise 49**). Then observe that

$$\mathsf{GL}\,(n, \mathbb{k}) = \mathbb{k}^{n\times n} \setminus \det^{-1}(0).$$

Since the set $\{0\}$ is closed (in $\mathbb{k}$), it follows that $\det^{-1}(0) = \det^{-1}(\{0\}) \subset \mathbb{k}^{n\times n}$ is also closed. [The preimage of a closed set under a continuous map is a closed set.] Hence $\mathsf{GL}\,(n, \mathbb{k})$ is open. [The complement of a closed set is an open set.] $\square$

We observe that the general linear group $\mathsf{GL}(n, \Bbbk)$ has more than just an *algebraic* structure: it has a *topological* structure as well (PROPOSITION 31). Thus we may naturally consider subsets which are not only closed in the algebraic sense (that is, subgroups), but in the topological sense as well.

DEFINITION 55. A **linear Lie group** is a closed subgroup of $\mathsf{GL}(n, \Bbbk)$.

Linear Lie groups are also known as *matrix* Lie groups. This terminology emphasizes the remarkable fact that *every closed linear group is a Lie group.*

NOTE : The condition that a set (group) of matrices $\mathsf{G} \subseteq \mathsf{GL}(n, \Bbbk)$ is a closed subset of (the metric space) $\mathsf{GL}(n, \Bbbk)$ means that the following condition is satisfied : *if* $(A_r)_{r\geq 0}$ *is any sequence of matrices in* G *and* $A_r \to A$, *then either* $A \in \mathsf{G}$ *or* A *is not invertible (i.e.* $A \notin \mathsf{GL}(n, \Bbbk)$*).* The condition that G be a *closed* subgroup, as opposed to merely a subgroup, should be regarded as a "technicality" since most of the *interesting* subgroups of $\mathsf{GL}(n, \Bbbk)$ have this property. Almost all of the matrix groups we will consider have the stronger property that if $(A_r)_{r\geq 0}$ is any sequence of matrices in G converging to some matrix A, then $A \in \mathsf{G}$.

We shall use the customary notation $\mathsf{G} \leq \mathsf{GL}(n, \Bbbk)$ to indicate that G is a subgroup of $\mathsf{GL}(n, \Bbbk)$.

EXAMPLE 56. The general linear group $\mathsf{GL}(n, \Bbbk)$ is a linear Lie group.

EXAMPLE 57. An example of a group of matrices which is <u>not</u> a linear Lie group is the set $\mathsf{GL}(n, \mathbb{Q})$ of all $n \times n$ invertible matrices all of whose entries are rational numbers. This is in fact a subgroup of $\mathsf{GL}(n, \mathbb{C})$ but not a closed subgroup; that is, one can (easily) have a sequence of invertible matrices with rational entries converging to an invertible matrix with some irrational entries.

NOTE : The *closure* of $\mathsf{GL}(2, \mathbb{Q})$ (in $\mathsf{GL}(2, \mathbb{C})$) can be thought of as (the direct product) $\mathbb{S}^1 \times \mathbb{S}^1$ and so <u>is</u> a linear Lie group (see **Exercise 61**).

PROPOSITION 32. *Let* G *be a linear Lie group and* H *a closed subgroup of* G. *Then* H *is a linear Lie group.*

Proof. Every sequence $(A_r)_{r\geq 0}$ in H with a limit in $\mathsf{GL}(n, \Bbbk)$ actually has its limit in G since each $A_r \in \mathsf{H} \subseteq \mathsf{G}$ and G is closed in $\mathsf{GL}(n, \Bbbk)$. Since H is closed in G, this means that $(A_r)_{r\geq 0}$ has a limit in H. So H is closed in $\mathsf{GL}(n, \Bbbk)$, showing it is a linear Lie group. $\square$

$\diamond$ **Exercise 51.** Prove that any *intersection* of linear Lie groups is a linear Lie group.

EXAMPLE 58. Denote by $\mathsf{SL}(n, \Bbbk)$ the set of all $n \times n$ matrices over $\Bbbk$, having determinant one. So

$$\mathsf{SL}(n, \Bbbk) := \{A \in \Bbbk^{n\times n} \,:\, \det A = 1\} \subset \mathsf{GL}(n, \Bbbk).$$

$\diamond$ **Exercise 52.** Show that $\mathsf{SL}(n, \Bbbk)$ is a closed subgroup of $\mathsf{GL}(n, \Bbbk)$ and hence is a linear Lie group.

$\mathsf{SL}(n, \Bbbk)$ is called the **special linear group** over $\Bbbk$. We will refer to $\mathsf{SL}(n, \mathbb{R})$ and $\mathsf{SL}(n, \mathbb{C})$ as the *real* and *complex* special linear group, respectively.

DEFINITION 59. A closed subgroup of a linear Lie group G is called a **linear Lie subgroup**.

EXAMPLE 60. We can consider $\mathsf{GL}(n, \Bbbk)$ as a subgroup of $\mathsf{GL}(n+1, \Bbbk)$ by *identifying* the $n \times n$ matrix $A = [\, a_{ij}\,]$ with

$$\begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & a_{11} & \dots & a_{1n} \\ 0 & a_{21} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \dots & a_{nn} \end{bmatrix}.$$

It is easy to verify that $\mathsf{GL}(n, \Bbbk)$ is closed in $\mathsf{GL}(n+1, \Bbbk)$ and hence $\mathsf{GL}(n, \Bbbk)$ is a linear Lie subgroup of $\mathsf{GL}(n+1, \Bbbk)$.

$\diamond$ **Exercise 53.** Show that $\mathsf{SL}(n, \Bbbk)$ is a linear Lie subgroup of $\mathsf{SL}(n+1, \Bbbk)$.

4.3. **Linear Lie groups: examples.** The vector space $\Bbbk^{n\times n}$ over $\Bbbk$ can be considered to be a *real* vector space, of dimension n^2 or $2n^2$, respectively. Explicitly, $\mathbb{R}^{n\times n}$ is (isomorphic to) $\mathbb{R}^{n^2}$, and $\mathbb{C}^{n\times n}$ is (isomorphic to) $\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$. Hence we may assume, without any loss of generality, that $\Bbbk^{n\times n}$ is some Euclidean space $\mathbb{R}^m$.

4.3.1. *The real general linear group* $\mathsf{GL}(n, \mathbb{R})$. We have seen that $\mathsf{GL}(n, \mathbb{R})$ is a linear Lie group and that it is an *open* subset of the vector space $\mathbb{R}^{n\times n}\left(=\mathbb{R}^{n^2}\right)$. Since the set $\mathsf{GL}(n, \mathbb{R})$ is not closed, it is *not* compact. [Any compact set is a closed set.] The determinant function $\det : \mathsf{GL}(n, \mathbb{R}) \to \mathbb{R}$ is continuous (in fact, *smooth*) and maps $\mathsf{GL}(n, \mathbb{R})$ onto the two *components* of $\mathbb{R}^\times$. Thus $\mathsf{GL}(n, \mathbb{R})$ is *not* connected. [The image of a connected set under a continuous map is a connected set.]

NOTE : A linear Lie group G is said to be **connected** if given any two matrices $A, B \in \mathsf{G}$, there exists a continuous *path* $\gamma : [a, b] \to \mathsf{G}$ with $\gamma(a) = A$ and $\gamma(b) = B$. This property is what is called **path-connectedness** in topology, which is not (in general) the same as connectedness. However, it is a fact (not particularly obvious at the moment) that *a linear Lie group is connected* *if and only if it is path-connected.* So in a slight abuse of terminology we shall continue to refer to the above property as *connectedness.*

A linear Lie group G which is not connected can be decomposed (uniquely) as a union of several pieces, called *components*, such that two elements of the same component can be joined by a continuous path, but two elements of different components cannot. The component of G containing the identity is a closed subgroup of G and hence a connected linear Lie group.

Consider the sets

$$\begin{aligned}\mathsf{GL}^{+}(n,\mathbb{R}) &:= \{A \in \mathsf{GL}(n,\mathbb{R}) \,:\, \det A > 0\}\\ \mathsf{GL}^{-}(n,\mathbb{R}) &:= \{B \in \mathsf{GL}(n,\mathbb{R}) \,:\, \det B < 0\}.\end{aligned}$$

These two disjoint subsets of $\mathsf{GL}(n,\mathbb{R})$ are *open* and satisfy

$$\mathsf{GL}^{+}(n,\mathbb{R}) \cup \mathsf{GL}^{-}(n,\mathbb{R}) = \mathsf{GL}(n,\mathbb{R}).$$

[The preimage of an open set under a continuous map is an open set.]

$\diamond$ **Exercise 54.** Show that $\mathsf{GL}^{+}(n,\mathbb{R})$ is a linear Lie subgroup of $\mathsf{GL}(n,\mathbb{R})$ but $\mathsf{GL}^{-}(n,\mathbb{R})$ is not.

The mapping

$$A \in \mathsf{GL}^{+}(n,\mathbb{R}) \mapsto SA \in \mathsf{GL}^{-}(n,\mathbb{R})$$

where $S = \operatorname{diag}(1,\ldots,1,-1)$, is a bijection (in fact, a *diffeomorphism*). The transformation $x \mapsto Sx$ may be thought of as a *reflection* in the hyperplane $\mathbb{R}^{n-1} = \mathbb{R}^{n-1}\times\{0\} \subset \mathbb{R}^{n}$.

NOTE : The group $\mathsf{GL}^{+}(n,\mathbb{R})$ is *connected*, which proves that $\mathsf{GL}^{+}(n,\mathbb{R})$ *is the connected component of the identity in* $\mathsf{GL}(n,\mathbb{R})$ and that $\mathsf{GL}(n,\mathbb{R})$ has two (connected) components.

4.3.2. *The real special linear group* $\mathsf{SL}(n,\mathbb{R})$*.* Recall that

$$\mathsf{SL}(n,\mathbb{R}) := \{A \in \mathsf{GL}(n,\mathbb{R}) \,:\, \det A = 1\} = \det{}^{-1}(1).$$

It follows that $\mathsf{SL}(n,\mathbb{R})$ is a closed subgroup of $\mathsf{GL}(n,\mathbb{R})$ and hence is a linear Lie group. [The preimage of a closed set under a continuous map is a closed set.] We introduce a new matrix norm on $\mathbb{R}^{n\times n}$, called the **Frobenius norm**, as follows :

$$\|A\|_F := \sqrt{\operatorname{tr}(A^{\top}A)} = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}.$$

NOTE : The Frobenius norm coincides with the Euclidean norm on $\mathbb{R}^{n^2}$, and is much easier to compute than the operator norm. However, *all* matrix norms on $\mathbb{R}^{n\times n}$ are *equivalent* (i.e., they generate the same metric topology).

We shall use this (matrix) norm to show that $\mathsf{SL}\,(n,\mathbb{R})$ is *not* compact. Indeed, all matrices of the form

$$\begin{bmatrix} 1 & 0 & \dots & t \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

are elements of $\mathsf{SL}\,(n,\mathbb{R})$ whose norm equals $\sqrt{n+t^2}$ for any $t \in \mathbb{R}$. Thus $\mathsf{SL}\,(n,\mathbb{R})$ is *not* a bounded subset of $\mathbb{R}^{n\times n}$ and hence is *not* compact. [In a metric space, any compact set is bounded.]

NOTE : The special linear group $\mathsf{SL}\,(n,\mathbb{R})$ is *connected.*

4.3.3. *The orthogonal and special orthogonal groups* $\mathsf{O}\,(n)$ *and* $\mathsf{SO}\,(n)$. The set

$$\mathsf{O}\,(n) := \{A \in \mathbb{R}^{n\times n} \,:\, A^\top A = \mathbf{1}\}$$

is the **orthogonal group**. Clearly, every *orthogonal matrix* $A \in \mathsf{O}\,(n)$ has an inverse, namely $A^\top$. Hence $\mathsf{O}\,(n) \subset \mathsf{GL}\,(n,\mathbb{R})$.

$\diamond$ **Exercise 55.** Verify that $\mathsf{O}\,(n)$ is a *subgroup* of the general linear group $\mathsf{GL}\,(n,\mathbb{R})$.

The single matrix equation $A^\top A = \mathbf{1}$ is equivalent to n^2 equations for the n^2 real numbers $a_{ij},\ i,j = \overline{1,n}$:

$$\sum_{k=1}^{n} a_{ki}a_{kj} = \delta_{ij}.$$

This means that $\mathsf{O}\,(n)$ is a *closed* subset of $\mathbb{R}^{n\times n}$ and hence of $\mathsf{GL}\,(n,\mathbb{R})$.

$\diamond$ **Exercise 56.** Prove that $\mathsf{O}\,(n)$ is a closed subset of $\mathbb{R}^{n^2}$.

Thus $\mathsf{O}\,(n)$ is a linear Lie group. The group $\mathsf{O}\,(n)$ is also *bounded* in $\mathbb{R}^{n\times n}$. Indeed, the (Frobenius) norm of $A \in \mathsf{O}\,(n)$ is

$$\|A\|_F = \sqrt{\operatorname{tr}\,(A^\top A)} = \sqrt{\operatorname{tr}\mathbf{1}} = \sqrt{n}.$$

Hence the group $\mathsf{O}\,(n)$ is *compact.* [A subset of $\mathbb{R}^{n\times n}$ is compact if and only if it is closed and bounded.] Let us consider the determinant function (restricted to $\mathsf{O}\,(n)$), $\det : \mathsf{O}\,(n) \to \mathbb{R}^\times$. Then for $A \in \mathsf{O}\,(n)$

$$\det \mathbf{1} = \det\,(A^\top A) = \det A^\top \cdot \det A = (\det A)^2.$$

Hence $\det A = \pm 1$. So we have

$$\mathsf{O}\,(n) = \mathsf{O}^+\,(n) \cup \mathsf{O}^-\,(n)$$

where

$$\mathsf{O}^+\,(n) := \{A \in \mathsf{O}\,(n) \,:\, \det A = 1\} \quad \text{and} \quad \mathsf{O}^-\,(n) := \{A \in \mathsf{O}\,(n) \,:\, \det A = -1\}.$$

NOTE : The group $\mathsf{O}^+(n)$ is *connected*, which proves that $\mathsf{O}^+(n)$ *is the connected component of the identity in* $\mathsf{O}(n)$.

The **special orthogonal group** is defined as

$$\mathsf{SO}(n) := \mathsf{O}(n) \cap \mathsf{SL}(n, \mathbb{R}).$$

That is,

$$\mathsf{SO}(n) = \{A \in \mathsf{O}(n) \,:\, \det A = 1\} = \mathsf{O}^+(n).$$

It follows that $\mathsf{SO}(n)$ is a closed subset of $\mathsf{O}(n)$ and hence is *compact*. [A closed subset of a compact set is compact.]

NOTE : One of the main reasons for the study of these groups $\mathsf{O}(n)$, $\mathsf{SO}(n)$ is their relationship with *isometries* (i.e., distance-preserving transformations on the Euclidean space $\mathbb{R}^n$). If such an isometry fixes the origin, then it is actually a linear transformation and so – with respect to the standard basis – corresponds to a matrix A. The isometry condition is equivalent to the fact that (for all $x, y \in \mathbb{R}^n$)

$$Ax \bullet Ay = x \bullet y$$

which in turn is equivalent to the condition that $A^\top A = \mathbf{1}$ (i.e., A is orthogonal). Elements of $\mathsf{SO}(n)$ are (identified with) *rotations* (or direct isometries); elements of $\mathsf{O}^-(n)$ are sometimes referred to as indirect isometries.

4.3.4. *The Lorentz group* $\mathsf{Lor}(1, n)$. Consider the inner product (i.e., non-degenerate symmetric bilinear form) $\odot$ on the vector space $\mathbb{R}^{n+1}$ given by (for $x, y \in \mathbb{R}^{n+1}$)

$$x \odot y := -x_1 y_1 + \sum_{i=2}^{n+1} x_i y_i$$

(the so-called **Minkowski product**). It is standard to denote this inner product space by $\mathbb{R}^{1,n}$.

$\diamond$ **Exercise 57.** Show that the group of all linear isometries (i.e., linear transformations on $\mathbb{R}^{1,n}$ that preserve the Minkowski product) is *isomorphic* to the matrix group

$$\mathsf{O}(1, n) := \left\{A \in \mathsf{GL}(n+1, \mathbb{R}) \,:\, A^\top S A = S\right\}$$

where

$$S = \operatorname{diag}(-1, 1, \ldots, 1) = \begin{bmatrix} -1 & 0 \\ 0 & \mathbf{1} \end{bmatrix} \in \mathsf{GL}(n+1, \mathbb{R}).$$

In a similar fashion, one can define more general matrix groups

$$\mathsf{O}(k, \ell) \leq \mathsf{GL}(k+\ell, \mathbb{R}) \quad \text{and} \quad \mathsf{SO}(k, \ell) \leq \mathsf{SL}(k+\ell, \mathbb{R})$$

usually called "pseudo-orthogonal" groups (this is PROBLEM 21).

NOTE : Since $\mathsf{O}(k, \ell)$ and $\mathsf{O}(\ell, k)$ are essentially the same group, we may assume (without any loss of generality) that $1 \leq k \leq \ell$. *The pseudo-orthogonal groups are neither compact nor* *connected.* The groups $\mathsf{O}\,(k,\ell)$ have four (connected) components, whereas the groups $\mathsf{SO}\,(k,\ell)$ have two components.

For each positive number $\rho > 0$, the *hyperboloid*

$$\mathcal{H}_{1,n}(\rho) := \left\{x \in \mathbb{R}^{1,n} \,:\, x \odot x = -\rho\right\}$$

has two (connected) components

$$\mathcal{H}^{+}_{1,n}(\rho) = \{x \in \mathcal{H}_{1,n}(\rho) \,:\, x_1 > 0\} \quad \text{and} \quad \mathcal{H}^{-}_{1,n}(\rho) = \{x \in \mathcal{H}_{1,n}(\rho) \,:\, x_1 < 0\}\,.$$

We define the **Lorentz group** $\mathsf{Lor}\,(1,n)$ to be the (closed) subgroup of $\mathsf{SO}\,(1,n)$ preserving each of the connected sets $\mathcal{H}^{\pm}_{1,n}(1)$. Thus

$$\mathsf{Lor}\,(1,n) := \left\{A \in \mathsf{SO}\,(1,n) \,:\, A\mathcal{H}^{\pm}_{1,n}(1) = \mathcal{H}^{\pm}_{1,n}(1)\right\} \leq \mathsf{SO}\,(1,n).$$

It turns out that $A \in \mathsf{Lor}\,(1,n)$ <u>if and only if</u> it preserves the hyperboloids $\mathcal{H}^{\pm}_{1,n}(\rho)$, $\rho > 0$ and the "light cones" $\mathcal{H}^{\pm}_{1,n}(0)$.

NOTE : The Lorentz group $\mathsf{Lor}\,(1,n)$ is *connected.*

Of particular interest in physics is the Lorentz group $\mathsf{Lor} = \mathsf{Lor}\,(1,3)$. That is,

$$\mathsf{Lor} = \left\{L \in \mathsf{SO}\,(1,3) \,:\, L\mathcal{H}^{\pm}_{1,3}(\rho) = \mathcal{H}^{\pm}_{1,3}(\rho),\ \rho \geq 0\right\} \leq \mathsf{SO}\,(1,3).$$

NOTE : We can write

$$\begin{aligned}
\mathsf{SO}\,(1,1) &= \mathsf{Lor}\,(1,1) \cup \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \mathsf{Lor}\,(1,1) \\
\mathsf{O}\,(1,1) &= \mathsf{SO}\,(1,1) \cup \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathsf{SO}\,(1,1).
\end{aligned}$$

(See also PROBLEM 22.)

4.3.5. *The real symplectic group* $\mathsf{Sp}\,(2n,\mathbb{R})$. Let

$$\mathbb{J} := \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix} \in \mathsf{SL}\,(2n,\mathbb{R}).$$

A matrix $A \in \mathbb{R}^{2n\times 2n}$ is called *symplectic* if

$$A^{\top}\mathbb{J}A = \mathbb{J}.$$

NOTE : The word *symplectic* was invented by HERMANN WEYL (1885-1955), who substituted Greek for Latin roots in the word *complex* to obtain a term which would describe a group (related to "line complexes" but which would not be confused with complex numbers).

Let $\mathsf{Sp}\,(2n,\mathbb{R})$ be the set of all $2n \times 2n$ symplectic matrices. Taking determinants of the condition $A^{\top}\mathbb{J}A = \mathbb{J}$ gives

$$1 = \det \mathbb{J} = (\det A^{\top}) \cdot (\det \mathbb{J}) \cdot (\det A) = (\det A)^2.$$

Hence $\det A = \pm 1$, and so $A \in \mathsf{GL}\,(2n, \mathbb{R})$. Furthermore, if $A, B \in \mathsf{Sp}\,(2n, \mathbb{R})$, then

$$(AB)^\top \mathbb{J}(AB) = B^\top A^\top \mathbb{J} AB = \mathbb{J}.$$

Hence $AB \in \mathsf{Sp}\,(2n, \mathbb{R})$. Now, if $A^\top \mathbb{J} A = \mathbb{J}$, then

$$\mathbb{J}A = (A^\top)^{-1}\mathbb{J} = (A^{-1})^\top \mathbb{J}$$

so

$$\mathbb{J} = (A^{-1})^\top \mathbb{J} A^{-1}.$$

It follows that $A^{-1} \in \mathsf{Sp}\,(2n, \mathbb{R})$ and hence $\mathsf{Sp}\,(2n, \mathbb{R})$ is a group. In fact, it is a *closed* subgroup of $\mathsf{GL}\,(2n, \mathbb{R})$, and thus a linear Lie group.

NOTE : The symplectic group $\mathsf{Sp}\,(2n, \mathbb{R})$ is *connected*. (It turns out that the determinant of a symplectic matrix must be positive; this fact is by no means obvious.)

◇ **Exercise 58.** Check that $\mathsf{Sp}\,(2, \mathbb{R}) = \mathsf{SL}\,(2, \mathbb{R})$. (In general, it is not true that $\mathsf{Sp}\,(2n, \mathbb{R}) = \mathsf{SL}\,(2n, \mathbb{R})$.)

All matrices of the form

$$\begin{bmatrix} \mathbf{1} & 0 \\ t\mathbf{1} & \mathbf{1} \end{bmatrix} \in \mathsf{SL}\,(2n, \mathbb{R})$$

are symplectic. However, the (Frobenius) norm of such a matrix is equal to $\sqrt{2n + t^2 n}$, which is *unbounded* if $t \in \mathbb{R}$. Therefore, $\mathsf{Sp}\,(2n, \mathbb{R})$ is not a bounded subset of $\mathbb{R}^{2n \times 2n}$ and hence is *not* compact.

◇ **Exercise 59.** Consider the skew-symmetric bilinear form on (the vector space) $\mathbb{R}^{2n}$ defined by

$$\Omega(x, y) := \sum_{i=1}^{n} (x_i y_{n+i} - x_{n+i} y_i)$$

(the standard *symplectic form* or the "canonical" symplectic structure). Show that a linear transformation (on $\mathbb{R}^{2n}$) $x \mapsto Ax$ preserves the symplectic form Ω if and only if $A^\top \mathbb{J} A = \mathbb{J}$ (i.e., the matrix A is symplectic). Such a structure-preserving transformation is called a *symplectic transformation*.

The group of all symplectic transformations on $\mathbb{R}^{2n}$ (equipped with the symplectic form Ω) is *isomorphic* to the linear Lie group $\mathsf{Sp}\,(2n, \mathbb{R})$.

NOTE : The symplectic group is related to *classical mechanics*. Consider a particle of mass m moving in a *potential field* V. Newton's second law states that the particle moves along a curve $t \mapsto x(t)$ in Euclidean space $\mathbb{R}^3$ in such a way that $m\ddot{x} = -\operatorname{grad} V(x)$. Introduce the conjugate *momenta* $p_i = m\dot{x}_i$, $i = 1, 2, 3$ and the *energy* (Hamiltonian)

$$H(x, p) := \frac{1}{2m} \sum_{i=1}^{3} p_i^2 + V(x).$$

Then

$$\frac{\partial H}{\partial x_i} = \frac{\partial V}{\partial x_i} = -m\ddot{x}_i = -\dot{p}_i \qquad \text{and} \qquad \frac{\partial H}{\partial p_i} = \frac{1}{m}p_i = \dot{x}_i$$

and hence *Newton's law* $\mathbf{F} = m\,a$ is equivalent to *Hamilton's equations*

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial x_i} \qquad (i = 1, 2, 3).$$

Writing $z = (x, p)$,

$$\mathbb{J} \cdot \operatorname{grad} H(z) = \begin{bmatrix} 0 & I_3 \\ -I_3 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \\ \frac{\partial H}{\partial p} \end{bmatrix} = (\dot{x}, \dot{p}) = \dot{z}$$

so Hamilton equations read $\dot{z} = \mathbb{J} \cdot \operatorname{grad} H(z)$. Now let

$$F : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$$

and write $w(t) = F(z(t))$. If $z(t)$ satisfies Hamilton's equations

$$\dot{z} = \mathbb{J} \cdot \operatorname{grad} H(z)$$

then $w(t) = F(z(t))$ satisfies $\dot{w} = A^\top \dot{z}$, where $A^\top = [\partial w^i / \partial z^j]$ is the Jacobian matrix of F. By the chain rule,

$$\dot{w} = A^\top \mathbb{J} \operatorname{grad}_z H(z) = A^\top \mathbb{J} A \operatorname{grad}_w H(z(w)).$$

Thus, the equations for $w(t)$ have the form of Hamilton's equations with energy $K(w) = H(z(w))$ if and only if $A^\top \mathbb{J} A = \mathbb{J}$; that is, if and only if A is symplectic. A nonlinear transformation F is *canonical* if and only if its Jacobian matrix is symplectic (or, if one prefers, its tangent mapping is a symplectic transformation).

As a special case, consider a (linear transformation) $A \in \mathsf{Sp}\,(2n, \mathbb{R})$ and let $w = Az$. Suppose H is quadratic (i.e., of the form $H(z) = \frac{1}{2} z^\top B z$ where B is a symmetric matrix). Then $\operatorname{grad} H(z) = Bz$ and thus the equations of motion become the linear equations $\dot{z} = \mathbb{J} B z$. Now

$$\dot{w} = A\dot{z} = A\mathbb{J}Bz = \mathbb{J}(A^\top)^{-1}Bz = \mathbb{J}(A^\top)^{-1}BA^{-1}Az = \mathbb{J}B'w$$

where $B' = (A^\top)^{-1} B A^{-1}$ is symmetric. For the new Hamiltonian we get

$$\begin{aligned} H'(w) &= \frac{1}{2} w^\top (A^\top)^{-1} B A^{-1} w = \frac{1}{2} (A^{-1}w)^\top B A^{-1} w \\ &= H(A^{-1}w) = H(z). \end{aligned}$$

Thus $\mathsf{Sp}\,(2n, \mathbb{R})$ *is the linear invariance group of classical mechanics.*

4.3.6. *The complex general linear group* $\mathsf{GL}\,(n, \mathbb{C})$. Many important matrix groups involve *complex* matrices. As in the real case,

$$\mathsf{GL}\,(n, \mathbb{C}) := \{A \in \mathbb{C}^{n \times n} \,:\, \det A \neq 0\}$$

is an *open* subset of $\mathbb{C}^{n \times n}$, and hence is *not* compact. Clearly $\mathsf{GL}\,(n, \mathbb{C})$ is a group under matrix multiplication.

Note : The general linear group $\mathsf{GL}\,(n, \mathbb{C})$ is *connected*. This is in contrast with the fact that $\mathsf{GL}\,(n, \mathbb{R})$ has two components.

4.3.7. *The complex special linear group* $\mathsf{SL}(n,\mathbb{C})$. This group is defined by

$$\mathsf{SL}(n,\mathbb{C}) := \{A \in \mathsf{GL}(n,\mathbb{C}) \,:\, \det A = 1\}$$

and is treated as in the real case. The matrix group $\mathsf{SL}(n,\mathbb{C})$ is *not* compact but *connected.*

4.3.8. *The unitary and special unitary groups* $\mathsf{U}(n)$ *and* $\mathsf{SU}(n)$. For $A = [a_{ij}] \in \mathbb{C}^{n\times n}$,

$$A^* := \bar{A}^\top = \overline{A^\top}$$

is the *Hermitian conjugate* (i.e., the conjugate transpose) matrix of A; thus, $(A^*)_{ij} = \bar{a}_{ji}$. The **unitary group** is defined as

$$\mathsf{U}(n) := \{A \in \mathsf{GL}(n,\mathbb{C}) \,:\, A^*A = \mathbf{1}\}.$$

$\diamond$ **Exercise 60.** Verify that $\mathsf{U}(n)$ is a *subgroup* of the general linear group $\mathsf{GL}(n,\mathbb{C})$.

The unitary condition amounts to n^2 equations for the n^2 complex numbers a_{ij}, $i,j = \overline{1,n}$

$$\sum_{k=1}^{n} \bar{a}_{ki}a_{kj} = \delta_{ij}.$$

By taking real and imaginary parts, these equations actually give $2n^2$ equations in the $2n^2$ real and imaginary parts of the a_{ij} (although there is some redundancy). This means that $\mathsf{U}(n)$ is a *closed* subset of $\mathbb{C}^{n\times n} = \mathbb{R}^{2n^2}$ and hence of $\mathsf{GL}(n,\mathbb{C})$. Thus $\mathsf{U}(n)$ is a complex linear Lie group.

NOTE : The unitary group $\mathsf{U}(n)$ is *compact* and *connected.*

Let $A \in \mathsf{U}(n)$. From $|\det A| = 1$, we see that the determinant function $\det : \mathsf{GL}(n,\mathbb{C}) \to \mathbb{C}$ maps $\mathsf{U}(n)$ onto the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} \,:\, |z| = 1\}$.

NOTE : In the special case $n = 1$, a complex linear mapping $\phi : \mathbb{C} \to \mathbb{C}$ is multiplication by some complex number z, and ϕ is an *isometry* if and only if $|z| = 1$. In this way, the unitary group $\mathsf{U}(1)$ is *identified* with the unit circle $\mathbb{S}^1$. The group $\mathsf{U}(1)$ is more commonly known as the *circle group* or the one-dimensional *torus*, and is also denoted by $\mathbb{T}^1$.

The dot product on $\mathbb{R}^n$ can be extended to $\mathbb{C}^n$ by setting (for $x, y \in \mathbb{C}^{n\times 1}$)

$$x \bullet y := x^*y = \bar{x}_1y_1 + \bar{x}_2y_2 + \cdots + \bar{x}_ny_n.$$

NOTE : This is *not* $\mathbb{C}$-linear but satisfies (for $x, y \in \mathbb{C}^{n\times 1}$ and $u, v \in \mathbb{C}$)

$$(ux) \bullet (vy) = \bar{u}v\,(x \bullet y).$$

This dot product allows us to define the *norm* of a complex vector $x \in \mathbb{C}^{n\times 1}$ by

$$\|x\| := \sqrt{x \bullet x}.$$

Then a matrix $A \in \mathbb{C}^{n\times n}$ is *unitary* if and only if

$$Ax \bullet Ay = x \bullet y \qquad (x, y \in \mathbb{C}^n).$$

$\diamond$ **Exercise 61.** If $\mathsf{G}_i \leq \mathsf{GL}\,(n_i, \Bbbk)$, $i = 1, 2$ are linear Lie groups, show that their (direct) *product* $\mathsf{G}_1 \times \mathsf{G}_2$ is also a linear Lie group (in $\mathsf{GL}\,(n_1 + n_2, \Bbbk)$). Observe, in particular, that the k-dimensional *torus*

$$\mathbb{T}^k := \mathbb{T}^1 \times \mathbb{T}^1 \times \cdots \times \mathbb{T}^1$$

is a linear Lie group (in $\mathsf{GL}\,(k, \mathbb{C})$). These groups are compact connected Abelian linear Lie groups. (In fact, they are the *only* linear Lie groups with these properties.)

The **special unitary group**

$$\mathsf{SU}\,(n) := \{A \in \mathsf{U}\,(n) \,:\, \det A = 1\}$$

is a closed subgroup of $\mathsf{U}\,(n)$ and hence a complex matrix group.

NOTE : The matrix group $\mathsf{SU}\,(n)$ is *compact* and *connected*. In the special case $n = 2$, $\mathsf{SU}\,(2)$ is *diffeomorphic* to the unit sphere $\mathbb{S}^3$ in $\mathbb{C}^2$ (or $\mathbb{R}^4$). The group $\mathsf{SU}\,(2)$ is used in the construction of the gauge group for the Yang-Mills equations in *elementary particle physics*. Also, there is a two-to-one surjection (in fact, a surjective *submersion*)

$$\pi : \mathsf{SU}\,(2) \to \mathsf{SO}\,(3)$$

which is of crucial importance in *computational mechanics* (it is related to the quaternionic representation of rotations in Euclidean space $\mathbb{R}^3$).

4.3.9. *The complex orthogonal groups* $\mathsf{O}\,(n, \mathbb{C})$ *and* $\mathsf{SO}\,(n, \mathbb{C})$. Consider the bilinear form on the vector space $\mathbb{C}$ defined by

$$(x, y) := x_1y_1 + x_2y_2 + \cdots + x_ny_n \qquad (x, y \in \mathbb{C}^n).$$

This form is *not* an inner product because of the lack of complex conjugation in the definition. The set of all complex $n \times n$ matrices which preserve this form (i.e., such that $(Ax, Ay) = (x, y)$ for all $x, y \in \mathbb{C}^n$) is the **complex orthogonal group** $\mathsf{O}\,(n, \mathbb{C})$. Thus

$$\mathsf{O}\,(n, \mathbb{C}) := \left\{A \in \mathsf{GL}\,(n, \mathbb{C}) \,:\, A^\top A = \mathbf{1}\right\} \subset \mathsf{GL}\,(n, \mathbb{C}).$$

It is easy to show that $\mathsf{O}\,(n, \mathbb{C})$ is a liner Lie group, and that $\det A = \pm 1$ for all $\mathsf{O}\,(n, \mathbb{C})$.

NOTE : The linear Lie group $\mathsf{O}\,(n, \mathbb{C})$ is not the same as the unitary group $\mathsf{U}\,(n)$.

The **complex special orthogonal group**

$$\mathsf{SO}\,(n, \mathbb{C}) := \{A \in \mathsf{O}\,(n, \mathbb{C}) \,:\, \det A = 1\}$$

is also a linear Lie group.

4.3.10. *The unipotent group* $\mathsf{UT}(n,\Bbbk)$. A matrix $A = [a_{ij}] \in \Bbbk^{n\times n}$ is *upper triangular* if all the entries below the main diagonal are equal to 0. Let $\mathsf{T}(n,\Bbbk)$ denote the set of all $n\times n$ *invertible* upper triangular matrices (over $\Bbbk$). Thus

$$\mathsf{T}(n,\Bbbk) := \{A \in \mathsf{GL}(n,\Bbbk) \;:\; a_{ij} = 0 \;\text{ for } i > j\}.$$

◇ **Exercise 62.** Show that $\mathsf{T}(n,\Bbbk)$ is a *closed* subgroup of the general linear group $\mathsf{GL}(n,\Bbbk)$ and hence a linear Lie group.

The group $\mathsf{T}(n,\Bbbk)$ is called the (upper) **triangular group**. This group is *not* compact.

NOTE : Likewise, one can define the *lower* triangular group

$$\widetilde{\mathsf{T}}(n,\Bbbk) := \{A \in \mathsf{GL}(n,\Bbbk) \;:\; a_{ij} = 0 \;\text{ for } i < j\}.$$

Clearly, $A \in \widetilde{\mathsf{T}}(n,\Bbbk)$ if and only if $A^\top \in \mathsf{T}(n,\Bbbk)$. The matrix groups $\mathsf{T}(n,\Bbbk)$ and $\widetilde{\mathsf{T}}(n,\Bbbk)$ are *isomorphic* and there is no need to distinguish between them.

◇ **Exercise 63.** Show that the **diagonal group**

$$\mathsf{D}(n,\Bbbk) := \{A \in \mathsf{GL}(n,\Bbbk) \;:\; a_{ij} = 0 \;\text{ for } i \neq j\}$$

is a closed subgroup of $\mathsf{T}(n,\Bbbk)$ and hence a *linear Lie group.*

◇ **Exercise 64.** For $k \leq n$, let $\mathsf{P}(k)$ denote the group of all linear transformations (i.e., invertible linear mappings) on $\mathbb{R}^n$ that preserve the subspace $\mathbb{R}^k = \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n$. Show that $\mathsf{P}(k)$ is (*isomorphic* to) the matrix group

$$\left\{ \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \;:\; A \in \mathsf{GL}(k,\mathbb{R}),\, B \in \mathsf{GL}(n-k,\mathbb{R}),\, X \in \mathbb{R}^{k\times(n-k)} \right\}.$$

An upper triangular matrix $A = [a_{ij}]$ is *unipotent* if it has all diagonal entries equal to 1. The (real or complex) **unipotent group** is (the subgroup of $\mathsf{GL}(n,\Bbbk)$)

$$\mathsf{UT}(n,\Bbbk) := \{A \in \mathsf{GL}(n,\Bbbk) \;:\; a_{ij} = 0 \;\text{ for } i > j \;\text{ and } a_{ii} = 1\}.$$

It is easy to see that the unipotent group $\mathsf{UT}(n,\Bbbk)$ is a closed subgroup of $\mathsf{GL}(n,\Bbbk)$ and hence a *liner Lie group.*

NOTE : $\mathsf{UT}(n,\Bbbk)$ is a closed subgroup of $\mathsf{T}(n,\Bbbk)$.

For the case

$$\mathsf{UT}(2,\Bbbk) = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \in \mathsf{GL}(n,\Bbbk) \;:\; t \in \Bbbk \right\}$$

the mapping

$$\theta : \Bbbk \to \mathsf{UT}(2,\Bbbk), \quad t \mapsto \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

is a *continuous* group homomorphism which is an isomorphism with continuous inverse. This allows us to *view* $\Bbbk$ *as a linear Lie group.*

NOTE : Given two linear Lie groups G and $\overline{\mathsf{G}}$, a group homomorphism $\theta : \mathsf{G} \to \overline{\mathsf{G}}$ is a *continuous homomorphism* if it is continuous and its image $\theta(\mathsf{G}) \leq \overline{\mathsf{G}}$ is a closed subset of $\overline{\mathsf{G}}$. For instance,

$$\theta : \mathsf{UT}\,(2,\mathbb{R}) \to \mathsf{U}\,(1), \qquad \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mapsto e^{2\pi t i}$$

is a continuous homomorphism of matrix groups, but (for $a \in \mathbb{R} \setminus \mathbb{Q}$)

$$\theta' : \mathsf{G} = \left\{ \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \in \mathsf{UT}\,(2,\mathbb{R}) \,:\, k \in \mathbb{Z} \right\} \to \mathsf{U}\,(1), \qquad \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \mapsto e^{2\pi k a i}$$

is *not* (since its image is a *dense* proper subset of $\mathsf{U}\,(1)$). Whenever we have a continuous homomorphism of linear Lie groups $\theta : \mathsf{G} \to \overline{\mathsf{G}}$ which is a *homeomorphism* (i.e., a continuous bijection with continuous inverse) we say that θ is a *continuous isomorphism* and regard G and $\overline{\mathsf{G}}$ as "identical" (as linear Lie groups).

The unipotent group $\mathsf{UT}\,(3,\mathbb{R})$ is the **Heisenberg group**

$$\mathsf{H}_3 := \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \,:\, a, b, c \in \mathbb{R} \right\}$$

which is particularly important in *quantum physics*; the *Lie algebra* of H_3 gives a realization of the *Heisenberg commutation relations* of quantum mechanics.

◇ **Exercise 65.** Verify that the 4×4 unipotent matrices A of the form

$$A = \begin{bmatrix} 1 & a_2 & a_3 & a_4 \\ 0 & 1 & a_1 & \frac{a_1^2}{2} \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

form a closed subgroup of $\mathsf{UT}\,(4,\mathbb{R})$ and hence a linear Lie group. Generalize (to $n \times n$ matrices).

Several other matrix groups are of great interest. We describe briefly some of them.

4.3.11. *The general affine group* $\mathsf{AGL}\,(n,\Bbbk)$. The **general affine group** (over $\Bbbk$) is the group

$$\mathsf{AGL}\,(n,\Bbbk) := \left\{ \begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix} \in \mathsf{GL}\,(n+1,\Bbbk) \,:\, c \in \Bbbk^{n\times 1} \text{ and } A \in \mathsf{GL}\,(n,\Bbbk) \right\}.$$

This is clearly a closed subgroup of the general linear group $\mathsf{GL}\,(n+1,\Bbbk)$ and hence a *linear Lie group*. The general affine group $\mathsf{AGL}\,(n,\Bbbk)$ is *not* compact. Likewise the case of the general linear group, the linear Lie group $\mathsf{AGL}\,(n,\mathbb{C})$ is connected but $\mathsf{AGL}\,(n,\mathbb{R})$ is <u>not</u>.

NOTE : If we identify the element $x \in \Bbbk^n$ with $\begin{bmatrix} 1 \\ x \end{bmatrix} \in \Bbbk^{(n+1)\times 1}$, then since

$$\begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ Ax + c \end{bmatrix}$$

we obtain an *action* of the group $\mathsf{AGL}\,(n, \Bbbk)$ on the vector space $\Bbbk^n$. Transformations on $\Bbbk^n$ having the form $x \mapsto Ax + c$ (with A invertible) are called *affine transformations* and they preserve *lines* (i.e., one-dimensional linear submanifolds of $\Bbbk^n$). The associated geometry is **affine geometry** that has $\mathsf{AGL}\,(n, \Bbbk)$ as its symmetry group.

The (additive group of the) vector space $\Bbbk^n$ (in fact, $\Bbbk^{n\times 1}$) can be viewed as (and identified with) the *translation subgroup* of $\mathsf{AGL}\,(n, \Bbbk)$

$$\left\{ \begin{bmatrix} 1 & 0 \\ c & \mathbf{1} \end{bmatrix} \in \mathsf{GL}\,(n+1, \Bbbk) \;:\; c \in \Bbbk^{n\times 1} \right\} \leq \mathsf{AGL}\,(n, \Bbbk)$$

and this is a closed subgroup.

The identity component of the general affine group $\mathsf{AGL}\,(n, \mathbb{R})$ is (the linear Lie group)

$$\mathsf{AGL}^+\,(n, \mathbb{R}) = \left\{ \begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix} \;:\; c \in \mathbb{R}^{n\times 1} \text{ and } A \in \mathsf{GL}^+\,(n, \mathbb{R}) \right\}.$$

In particular,

$$\mathsf{AGL}^+\,(1, \mathbb{R}) = \left\{ \begin{bmatrix} 1 & 0 \\ c & e^a \end{bmatrix} \;:\; a, c \in \mathbb{R} \right\}$$

is a *connected* linear Lie group (of "dimension" 2). Its elements are (in fact, can be identified with) transformations of the real line $\mathbb{R}$ having the form $x \mapsto bx + c$ (with $b, c \in \mathbb{R}$ and $b > 0$).

4.3.12. *The Euclidean group* $\mathsf{E}\,(n)$. This is the matrix group

$$\mathsf{E}\,(n) := \left\{ \begin{bmatrix} 1 & 0 \\ c & A \end{bmatrix} \in \mathsf{GL}\,(n+1, \mathbb{R}) \;:\; c \in \mathbb{R}^{n\times 1} \text{ and } A \in \mathsf{O}\,(n) \right\}.$$

The Euclidean group $\mathsf{E}\,(n)$ is a closed subgroup of the general affine group $\mathsf{AGL}\,(n, \mathbb{R})$ and also is neither compact nor connected. It can be viewed as (and thus identified with) the group of all *isometries* (i.e., *rigid motions*) of the Euclidean space $\mathbb{R}^n$.

4.3.13. *The special Euclidean group* $\mathsf{SE}\,(n)$. The **special Euclidean group** $\mathsf{SE}\,(n)$ is (the linear Lie group) defined by

$$\mathsf{SE}\,(n) := \left\{ \begin{bmatrix} 1 & 0 \\ c & R \end{bmatrix} \in \mathsf{GL}\,(n+1, \mathbb{R}) \;:\; c \in \mathbb{R}^{n\times 1} \text{ and } R \in \mathsf{SO}\,(n) \right\}.$$

This group is *isomorphic* to the group of all *orientation-preserving isometries* (i.e., *proper rigid motions*) on the Euclidean space $\mathbb{R}^n$. It is *not* compact but *connected*.

4.3.14. *Further examples.* Several important groups which are not naturally groups of matrices can be viewed as linear Lie groups. We have seen that the multiplicative groups $\mathbb{R}^\times$ and $\mathbb{C}^\times$ (of non-zero real numbers and complex numbers, respectively) are *isomorphic* to the linear Lie groups $\mathsf{GL}\,(1,\mathbb{R})$ and $\mathsf{GL}\,(1,\mathbb{C})$, respectively. Also, the *circle group* $\mathbb{S}^1$ (of complex numbers with absolute value one) is *isomorphic* to $\mathsf{U}\,(1)$. The *n-torus* (the direct product of n copies of $\mathbb{S}^1$)

$$\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1 \leq \mathsf{GL}\,(n,\mathbb{C})$$

is *isomorphic* to the linear Lie group of $n \times n$ diagonal matrices with complex entries of modulus one. ($\mathbb{T}^n$ can also be realized as the *quotient group* $\mathbb{R}^n/\mathbb{Z}^n$: an element $(\theta_1,\dots,\theta_n) \mod \mathbb{Z}^n$ of $\mathbb{R}^n/\mathbb{Z}^n$ can be identified with the diagonal matrix $\operatorname{diag}\left(e^{2\pi i\theta_1},\dots,e^{2\pi i\theta_n}\right)$.)

Note : If $\theta : \mathsf{G} \to \overline{\mathsf{G}}$ is a continuous homomorphism of linear Lie groups, then its *kernel* $\operatorname{Ker}\theta \leq \mathsf{G}$ is a linear Lie group. Moreover, the *quotient group* $\mathsf{G}/\operatorname{Ker}\theta$ can be identified with the linear Lie group $\operatorname{Im}\theta$ by the usual quotient isomorphism $\widetilde{\theta} : \mathsf{G}/\operatorname{Ker}\theta \to \operatorname{Im}\theta$. However, it is important to realize that *not every normal matrix subgroup* H *of the linear Lie group* G *gives rise to a linear Lie group* G/H; there are examples for which G/H is a *Lie group* but not a linear Lie group. (It is true, but by no means obvious, that every linear Lie group is in fact a Lie group.)

Recall that the (additive) groups $\mathbb{R}$ and $\mathbb{C}$ are *isomorphic* to the unipotent groups $\mathsf{UT}\,(2,\mathbb{R})$ and $\mathsf{UT}\,(2,\mathbb{C})$, respectively.

◇ **Exercise 66.** Verify that the map

$$x \in \mathbb{R} \mapsto [e^x] \in \mathsf{GL}^+\,(1,\mathbb{R})$$

is a *continuous isomorphism* of linear Lie groups, and then show that the additive group $\mathbb{R}^n$ is *isomorphic* to the linear Lie group of all $n \times n$ diagonal matrices with positive entries.

The *symmetric group* $\mathfrak{S}_n$ of permutations on n elements may be considered as well as a linear Lie group. Indeed, we can make $\mathfrak{S}_n$ to *act* on $\mathbb{k}^n$ by linear transformations :

$$\sigma \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{\sigma^{-1}(1)} \\ x_{\sigma^{-1}(2)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{bmatrix}.$$

Thus (for the standard unit vectors $e_1, e_2, \dots, e_n$) $\sigma \cdot e_i = e_{\sigma(i)}, \quad i = \overline{1,n}$.

The matrix $[\sigma]$ of the linear transformation induced by $\sigma \in \mathfrak{S}_n$ (with respect to the standard basis) has all its entries 0 or 1, with exactly one 1 in each row and column. Such a matrix is usually called a *permutation matrix*.

$\diamond$ **Exercise 67.** Write down the permutations matrices induces by the elements (permutations) of $\mathfrak{S}_3$.

When $\Bbbk = \mathbb{R}$ each of these permutation matrices is orthogonal, while when $\Bbbk = \mathbb{C}$ it is unitary. So, for a given $n \geq 1$, *the symmetric group* $\mathfrak{S}_n$ *is (isomorphic to) a closed subgroup of* $\mathsf{O}(n)$ *or* $\mathsf{U}(n)$.

NOTE : Any *finite* group is (isomorphic to) a linear Lie subgroup of some orthogonal group $\mathsf{O}(n)$.

4.4. **Complex matrix groups as real matrix groups.** Recall that the (complex) vector space $\mathbb{C}$ can be viewed as a *real* two-dimensional vector space (with basis $\{1, i\}$, for example).

$\diamond$ **Exercise 68.** Show that the mapping

$$\rho : \mathbb{C} \to \mathbb{R}^{2\times 2}, \quad z = x + iy \mapsto \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

is an injective *ring homomorphism* (i.e., a one-to-one mapping such that, for $z, z' \in \mathbb{C}$,

$$\rho(z + z') = \rho(z) + \rho(z') \quad \text{and} \quad \rho(zz') = \rho(z)\rho(z').)$$

We can view $\mathbb{C}$ as a *subring* of $\mathbb{R}^{2\times 2}$. In other words, we can *identify* the complex number $z = x + iy$ with the 2×2 real matrix $\rho(z)$.

NOTE : This can also be expressed as

$$\rho(x + iy) = xI_2 - yJ_2, \quad \text{where} \quad J_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Also, for $z \in \mathbb{C}$,

$$\rho(\bar{z}) = \rho(z)^\top$$

(complex conjugation corresponds to transposition).

More generally, given $Z = [z_{rs}] \in \mathbb{C}^{n\times n}$ with $z_{rs} = x_{rs} + iy_{rs}$, we can write

$$Z = X + iY,$$

where $X = [x_{rs}]$, $Y = [y_{rs}] \in \mathbb{R}^{n\times n}$.

$\diamond$ **Exercise 69.** Show that the mapping

$$\rho_n : \mathbb{C}^{n\times n} \to \mathbb{R}^{2n\times 2n}, \qquad Z = X + iY \mapsto \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}$$

is an injective *ring homomorphism*.

Hence we can *identify* the complex matrix $Z = X + iY$ with the $2n \times 2n$ real matrix $\rho_n(Z)$. Let

$$\mathbb{J} = \mathbb{J}_{2n} := \begin{bmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{bmatrix} \in \mathsf{SL}\,(2n, \mathbb{R}).$$

Then we can write

$$\rho_n(Z) = \rho_n(X + iY) = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} - \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix} \mathbb{J}.$$

◇ **Exercise 70.** First verify that

$$\mathbb{J}^2 = -I_{2n} \quad \text{and} \quad \mathbb{J}^\top = -\mathbb{J}$$

and then show that, for $Z \in \mathbb{C}^{n\times n}$,

$$\rho_n(\bar{Z}) = \rho_n(Z)^\top \iff X = X^\top \quad \text{and} \quad Y = Y^\top.$$

We see that $\rho_n(\mathsf{GL}\,(n, \mathbb{C}))$ is a closed subgroup of $\mathsf{GL}\,(2n, \mathbb{R})$, so *any linear Lie subgroup* G *of* $\mathsf{GL}\,(n, \mathbb{C})$ *can be viewed as a linear Lie subgroup of* $\mathsf{GL}\,(2n, \mathbb{R})$ (by identifying it with its image $\rho_n(\mathsf{G})$ under ρ_n). The following characterizations are sometimes useful:

$$\begin{aligned} \rho_n(\mathbb{C}^{n\times n}) &= \left\{A \in \mathbb{R}^{n\times n} \,:\, A\mathbb{J} = \mathbb{J}A\right\} \\ \rho_n(\mathsf{GL}\,(n, \mathbb{C})) &= \left\{A \in \mathsf{GL}\,(2n, \mathbb{R}) \,:\, A\mathbb{J} = \mathbb{J}A\right\}. \end{aligned}$$

Note : In a slight abuse of notation, the real symplectic group $\mathsf{Sp}\,(2n, \mathbb{R})$ is related to the unitary group $\mathsf{U}\,(n)$ by

$$\mathsf{Sp}\,(2n, \mathbb{R}) \cap \mathsf{O}\,(2n) = \mathsf{U}\,(n).$$

Problems (16–25)

(16) Consider a matrix $A \in \Bbbk^{n\times n}$.

(a) Assume that $\operatorname{rank} A = k$; show that there exist matrices $P, Q \in \mathsf{GL}\,(n, \Bbbk)$ such that

$$A = P \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} Q.$$

(b) Verify that the sequence $(A_r)_{r\in\mathbb{N}}$ in $\mathsf{GL}\,(n, \Bbbk)$ with

$$A_r = P \begin{bmatrix} I_k & 0 \\ 0 & \frac{1}{r} I_{n-k} \end{bmatrix} Q$$

converges to A. Hence deduce that the set $\mathsf{GL}\,(n, \Bbbk)$ is *dense* in $\Bbbk^{n\times n}$. (A set whose closure is the whole space is said to be **dense** in the space.)

(17) Let $A, B \in \mathbb{k}^{n\times n}$. By using the result of PROBLEM 16 or otherwise, prove that the matrices AB and BA have the same *characteristic polynomial* and hence the same eigenvalues. (The **characteristic polynomial** of A is defined by $\text{char}_A(\lambda) := \det(\lambda\mathbf{1} - A) \in \mathbb{k}[\lambda]$.)
[HINT: For an alternative proof, compare the determinants of the two products of the block matrices $\begin{bmatrix} A & -\lambda I_n \\ I_n & 0 \end{bmatrix}$ and $\begin{bmatrix} B & -\lambda I_n \\ -I_n & A \end{bmatrix}$.]

(18) (a) Determine the *center* $\mathsf{Z}(\mathsf{GL}(n,\mathbb{k}))$ of the general linear group $\mathsf{GL}(n,\mathbb{k})$.
(b) Show that
(i) $\mathsf{Z}(\mathsf{GL}(n,\mathbb{k}))$ and $\mathsf{SL}(\mathsf{n},\mathbb{k})$ are *normal* subgroups of $\mathsf{GL}(n,\mathbb{k})$.
(ii) $\mathsf{GL}^+(n,\mathbb{R})$ is a *normal* subgroup of $\mathsf{GL}(n,\mathbb{R})$ (see section 4.3.1).
(iii) for each subset $\mathsf{M} \subseteq \mathbb{k}^{n\times n}$, the **centralizer**

$$\mathsf{Z}_{\mathsf{GL}(n,\mathbb{k})}(\mathsf{M}) := \{A \in \mathsf{GL}(n,\mathbb{k}) \,:\, AX = XA \text{ for all } X \in \mathsf{M}\}$$

is a *closed* subgroup of $\mathsf{GL}(n,\mathbb{k})$.

(19) Let $A \in \mathsf{GL}(n,\mathbb{R})$.
(a) Show that the symmetric matrix $S = A^\top A$ is *positive definite* (i.e., its eigenvalues are all positive real numbers). Deduce that S has a positive definite (real) symmetric square root, i.e., there is a positive definite symmetric matrix S_1 such that $S_1^2 = S$.
(b) Show that the matrix $S_1^{-1}A$ is orthogonal.
(c) If $PR = QS$, where P, Q are positive definite symmetric matrices and $R, S \in \mathsf{O}(n)$, show that $P^2 = Q^2$.
(d) Let S_2 be a positive definite symmetric matrix for which $S_2^2 = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Show that $S_2 = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$.
(e) Show that A can be *uniquely* expressed as $A = PR$, where P is a positive definite symmetric matrix and $R \in \mathsf{O}(n)$. If $\det A > 0$, show that $R \in \mathsf{SO}(n)$. (Such factorization is called **polar decomposition** of A.)

(20) Let $a \in \mathbb{R} \setminus \mathbb{Q}$. Show that

$$\mathsf{G} = \left\{ \begin{bmatrix} e^{it} & 0 \\ 0 & e^{iat} \end{bmatrix} \,:\, t \in \mathbb{R} \right\}$$

is a subgroup of $\mathsf{GL}(2,\mathbb{C})$, and then find a sequence of matrices in G which converges to $-I_2 \notin \mathsf{G}$. This means that G is <u>not</u> a linear Lie group.
[HINT : By taking $t = (2n+1)\pi$ for a suitably chosen $n \in \mathbb{Z}$, we can make ta *arbitrarily close* to an odd integer multiple of π, $(2m+1)\pi$ say. It is sufficient to show that for any positive integer N, there exist $n, m \in \mathbb{Z}$ such that $|(2n+1)a - (2m+1)| < \frac{1}{N}$.]

(21) Define the inner product $\langle \cdot, \cdot \rangle_{k,\ell}$ on $\mathbb{R}^{k+\ell}$ by the formula

$$\langle x, y \rangle_{k,\ell} := -x_1 y_1 - \cdots - x_k y_k + x_{k+1} y_{k+1} + \cdots + x_{k+\ell} y_{k+\ell}.$$

The **pseudo-orthogonal group** $\mathsf{O}(k, \ell)$ consists of all matrices $A \in \mathsf{GL}(k+\ell, \mathbb{R})$ which preserve this inner product (i.e., such that $\langle Ax, Ay \rangle_{k,\ell} = \langle x, y \rangle_{k,\ell}$ for all $x, y \in \mathbb{R}^{k+\ell}$).

(a) Verify that $\mathsf{O}(k, \ell)$ is a *linear Lie subgroup* of $\mathsf{GL}(k+\ell, \mathbb{R})$.

(b) Let

$$Q = \operatorname{diag}(-1, \ldots, -1, 1, \ldots, 1) = \begin{bmatrix} -I_k & 0 \\ 0 & I_\ell \end{bmatrix}.$$

Prove that a matrix $A \in \mathsf{GL}(k+\ell, \mathbb{R})$ is in $\mathsf{O}(k, \ell)$ <u>if and only if</u> $A^\top Q A = Q$. Hence deduce that $\det A = \pm 1$.

(c) Verify that $\mathsf{SO}(k, \ell) := \mathsf{O}(k, \ell) \cap \mathsf{SL}(k+\ell, \mathbb{R})$ is a *linear Lie subgroup* of $\mathsf{SL}(k+\ell, \mathbb{R})$.

(22) Show that

(a) The matrix $A = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$ is in $\mathsf{SO}(1, 1)$.

(b) For every $s, t \in \mathbb{R}$

$$\begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix} \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} = \begin{bmatrix} \cosh(s+t) & \sinh(s+t) \\ \sinh(s+t) & \cosh(s+t) \end{bmatrix}.$$

(c) Every element (matrix) of $\mathsf{O}(1, 1)$ can be written in one of the four forms

$$\begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \begin{bmatrix} -\cosh t & \sinh t \\ \sinh t & -\cosh t \end{bmatrix}, \begin{bmatrix} \cosh t & -\sinh t \\ \sinh t & -\cosh t \end{bmatrix}, \begin{bmatrix} -\cosh t & -\sinh t \\ \sinh t & \cosh t \end{bmatrix}.$$

(Since $\cosh t$ is always positive, there is no overlap among the four cases. Matrices of the first two forms have determinant one; matrices of the last two forms have determinant minus one.)

(23) Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathsf{GL}(2n, \mathbb{R})$, show that $A \in \mathsf{Sp}(2n, \mathbb{R})$ <u>if and only if</u> $a^\top c$ and $b^\top d$ are symmetric and $a^\top d - c^\top b = \mathbf{1}$.

(24) Let $\mathbb{Z}^n \leq \mathbb{R}^n$ be the *discrete* subgroup of vectors with integer entries and set

$$\mathsf{GL}(n, \mathbb{Z}) := \{A \in \mathsf{GL}(n, \mathbb{R}) \ : \ A(\mathbb{Z}^n) = \mathbb{Z}^n\}.$$

Show that $\mathsf{GL}(n, \mathbb{Z})$ is a linear Lie group. (This linear group consists of $n \times n$ matrices over (the ring) $\mathbb{Z}$ with determinant ± 1.)

(25) Verify the folowing set of equalities :

$$\begin{aligned}
\rho_n(\mathsf{U}\,(n)) &= \mathsf{O}\,(n) \cap \rho_n(\mathsf{GL}\,(n, \mathbb{C})) \\
&= \mathsf{O}\,(n) \cap \mathsf{Sp}\,(2n, \mathbb{R}) \\
&= \rho_n(\mathsf{GL}\,(n, \mathbb{C})) \cap \mathsf{Sp}\,(2n, \mathbb{R}).
\end{aligned}$$