6. Lie Algebras

Tangent space to a linear Lie group • Lie algebras • Homomorhisms of Lie algebras • Lie algebras of linear Lie groups: examples.

6.1. Tangent space to a linear Lie group.

Let $\mathsf{G} \leq \mathsf{GL}(n, \Bbbk)$ be a linear Lie group.

Definition 62. A **one-parameter subgroup** of G is a continuous mapping $\gamma : \mathbb{R} \to \mathsf{G}$ which satisfies (the homomorphism property)

$$\gamma(s+t) = \gamma(s)\gamma(t) \qquad (t, s \in \mathbb{R}).$$

Note : Recall that $\mathbb{R}$ can be viewed as a linear Lie group. Hence (the one-parameter subgroup) γ is a *continuous homomorphism* of linear Lie groups. It can be shown that *every one-parameter subgroup of* G *is differentiable at* 0 (in fact, differentiable at every $t \in \mathbb{R}$).

A one-parameter subgroup $\gamma : \mathbb{R} \to \mathsf{G}$ can be viewed as a *collection* $(\gamma(t))_{t \in \mathbb{R}}$ of linear transformations on $\Bbbk^n$ such that (for $t, s \in \mathbb{R}$)

- $\gamma(0) = \mathrm{id}_{\Bbbk^n}$.
- $\gamma(s+t) = \gamma(s)\gamma(t)$.
- $\gamma(t) \in \mathsf{G}$ depends continuously on t.

In other words, γ is a linear representation of the (Abelian) group $\mathbb{R}$ on (the vector space) $\Bbbk^n$. (So γ defines a continuous action of $\mathbb{R}$ on $\Bbbk^n$.) On the other hand, the (parametrized) curve $\gamma : \mathbb{R} \to \mathsf{G}$ has a *tangent vector* $\dot{\gamma}(0) \in \Bbbk^{n \times n}$ at $\gamma(0) = \mathbf{1}$.

Proposition 43. *Let* $\gamma : \mathbb{R} \to \mathsf{G}$ *be a one-parameter subgroup of* G. *Then* γ *is differentiable at every* $t \in \mathbb{R}$ *and*

$$\dot{\gamma}(t) = \dot{\gamma}(0)\,\gamma(t) = \gamma(t)\,\dot{\gamma}(0).$$

Proof. We have (for $t, h \in \mathbb{R}$)

$$\begin{aligned} \dot{\gamma}(t) &= \lim_{h \to 0} \frac{1}{h}\left(\gamma(t+h) - \gamma(t)\right) \\ &= \lim_{h \to 0} \frac{1}{h}\left(\gamma(h)\gamma(t) - \gamma(t)\right) \\ &= \left(\lim_{h \to 0} \frac{1}{h}\left(\gamma(h) - \mathbf{1}\right)\right)\gamma(t) \\ &= \dot{\gamma}(0)\,\gamma(t) \end{aligned}$$

and similarly

$$\dot{\gamma}(t) = \gamma(t)\,\dot{\gamma}(0).$$

□

We can now determine the form of all one-parameter subgroups of G.

THEOREM 44. *Let* $\gamma : \mathbb{R} \to \mathsf{G}$ *be a one-parameter subgroup of* G. *Then it has the form*

$$\gamma(t) = \exp(tA)$$

for some $A \in \Bbbk^{n\times n}$.

Proof. Let $A = \dot\gamma(0)$. This means that γ satisfies (the differential equation)

$$\dot\gamma(t) = A\,\gamma(t)$$

and is subject to (the initial condition)

$$\gamma(0) = \mathbf{1}.$$

This initial value problem (IVP) has the unique solution $\gamma(t) = \exp(tA)$. $\square$

We cannot yet reverse this process and decide for which $A \in \Bbbk^{n\times n}$ the one-parameter subgroup

$$\gamma : \mathbb{R} \to \mathsf{GL}(n, \Bbbk), \quad t \mapsto \exp(tA)$$

actually takes values in G. (The answer involves the *Lie algebra* of G.)

NOTE : We have a curious phenomenon in the fact that although the definition of a one-parameter group only involves first order differentiability, the general form $\exp(tA)$ is always infinitely differentiable (and indeed *analytic*) as a function of t. This is an important characteristic of much of the *Lie theory*, namely that conditions of first order differentiability (and even continuity) often lead to much stronger conditions.

Let $\mathsf{G} \leq \mathsf{GL}(n, \Bbbk)$ be a linear Lie group. Recall that $\Bbbk^{n\times n}$ may be considered to be some Euclidean space $\mathbb{R}^m$.

DEFINITION 63. A (parametrized) **curve** in G is a *differentiable* mapping $\gamma : (a, b) \subseteq \mathbb{R} \to \Bbbk^{n\times n}$ such that

$$\gamma(t) \in \mathsf{G} \quad \text{for all } t \in (a, b).$$

The derivative

$$\dot\gamma(t) := \lim_{h\to 0} \frac{1}{h}\left(\gamma(t+h) - \gamma(t)\right) \in \Bbbk^{n\times n}$$

is called the **tangent vector** to γ at $\gamma(t)$. We will usually assume that $a < 0 < b$.

◇ **Exercise 83.** Given two curves $\gamma, \sigma : (a, b) \to \mathsf{G}$, we define a new curve, the **product curve**, by

$$(\gamma\,\sigma)(t) := \gamma(t)\,\sigma(t).$$

Show that (for $t \in (a, b)$)

$$(\gamma\,\sigma)^{\cdot}(t) = \gamma(t)\,\dot\sigma(t) + \dot\gamma(t)\,\sigma(t).$$

◇ **Exercise 84.**

(a) Let $\gamma : (-1,1) \to \mathbb{R}^{3\times 3}$ be given by

$$\gamma(t) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}.$$

Show that γ is a curve in $\mathsf{SO}\,(3)$ and find $\dot\gamma(0)$. Show that

$$(\gamma^2)^{\cdot}(0) = 2\dot\gamma(0).$$

(b) Let $\sigma : (-1,1) \to \mathbb{R}^{3\times 3}$ be given by

$$\sigma(t) := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}.$$

Calculate $\dot\sigma(0)$. Write the matrix $\gamma(t)\,\sigma(t)$ and verify that

$$(\gamma\,\sigma)^{\cdot}(0) = \dot\gamma(0) + \dot\sigma(0).$$

◇ **Exercise 85.** Let $\alpha : (-1,1) \to \mathbb{C}^{n\times n}$ be given by

$$\alpha(t) := \begin{bmatrix} e^{i\pi t} & 0 & 0 \\ 0 & e^{i\frac{\pi t}{2}} & 0 \\ 0 & 0 & e^{i\frac{\pi t}{2}} \end{bmatrix}.$$

Show that α is a curve in $\mathsf{U}\,(3)$. Calculate $\dot\alpha(0)$.

Definition 64. The **tangent space** to G at $A \in \mathsf{G}$ is the set

$$T_A\,\mathsf{G} := \{\dot\gamma(0) \in \mathbb{k}^{n\times n} \;:\; \gamma \text{ is a curve in } \mathsf{G} \text{ with } \gamma(0) = A\}.$$

Proposition 45. *The set $T_A\,\mathsf{G}$ is a real vector subspace of $\mathbb{k}^{n\times n}$.*

Proof. Let $\alpha, \beta : (a,b) \to \mathbb{k}^{n\times n}$ be two curves in G through A (i.e., $\alpha(0) = \beta(0) = A$). Then

$$\gamma : (a,b) \to \mathbb{k}^{n\times n}, \quad t \mapsto \alpha(t)\,A^{-1}\,\beta(t)$$

is also a curve in G with $\gamma(0) = A$. We have

$$\dot\gamma(t) = \dot\alpha(t)\,A^{-1}\,\beta(t) + \alpha(t)\,A^{-1}\,\dot\beta(t)$$

and hence

$$\dot\gamma(0) = \dot\alpha(0)\,A^{-1}\,\beta(0) + \alpha(0)\,A^{-1}\,\dot\beta(0) = \dot\alpha(0) + \dot\beta(0)$$

which shows that $T_A\,\mathsf{G}$ is closed under (vector) addition.

Similarly, if $\lambda \in \mathbb{R}$ and $\alpha : (a,b) \to \mathbb{k}^{n\times n}$ is a curve in G with $\alpha(0) = A$, then

$$\eta : (a,b) \to \mathbb{k}^{n\times n}, \quad t \mapsto \alpha(\lambda t)$$

is another such curve. Since $\dot\eta(0) = \lambda\,\dot\alpha(0)$, we see that $T_A\,\mathsf{G}$ is closed under (real) scalar multiplication. So $T_A\,\mathsf{G}$ is a (real) vector subspace of $\mathbb{k}^{n\times n}$. □

NOTE : Since the vector space $\Bbbk^{n\times n}$ is *finite dimensional*, so is (the tangent space) $T_A\,\mathsf{G}$.

DEFINITION 65. If $\mathsf{G} \leq \mathsf{GL}\,(n, \Bbbk)$ is a linear Lie group, its **dimension** is the dimension of the (real) vector space $T_{\mathbf{1}}\,\mathsf{G}$ ($\mathbf{1}$ is the identity matrix). So

$$\dim\,\mathsf{G} := \dim_{\mathbb{R}}\,T_{\mathbf{1}}\,\mathsf{G}.$$

NOTE : If the linear Lie group G is complex, then its *complex dimension* is

$$\dim_{\mathbb{C}}\,\mathsf{G} := \dim_{\mathbb{C}}\,T_{\mathbf{1}}\,\mathsf{G}.$$

$\diamond$ **Exercise 86.** Show that the matrix group $\mathsf{U}\,(1)$ has dimension 1.

NOTE : The *only* connected linear Lie groups (up to isomorphism) of dimension 1 are $\mathbb{T}^1 = \mathsf{U}\,(1)$ and $\mathbb{R}$, and of dimension 2 are $\mathbb{R}^2, \mathbb{T}^1 \times \mathbb{R}, \mathbb{T}^2$, and $\mathsf{AGL}^+\,(1,\mathbb{R})$.

EXAMPLE 66. *The real general linear group* $\mathsf{GL}\,(n,\mathbb{R})$ *has dimension* n^2. The determinant function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is *continuous* and $\det\,(\mathbf{1}) = 1$. So there is some ϵ-ball about $\mathbf{1}$ in $\mathbb{R}^{n\times n}$ such that, for each A in this ball, $\det A \neq 0$ (i.e., $A \in \mathsf{GL}\,(n,\mathbb{R})$). If $B \in \mathbb{R}^{n\times n}$, then define a curve σ in $\mathbb{R}^{n\times n}$ by

$$\sigma(t) := \mathbf{1} + tB.$$

Then $\sigma(0) = \mathbf{1}$ and $\dot{\sigma}(0) = B$, and (for small t) $\sigma(t) \in \mathsf{GL}\,(n,\mathbb{R})$. Hence the tangent space $T_{\mathbf{1}}\,\mathsf{GL}\,(n,\mathbb{R})$ is all of $\mathbb{R}^{n\times n}$ which has dimension n^2. So $\dim\,\mathsf{GL}\,(n,\mathbb{R}) = n^2$.

$\diamond$ **Exercise 87.** Show that the dimension of the *complex* general linear group $\mathsf{GL}\,(n,\mathbb{C})$ is $2n^2$.

PROPOSITION 46. *Let* $\mathsf{Sk\text{-}sym}\,(n)$ *denote the set of all skew-symmetric matrices in* $\mathbb{R}^{n\times n}$. *Then* $\mathsf{Sk\text{-}sym}\,(n)$ *is a linear subspace of* $\mathbb{R}^{n\times n}$ *and its dimension is* $\frac{n(n-1)}{2}\cdot$

Proof. If $A, B \in \mathsf{Sk\text{-}sym}\,(n)$, then

$$(A+B)^\top + (A+B) = A^\top + A + B^\top + B = 0$$

so that $\mathsf{Sk\text{-}sym}\,(n)$ is closed under (vector) addition.

It is also closed under scalar multiplication, for if $A \in \mathsf{Sk\text{-}sym}\,(n)$ and $\lambda \in \mathbb{R}$, then $(\lambda\,A)^\top = \lambda\,A^\top$ so that

$$(\lambda\,A)^\top + \lambda\,A = \lambda\,(A^\top + A) = 0.$$

To check the dimension of $\mathsf{Sk\text{-}sym}\,(n)$ we construct a basis. Let E_{ij} denote the matrix whose entries are all zero except the ij-entry, which is 1, and the ji-entry, which is -1. If we define these E_{ij} *only* for $i < j$, we can see that they form a *basis* for $\mathsf{Sk\text{-}sym}\,(n)$. It is easy to compute that there are

$$(n-1) + (n-2) + \cdots + 2 + 1 = \frac{n(n-1)}{2}$$

of them. □

◇ **Exercise 88.** Show that if σ is a curve through the identity (i.e., $\sigma(0) = \mathbf{1}$) in the orthogonal group $\mathsf{O}\,(n)$, then $\dot\sigma(0)$ is skew-symmetric.

NOTE : It follows that $\dim \mathsf{O}\,(n) \leq \frac{n(n-1)}{2}\cdot$ (Later we will show that this estimation is an equality.)

6.2. **Lie algebras.** We will adopt the notation $\mathfrak{g} := T_{\mathbf{1}}\,\mathsf{G}$ for this real vector subspace of $\Bbbk^{n\times n}$. In fact, $\mathfrak{g}$ has a more interesting algebraic structure, namely that of a *Lie algebra.*

NOTE : It is customary to use *lower case Gothic* (Fraktur) characters (such as $\mathfrak{a}, \mathfrak{g}$ and $\mathfrak{h}$) to refer to Lie algebras.

DEFINITION 67. A (real) **Lie algebra** $\mathfrak{a}$ is a real vector space equipped with a product

$$[\cdot,\cdot] : \mathfrak{a}\times\mathfrak{a}\to\mathfrak{a}, \qquad (x,y)\mapsto [x,y]$$

such that (for $\lambda,\mu\in\mathbb{R}$ and $x,y,z\in\mathfrak{a}$)

(LA1) $[x,y] = -[y,x]$.
(LA2) $[\lambda x+\mu y, z] = \lambda\,[x,z]+\mu\,[y,z]$.
(LA3) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$.

The product $[\cdot,\cdot]$ is called the *Lie bracket* of the Lie algebra $\mathfrak{a}$.

NOTE : (1) Condition (LA3) is called the *Jacobi identity.* So the Lie bracket $[\cdot,\cdot]$ of (the Lie algebra) $\mathfrak{a}$ is a *skew-symmetric bilinear* mapping (on $\mathfrak{a}$) which satisfies the Jacobi identity. Hence Lie algebras are *non-associative* algebras. The Lie bracket plays for Lie algebras the same role that the associative law plays for associative algebras.

(2) While we can define *complex* Lie algebras (or, more generally, Lie algebras over any field), we shall only consider Lie algebras over $\mathbb{R}$.

EXAMPLE 68. Let $\mathfrak{a} = \mathbb{R}^n$ and set (for all $x,y\in\mathbb{R}^n$)

$$[x,y] := 0.$$

The *trivial product* is a skew-symmetric bilinear multiplication (on $\mathbb{R}^n$) which satisfies the Jacobi identity and hence is a Lie bracket. $\mathbb{R}^n$ equipped with this product (Lie bracket) is a Lie algebra. Such a Lie algebra is called an *Abelian Lie algebra.*

◇ **Exercise 89.** Show that the *only* Lie algebra structure on (the vector space) $\mathbb{R}$ is the trivial one.

EXAMPLE 69. Let $\mathfrak{a} = \mathbb{R}^3$ and set (for $x,y\in\mathbb{R}^3$)

$$[x,y] := x\times y \qquad \text{(the cross product).}$$

For the *standard unit vectors* e_1, e_2, e_3 we have

$$[e_1, e_2] = -[e_2, e_1] = e_3, \quad [e_2, e_3] = -[e_3, e_2] = e_1, \quad [e_3, e_1] = -[e_1, e_3] = e_2.$$

Then $\mathbb{R}^3$ equipped with this bracket operation is a Lie algebra. In fact, as we will see later, this is the Lie algebra of (the matrix group) $\mathsf{SO}\,(3)$ and also of $\mathsf{SU}\,(2)$ in disguise.

Given two matrices $A, B \in \Bbbk^{n\times n}$, their **commutator** is

$$[A, B] := AB - BA.$$

A and B *commute* (i.e., $AB = BA$) if and only if $[A, B] = 0$. The commutator $[\cdot, \cdot]$ is a product on $\Bbbk^{n\times n}$ satisfying conditions (LA1) – (LA3).

$\diamond$ **Exercise 90.** Verify the *Jacobi identity* for the commutator $[\cdot, \cdot]$.

The *real* vector space $\Bbbk^{n\times n}$ equipped with the commutator $[\cdot, \cdot]$ is a Lie algebra.

NOTE : The procedure to give $\Bbbk^{n\times n}$ a Lie algebra structure can be extended to any *associative* algebra. A Lie bracket can be defined in any associative algebra by the commutator $[x, y] = xy - yx$, making it a Lie algebra. Here the skew-symmetry condition (axiom) is clearly satisfied, and one can check easily that in this case the Jacobi identity for the commutator follows from the associativity law for the ordinary product.

There is another way in which Lie algebras arise in the study of algebras. A **derivation** d of a *non-associative* algebra $\mathcal{A}$ (i.e., a vector space endowed with a bilinear mapping $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$) is a linear mapping $\mathcal{A} \to \mathcal{A}$ satisfying the formal analogue of the Leibniz rule for differentiating a product (for all $x, y \in \mathcal{A}$)

$$d(xy) = (dx)y + x(dy).$$

(The concept of a derivation is an abstraction of the idea of a *first-order differential operator*.) The set of all derivations on $\mathcal{A}$ is clearly a vector subspace of the algebra $\mathsf{End}\,(\mathcal{A})$ of all linear mappings $\mathcal{A} \to \mathcal{A}$. Although the product of derivations is in general *not* a derivation, the commutator $d_1 \circ d_2 - d_2 \circ d_1$ of two derivations is again a derivation. Thus the set of all derivations of a non-associative algebra is a Lie algebra, called the *derivation algebra* of the given non-associative algebra.

Suppose that $\mathfrak{a}$ is a vector subspace of the Lie algebra $\Bbbk^{n\times n}$. Then $\mathfrak{a}$ is a **Lie subalgebra** of $\Bbbk^{n\times n}$ if it is closed under taking commutators of pairs of alements in $\mathfrak{a}$; that is,

$$A, B \in \mathfrak{a} \Rightarrow [A, B] \in \mathfrak{a}.$$

Of course, $\Bbbk^{n\times n}$ is a Lie subalgebra of itself.

THEOREM 47. *If* $\mathsf{G} \leq \mathsf{GL}\,(n, \Bbbk)$ *is a linear Lie group, then the tangent space* $\mathfrak{g} = T_{\mathbf{1}}\,\mathsf{G}$ *is a Lie subalgebra of* $\Bbbk^{n\times n}$.

Proof. We will show that two curves α, β in G with $\alpha(0) = \beta(0) = \mathbf{1}$, there is such a curve γ with $\dot\gamma(0) = [\dot\alpha(0),\, \dot\beta(0)]$, where $[\cdot,\cdot]$ is the matrix commutator.

Consider the mapping

$$F : (s,t) \mapsto F(s,t) := \alpha(s)\,\beta(t)\,\alpha(s)^{-1}.$$

This is clearly differentiable with respect to each of the variables s, t. For each s (in the domain of α), $F(s,\cdot)$ is a curve in G with $F(s,0) = \mathbf{1}$. Differentiating gives

$$\frac{d}{dt}F(s,t)\bigg|_{t=0} = \alpha(s)\,\dot\beta(0)\,\alpha(s)^{-1}$$

and so

$$\alpha(s)\,\dot\beta(0)\,\alpha(s)^{-1} \in \mathfrak{g}.$$

Since $\mathfrak{g}$ is a *closed subspace* of $\Bbbk^{n\times n}$ (any vector subspace is an intersection of hyperplanes), whenever this limit exists we also have

$$\lim_{s\to 0}\frac{1}{s}\Big(\alpha(s)\,\dot\beta(0)\,\alpha(s)^{-1} - \dot\beta(0)\Big) \in \mathfrak{g}.$$

◇ **Exercise 91.** Verify the following matrix version of the usual rule for differentiating an inverse :

$$\frac{d}{dt}\left(\alpha(t)^{-1}\right) = -\alpha(t)^{-1}\,\dot\alpha(t)\,\alpha(t)^{-1}.$$

We have

$$\begin{aligned}
\lim_{s\to 0}\frac{1}{s}\Big(\alpha(s)\,\dot\beta(0)\,\alpha(s)^{-1} - \dot\beta(0)\Big) &= \frac{d}{ds}\alpha(s)\,\dot\beta(0)\,\alpha(s)^{-1}\bigg|_{s=0}\\
&= \dot\alpha(0)\,\dot\beta(0)\,\alpha(0) - \alpha(0)\,\dot\beta(0)\,\alpha(0)^{-1}\dot\alpha(0)\,\alpha(0)^{-1}\\
&= \dot\alpha(0)\,\dot\beta(0)\,\alpha(0) - \alpha(0)\,\dot\beta(0)\,\dot\alpha(0)\\
&= \dot\alpha(0)\,\dot\beta(0) - \dot\beta(0)\,\dot\alpha(0)\\
&= [\dot\alpha(0),\, \dot\beta(0)].
\end{aligned}$$

This shows that $[\dot\alpha(0), \dot\beta(0)] \in \mathfrak{g}$, hence it must be of the form $\dot\gamma(0)$ for some curve γ. □

So, for each linear Lie group G, there is a Lie algebra $\mathfrak{g} = T_{\mathbf{1}}\,\mathsf{G}$. We call $\mathfrak{g}$ the **Lie algebra** of G.

NOTE : The essential phenomenon behind Lie theory is that *one may associate, in a natural way, to a linear Lie group* G *its Lie algebra* $\mathfrak{g}$. The Lie algebra is first of all a (real) vector space and secondly is endowed with a skew-symmetric bilinear product (the Lie bracket). Amazingly, *the group* G *is almost completely determined by* $\mathfrak{g}$ *and its Lie bracket.* Thus, for many purposes, one can replace G with $\mathfrak{g}$. Since G is a complicated nonlinear object and $\mathfrak{g}$ is just a vector space, it is usually vastly simpler to work with $\mathfrak{g}$. Otherwise intractable computations may become straightforward linear algebra; this is one source of the power of Lie theory.

6.3. **Homomorphisms of Lie algebras.** A suitable type of homomorphism $\mathsf{G} \to \mathsf{H}$ between linear Lie groups gives rise to a linear mapping $\mathfrak{g} \to \mathfrak{h}$ respecting the Lie algebra structures.

Definition 70. Let $\mathsf{G} \leq \mathsf{GL}\,(n, \Bbbk)$, $\mathsf{H} \leq \mathsf{GL}\,(m, \Bbbk)$ be linear Lie groups and let $\Phi : \mathsf{G} \to \mathsf{H}$ be a continuous mapping. Then Φ is said to be **differentiable** if for every (differentiable) curve $\gamma : (a, b) \to \mathsf{G}$, the composite mapping $\Phi \circ \gamma : (a, b) \to \mathsf{H}$ is a (differentiable) curve with derivative

$$(\Phi \circ \gamma)^{\cdot}(t) = \frac{d}{dt}\Phi(\gamma(t))$$

and, if two (differentiable) curves $\alpha, \beta : (a, b) \to \mathsf{G}$ both satisfy the conditions

$$\alpha(0) = \beta(0) \quad \text{and} \quad \dot{\alpha}(0) = \dot{\beta}(0),$$

then $(\Phi \circ \alpha)^{\cdot}(0) = (\Phi \circ \beta)^{\cdot}(0)$.

Such a mapping Φ is a *differentiable homomorphism* if it is also a group homomorphism. A continuous homomorphism of matrix groups that is also *differentiable* is called a **Lie homomorphism**.

Note : The "technical restriction" in the definition of a Lie homomorphism is, in fact, *unnecessary*. (It turns out, but by no means easy to prove, that *every continuous homomorphism between Lie groups is differentiable* – in fact, analytic.)

If $\Phi : \mathsf{G} \to \mathsf{H}$ is the restriction of a differentiable mapping $\mathsf{GL}\,(n, \Bbbk) \to \mathsf{GL}\,(m, \Bbbk)$, then Φ is also a differentiable mapping.

Proposition 48. *Let* G, H, K *be linear Lie groups and let* $\Phi : \mathsf{G} \to \mathsf{H}, \Psi : \mathsf{H} \to \mathsf{K}$ *be differentiable homomorphisms.*

(a) *For each* $A \in \mathsf{G}$ *there is a linear mapping* $d\Phi_A : T_A\,\mathsf{G} \to T_{\Phi(A)}\,\mathsf{H}$ *given by*

$$d\Phi_A(\dot{\gamma}(0)) = (\Phi \circ \gamma)^{\cdot}(0).$$

(b) *We have*

$$d\Psi_{\Phi(A)} \circ d\Phi_A = d(\Psi \circ \Phi)_A.$$

(c) *For the identity mapping* $\mathbf{1}_{\mathsf{G}} : \mathsf{G} \to \mathsf{G}$ *and* $A \in \mathsf{G}$,

$$d\,(\mathbf{1}_{\mathsf{G}})_A = \mathbf{1}_{T_A\,\mathsf{G}}.$$

Proof. (a) The definition of $d\Phi_A$ makes sense since (by the definition of differentiability), given $X \in T_A\,\mathsf{G}$, for any curve γ with $\gamma(0) = A$ and $\dot{\gamma}(0) = X$, $(\Phi \circ \gamma)^{\cdot}(0)$ depends *only* on X and *not* on γ. The identities (b) and (c) are straightforward to verify. $\square$

$\diamond$ **Exercise 92.** Verify that the map $d\Phi_A : T_A\,\mathsf{G} \to T_{\Phi(A)}\,\mathsf{H}$ is *linear*.

If $\Phi : \mathsf{G} \to \mathsf{H}$ is a differentiable homomorphism, then (since $\Phi(\mathbf{1}) = \mathbf{1}$) $d\Phi_{\mathbf{1}} : T_{\mathbf{1}}\,\mathsf{G} \to T_{\mathbf{1}}\,\mathsf{H}$ is a linear mapping, called the **derivative** of Φ and usually denoted by

$$d\Phi : \mathfrak{g} \to \mathfrak{h}.$$

DEFINITION 71. Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. A linear mapping $\phi : \mathfrak{g} \to \mathfrak{h}$ is a **homomorphism of Lie algebras** if (for $x, y \in \mathfrak{g}$)

$$\phi\,([x, y]) = [\phi\,(x), \phi\,(y)].$$

THEOREM 49. *Let* G, H *be linear Lie groups and* $\Phi : \mathsf{G} \to \mathsf{H}$ *be a Lie homomorphism. Then the derivative* $d\Phi : \mathfrak{g} \to \mathfrak{h}$ *is a homomorphism of Lie algebras.*

Following ideas and notation in the proof of THEOREM 47, for curves α, β in G with $\alpha(0) = \beta(0) = \mathbf{1}$, we can use the composite mapping

$$\Phi \circ F : (s, t) \mapsto \Phi(F(s, t)) = \Phi(\alpha(s))\Phi(\beta(t))\Phi(\alpha(s))^{-1}$$

to deduce $d\Phi([\dot\alpha(0), \dot\beta(0)]) = [d\Phi(\dot\alpha(0)), d\Phi(\dot\beta(0))]$.

◇ **Exercise 93.** Write down a full proof of THEOREM 49.

COROLLARY 50. *Let* G, H *be linear Lie groups and* $\Phi : \mathsf{G} \to \mathsf{H}$ *be a Lie isomorphism of linear Lie groups. Then the derivative* $d\Phi : \mathfrak{g} \to \mathfrak{h}$ *is an isomorphism of Lie algebras.*

Proof. $\Phi^{-1} \circ \Phi$ is the identity, so

$$d\Phi^{-1} \circ d\Phi : T_{\mathbf{1}}\,\mathsf{G} \to T_{\mathbf{1}}\,\mathsf{G}$$

is the identity. Thus $d\Phi^{-1}$ is surjective and $d\Phi$ is injective.

Likewise, $\Phi \circ \Phi^{-1}$ is the identity, so $d\Phi \circ d\Phi^{-1}$ is the identity. Thus $d\Phi^{-1}$ is injective, and $d\Phi$ is surjective. The result now follows. □

NOTE : Isomorphic linear Lie groups have isomorphic Lie algebras. The converse (i.e., linear Lie groups with isomorphic Lie algebras are isomorphic) is *false*. For example, the rotation group $\mathsf{SO}\,(2)$ and the diagonal group

$$\mathsf{D}_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & e^a \end{bmatrix} : a \in \mathbb{R} \right\} \leq \mathsf{AGL}^+\,(1, \mathbb{R})$$

both have Lie algebras isomorphic to $\mathbb{R}$ (the only Lie algebra structure on $\mathbb{R}$), but $\mathsf{SO}\,(2)$ is homeomorphic to a circle, while D_1 is homeomorphic to $\mathbb{R}$, so they are certainly *not* isomorphic.

However, the converse is – in a sense – *almost* true, so that the bracket operation on $\mathfrak{g}$ *almost* determines G as a group. After the existence of the Lie algebra, this fact is the most remarkable in Lie theory. Its precise formulation is known as *Lie's Third Theorem*.

6.4. **Lie algebras of linear Lie groups: examples.**

6.4.1. *The Lie algebras of* $\mathsf{GL}\,(n,\mathbb{R})$ *and* $\mathsf{GL}\,(n,\mathbb{C})$. Let us start with the *real* general linear group $\mathsf{GL}\,(n,\mathbb{R}) \subset \mathbb{R}^{n\times n}$. We have shown (see Example 90) that $T_{\mathbf{1}}\,\mathsf{GL}\,(n,\mathbb{R}) = \mathbb{R}^{n\times n}$. Hence *the Lie algebra* $\mathfrak{gl}(n,\mathbb{R})$ *of* $\mathsf{GL}\,(n,\mathbb{R})$ *consists of all* $n\times n$ *matrices (with real entries)*, with the commutator as the Lie bracket. Thus $\mathfrak{gl}\,(n,\mathbb{R}) = \mathbb{R}^{n\times n}$. It follows that

$$\dim\,\mathsf{GL}\,(n,\mathbb{R}) = \dim\,\mathfrak{gl}\,(n,\mathbb{R}) = n^2.$$

Similarly, the Lie algebra of the *complex* general linear group $\mathsf{GL}\,(n,\mathbb{C})$ is $\mathfrak{gl}\,(n,\mathbb{C}) = \mathbb{C}^{n\times n}$ and

$$\dim\,\mathsf{GL}\,(n,\mathbb{C}) = \dim_{\mathbb{R}}\,\mathfrak{gl}\,(n,\mathbb{C}) = 2n^2.$$

6.4.2. *The Lie algebras of* $\mathsf{SL}\,(n,\mathbb{R})$ *and* $\mathsf{SL}\,(n,\mathbb{C})$. For $\mathsf{SL}\,(n,\mathbb{R}) \leq \mathsf{GL}\,(n,\mathbb{R})$, suppose that $\alpha : (a,b) \to \mathsf{SL}\,(n,\mathbb{R})$ is a curve in $\mathsf{SL}\,(n,\mathbb{R})$ with $\alpha(0) = \mathbf{1}$. For $t \in (a,b)$ we have $\det\,\alpha(t) = 1$ and so

$$\frac{d}{dt}\det\,\alpha(t) = 0.$$

Using Proposition 37, it follows that $\operatorname{tr}\dot\alpha(0) = 0$ and thus

$$T_{\mathbf{1}}\,\mathsf{SL}\,(n,\mathbb{R}) \subseteq \operatorname{Ker}\operatorname{tr} := \left\{A \in \mathbb{R}^{n\times n} \,:\, \operatorname{tr} A = 0\right\}.$$

If $A \in \operatorname{Ker}\operatorname{tr} \subseteq \mathbb{R}^{n\times n}$, the curve $\alpha : (a,b) \to \mathbb{R}^{n\times n}, \quad t \mapsto \exp\,(tA)$ satisfies (the boundary conditions)

$$\alpha(0) = \mathbf{1} \quad \text{and} \quad \dot\alpha(0) = A.$$

Moreover, using Liouville's Formula, we get

$$\det\,\alpha(t) = \det\,\exp\,(tA) = e^{t\operatorname{tr} A} = 1.$$

Hence *the Lie algebra* $\mathfrak{sl}\,(n,\mathbb{R})$ *of* $\mathsf{SL}\,(n,\mathbb{R})$ *consists of all* $n\times n$ *matrices (with real entries) having trace zero*, with the commutator as the Lie bracket. Thus

$$\mathfrak{sl}\,(n,\mathbb{R}) = T_{\mathbf{1}}\,\mathsf{SL}\,(n,\mathbb{R}) = \{A \in \mathfrak{gl}\,(n,\mathbb{R}) \,:\, \operatorname{tr} A = 0\}.$$

Since $\operatorname{tr} A = 0$ imposes one condition on A, it follows that

$$\dim \mathsf{SL}\,(n,\mathbb{R}) = \dim_{\mathbb{R}}\,\mathfrak{sl}\,(n,\mathbb{R}) = n^2 - 1.$$

Similarly, *the Lie algebra of the complex special linear group* $\mathsf{SL}\,(n,\mathbb{C})$ *is*

$$\mathfrak{sl}\,(n,\mathbb{C}) = T_{\mathbf{1}}\,\mathsf{SL}\,(n,\mathbb{C}) = \{A \in \mathfrak{gl}\,(n,\mathbb{C}) \,:\, \operatorname{tr} A = 0\}$$

and

$$\dim\,\mathsf{SL}\,(n,\mathbb{C}) = \dim_{\mathbb{R}}\,\mathfrak{sl}\,(n,\mathbb{C}) = 2(n^2 - 1).$$

6.4.3. *The Lie algebras of* $\mathsf{O}(n)$ *and* $\mathsf{SO}(n)$. First, consider the orthogonal group $\mathsf{O}(n)$; that is,

$$\mathsf{O}(n) = \left\{A \in \mathsf{GL}(n,\mathbb{R}) \,:\, A^\top A = \mathbf{1}\right\} \leq \mathsf{GL}(n,\mathbb{R}).$$

Given a curve $\alpha : (a,b) \to \mathsf{O}(n)$ with $\alpha(0) = \mathbf{1}$, we have

$$\frac{d}{dt}\alpha(t)^T\alpha(t) = 0$$

and so

$$\dot{\alpha}(t)^\top\alpha(t) + \alpha(t)^\top\dot{\alpha}(t) = 0$$

which implies

$$\dot{\alpha}(0)^\top + \dot{\alpha}(0) = 0.$$

Thus we must have $\dot{\alpha}(0) \in \mathbb{R}^{n\times n}$ is *skew-symmetric*. So

$$T_{\mathbf{1}}\,\mathsf{O}(n) \subseteq \mathsf{Sk\text{-}sym}(n) = \left\{A \in \mathbb{R}^{n\times n} \,:\, A^\top + A = 0\right\}$$

(the set of all $n \times n$ skew-symmetric matrices in $\mathbb{R}^{n\times n}$).

On the other hand, if $A \in \mathsf{Sk\text{-}sym}(n) \subseteq \mathbb{R}^{n\times n}$, we consider the curve

$$\alpha : (a,b) \to \mathsf{GL}(n,\mathbb{R}), \quad t \mapsto \exp(tA).$$

Then

$$\begin{aligned}
\alpha(t)^\top\alpha(t) &= \exp(tA)^\top \exp(tA) \\
&= \exp(tA^\top)\exp(tA) \\
&= \exp(-tA)\exp(tA) \\
&= \mathbf{1}.
\end{aligned}$$

Hence we can view α as a curve in $\mathsf{O}(n)$. Since $\dot{\alpha}(0) = A$, this shows that

$$\mathsf{Sk\text{-}sym}(n) \subseteq T_{\mathbf{1}}\,\mathsf{O}(n)$$

and hence *the Lie algebra* $\mathfrak{o}(n)$ *of the orthogonal group* $\mathsf{O}(n)$ *consists of all* $n \times n$ *skew-symmetric matrices*, with the usual commutator as the Lie bracket. Thus

$$\mathfrak{o}(n) = T_{\mathbf{1}}\,\mathsf{O}(n) = \mathsf{Sk\text{-}sym}(n) = \left\{A \in \mathbb{R}^{n\times n} \,:\, A^\top + A = 0\right\}.$$

It follows that (see Proposition 46)

$$\dim \mathsf{O}(n) = \dim \mathfrak{o}(n) = \frac{n(n-1)}{2}\cdot$$

◇ **Exercise 94.** Show that if $A \in \mathsf{Sk\text{-}sym}(n)$, then $\operatorname{tr} A = 0$.

By Liouville's Formula, we have

$$\det \alpha(t) = \det \exp(tA) = 1$$

and hence $\alpha : (a,b) \to \mathsf{SO}(n)$, where $\mathsf{SO}(n)$ is the *special orthogonal group*. We have actually shown that *the Lie algebra of the special orthogonal group* $\mathsf{SO}(n)$ *is*

$$\mathfrak{so}(n) = \mathfrak{o}(n) = \left\{ A \in \mathbb{R}^{n\times n} \;:\; A^\top + A = 0 \right\}.$$

6.4.4. *The Lie algebra of* $\mathsf{SO}(3)$. We will discuss the Lie algebra $\mathfrak{so}(3)$ of the *rotation group* $\mathsf{SO}(3)$ in some detail.

◇ **Exercise 95.** Show that

$$\mathfrak{so}(3) = \left\{ \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3} \;:\; a,b,c \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{so}(3)$ is a three-dimensional real vector space. Consider the *rotations*

$$R_1(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}, \; R_2(t) = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix}, \; R_3 = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then the mappings

$$\rho_i : t \mapsto R_i(t), \quad i = 1,2,3$$

are curves in $\mathsf{SO}(3)$ and clearly $\rho_i(0) = \mathbf{1}$. It follows that

$$\dot\rho_i(0) := A_i \in \mathfrak{so}(3), \quad i = 1,2,3.$$

These elements (matrices) are

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

◇ **Exercise 96.** Verify that the matrices A_1, A_2, A_3 form a basis for $\mathfrak{so}(3)$. (We shall refer to this basis as the *standard basis*.)

◇ **Exercise 97.** Compute all the Lie brackets (commutators) $[A_i, A_j]$, $i,j = 1,2,3$ and then determine the coefficients c_{ij}^k defined by

$$[A_i, A_j] = c_{ij}^1 A_1 + c_{ij}^2 A_2 + c_{ij}^3 A_3, \quad i,j = 1,2,3.$$

(These coefficients are called the **structure constants** of the Lie algebra. They completely determine the Lie bracket $[\cdot,\cdot]$.)

The Lie algebra $\mathfrak{so}(3)$ may be *identified* with (the Lie algebra) $\mathbb{R}^3$ as follows. We define the mapping

$$\widehat{\ }: \mathbb{R}^3 \to \mathfrak{so}\,(3), \quad x = (x_1, x_2, x_3) \mapsto \widehat{x} := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

This mapping is called the **hat map**.

$\diamond$ **Exercise 98.** Show that the hat map $\widehat{\ }: \mathbb{R}^3 \to \mathfrak{so}\,(3)$ is an *isomorphism* of vector spaces.

$\diamond$ **Exercise 99.** Show that (for $x, y \in \mathbb{R}^3$)

(a) $x \times y = \widehat{x}\, y$.
(b) $\widehat{x \times y} = [\widehat{x}, \widehat{y}]$.
(c) $x \bullet y = -\frac{1}{2}\mathrm{tr}\,(\widehat{x}\,\widehat{y})$.

Formula (b) says that the hat map is in fact an isomorphism of Lie algebras and so *we can identify the Lie algebra* $\mathfrak{so}(3)$ *with (the Lie algebra)* $\mathbb{R}^3$.

NOTE : For $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$, the matrix exponential $\exp\,(t\,\widehat{x})$ is a *rotation* about (the axis) x through the angle $t\|x\|$. The following explicit formula for $\exp\,(\widehat{x})$ is known as RODRIGUES' FORMULA:

$$\exp\,(\widehat{x}) = \mathbf{1} + \frac{\sin\|x\|}{\|x\|}\,\widehat{x} + \frac{1}{2}\left[\frac{\sin\left(\frac{\|x\|}{2}\right)}{\frac{\|x\|}{2}}\right]^2 \widehat{x}^2.$$

This result says that *the exponential map*

$$\exp : \mathfrak{so}\,(3) \to \mathsf{SO}\,(3)$$

is onto. RODRIGUES' FORMULA is useful in *computational solid mechanics*, along with its quaternionic counterpart.

6.4.5. *The Lie algebras of* $\mathsf{U}\,(n)$ *and* $\mathsf{SU}\,(n)$. Consider the unitary group $\mathsf{U}\,(n)$; that is,

$$\mathsf{U}\,(n) = \{A \in \mathsf{GL}\,(n, \mathbb{C}) \;:\; A^*A = \mathbf{1}\}.$$

For a curve α in $\mathsf{U}\,(n)$ with $\alpha(0) = \mathbf{1}$, we obtain

$$\dot{\alpha}(0)^* + \dot{\alpha}(0) = 0$$

and so $\dot{\alpha}(0) \in \mathbb{C}^{n\times n}$ is skew-Hermitian. So

$$T_{\mathbf{1}}\,\mathsf{U}\,(n) \subseteq \mathsf{Sk\text{-}Herm}\,(n) = \left\{A \in \mathbb{C}^{n\times n} \;:\; A^* + A = 0\right\}$$

(the set of all $n \times n$ skew-Hermitian matrices in $\mathbb{C}^{n\times n}$).

If $H \in \mathsf{Sk\text{-}Herm}\,(n)$, then the curve

$$\alpha : (a, b) \to \mathsf{GL}\,(n, \mathbb{C}), \quad t \mapsto \exp\,(tH)$$

satisfies

$$\begin{aligned}\alpha(t)^*\alpha(t) &= \exp(tH)^* \exp(tH) \\ &= \exp(tH^*)\exp(tH) \\ &= \exp(-tH)\exp(tH) \\ &= \mathbf{1}.\end{aligned}$$

Hence we can view α as a curve in $\mathsf{U}(n)$. Since $\dot\alpha(0) = H$, this shows that

$$\mathsf{Sk\text{-}Herm}(n) \subseteq T_{\mathbf{1}}\,\mathsf{U}(n)$$

and hence *the Lie algebra* $\mathfrak{u}(n)$ *of the unitary group* $\mathsf{U}(n)$ *consists of all* $n \times n$ *skew-Hermitian matrices, with the usual commutator as the Lie bracket.* Thus

$$\mathfrak{u}(n) = T_{\mathbf{1}}\,\mathsf{U}(n) = \mathsf{Sk\text{-}Herm}(n) = \left\{H \in \mathbb{C}^{n\times n} \;:\; H^* + H = 0\right\}.$$

It follows that (see **Problem 39**)

$$\dim \mathsf{U}(n) = \dim_{\mathbb{R}} \mathfrak{u}(n) = n^2.$$

The special unitary group $\mathsf{SU}(n)$ can be handled in a similar way. Again we have

$$\mathfrak{su}(n) = T_{\mathbf{1}}\,\mathsf{SU}(n) \subseteq \mathsf{Sk\text{-}Herm}(n).$$

If $\alpha : (a,b) \to \mathsf{SU}(n)$ is a curve with $\alpha(0) = \mathbf{1}$ then, as in the analysis for $\mathsf{SL}(n,\mathbb{R})$, we have $\operatorname{tr}\dot\alpha(0) = 0$. Writing

$$\mathsf{Sk\text{-}Herm}^0(n) := \{H \in \mathsf{Sk\text{-}Herm}(n) \;:\; \operatorname{tr} H = 0\}$$

this gives $\mathfrak{su}(n) \subseteq \mathsf{Sk\text{-}Herm}^0(n)$. On the other hand, if $H \in \mathsf{Sk\text{-}Herm}^0(n)$ then the curve

$$\alpha : (a,b) \to \mathsf{U}(n), \quad t \mapsto \exp(tH)$$

takes values in $\mathsf{SU}(n)$ and $\dot\alpha(0) = H$. Hence

$$\mathfrak{su}(n) = T_{\mathbf{1}}\,\mathsf{SU}(n) = \mathsf{Sk\text{-}Herm}^0(n) = \left\{H \in \mathbb{C}^{n\times n} \;:\; H^* + H = 0 \text{ and } \operatorname{tr} H = 0\right\}.$$

NOTE : For a linear Lie group $\mathsf{G} \leq \mathsf{GL}(n,\mathbb{R})$ (with Lie algebra $\mathfrak{g}$), the following are true (and can be used in determining Lie algebras of linear Lie groups).

- The mapping

  $$\exp_{\mathsf{G}} : \mathfrak{g} \to \mathsf{GL}(n,\mathbb{R}), \quad X \mapsto \exp(X)$$

  has image contained in G, $\exp_{\mathsf{G}}(\mathfrak{g}) \subseteq \mathsf{G}$. We will normally write $\exp_{\mathsf{G}} : \mathfrak{g} \to \mathsf{G}$ for the exponential mapping on G (and sometimes even just $\exp$). In general, the exponential mapping $\exp_{\mathsf{G}}$ is neither one-to-one nor onto.
- If G is compact and connected, then $\exp_{\mathsf{G}}$ is onto.
- The mapping $\exp_{\mathsf{G}}$ maps a neighborhood of 0 in $\mathfrak{g}$ bijectively onto a neighborhood of $\mathbf{1}$ in G.

◇ **Exercise 100.** Verify that the exponential map

$$\exp_{\mathsf{U}(1)} : \mathbb{R} \to \mathsf{U}(1) = \mathbb{S}^1, \qquad t \mapsto e^{it}$$

is onto but *not* one-to-one.

Example 72. The exponential map

$$\exp_{\mathsf{SL}(2,\mathbb{R})} : \mathfrak{sl}(2,\mathbb{R}) \to \mathsf{SL}(2,\mathbb{R})$$

is *not* onto. Let

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \quad \text{with } \lambda < -1.$$

We see that $A \in \mathsf{SL}(2,\mathbb{R})$ and we shall show that A is *not* of the form $\exp(X)$ with $X \in \mathfrak{sl}(2,\mathbb{R})$. If $A = \exp(X)$, then the eigenvalues of A are of the form e^a and e^b, where a and b are the eigenvalues of X. Suppose $\lambda = e^a$ and $\frac{1}{\lambda} = e^b$. Then

$$a = -b + 2k\pi\, i, \quad k \in \mathbb{Z}.$$

However, since λ is negative, a is actually complex and therefore its conjugate is also an eigenvalue; that is, $b = \bar{a}$. This gives a as pure imaginary. Then

$$1 = |e^a| = |\lambda| = -\lambda$$

which contradicts the assumption that $\lambda < -1$.

6.4.6. *The Lie algebra of* $\mathsf{SU}(2)$. We will discuss the Lie algebra $\mathfrak{su}(2)$ in some detail.

◇ **Exercise 101.** Show that

$$\mathfrak{su}(2) = \left\{ \begin{bmatrix} ci & -b+ai \\ b+ai & -ci \end{bmatrix} \in \mathbb{C}^{2\times 2} \; : \; a,b,c \in \mathbb{R} \right\}.$$

The Lie algebra $\mathfrak{su}(2)$ is a three-dimensional *real* vector space. Consider the matrices

$$H_1 = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad H_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad H_3 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Clearly,

$$H_i \in \mathfrak{su}(2), \quad i = 1,2,3.$$

◇ **Exercise 102.** Verify that the matrices H_1, H_2, H_3 form a basis for $\mathfrak{su}(2)$.

◇ **Exercise 103.** Compute all the Lie brackets (commutators) $[H_i, H_j]$, $i,j = 1,2,3$ and then determine the structure constants of the Lie algebra $\mathfrak{su}(2)$.

Consider the mapping

$$\phi : \mathbb{R}^3 \to \mathfrak{su}(2), \quad x = (x_1, x_2, x_3) \mapsto x_1 H_1 + x_2 H_2 + x_3 H_3.$$

◇ **Exercise 104.** Show that the mapping $\phi : \mathbb{R}^3 \to \mathfrak{su}(2)$ is an *isomorphism* of Lie algebras ($\mathbb{R}^3$ with the cross product).

Thus *we can identify the Lie algebra* $\mathfrak{su}(2)$ *with (the Lie algebra)* $\mathbb{R}^3$.

NOTE : The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ look the same algebraically (they are isomorphic). An explicit isomorphism (of Lie algebras) is given by

$$\psi : x_1H_1 + x_2H_2 + x_3H_3 \mapsto x_1A_1 + x_2A_2 + x_3A_3.$$

This suggests that there might be a close relationship between the matrix groups themselves. Indeed there is a (surjective) *Lie homomorphism* $\mathsf{SU}(2) \to \mathsf{SO}(3)$ whose derivative (at $\mathbf{1}$) is ψ. Recall the adjoint representation

$$\mathrm{Ad} : \mathsf{SU}(2) \to \mathsf{Aut}(\mathfrak{su}(2)), \quad A \mapsto \mathrm{Ad}_A (: U \mapsto AUA^*).$$

Each Ad_A is a *linear isomorphism* of $\mathfrak{su}(2)$. Ad_A is actually an *orthogonal transformation* on $\mathfrak{su}(2)$ (the mapping $(X, Y) \mapsto -\mathrm{tr}(XY)$ is an inner product on $\mathfrak{su}(2)$) and so Ad_A corresponds to an element of $\mathsf{O}(3)$ (in fact, $\mathsf{SO}(3)$). The mapping

$$\overline{\mathrm{Ad}} : \mathsf{SU}(2) \to \mathsf{SO}(3), \quad A \mapsto \mathrm{Ad}_A$$

turns out to be a continuous homomorphism of matrix groups that is differentiable (i.e., a Lie homomorphism) and such that its derivative $d\,\overline{\mathrm{Ad}} : \mathfrak{su}(2) \to \mathfrak{so}(3)$ is ψ.

6.4.7. *The Lie algebras of* $\mathsf{T}(n, \Bbbk)$ *and* $\mathsf{UT}(n, \Bbbk)$. Let $\alpha : (a, b) \to \mathsf{T}(n, \Bbbk)$ be a curve in $\mathsf{T}(n, \Bbbk)$ with $\alpha(0) = \mathbf{1}$. Then $\dot{\alpha}(0)$ is upper triangular. Moreover, using the argument for $\mathsf{GL}(n, \Bbbk)$ we see that given any upper triangular matrix $A \in \Bbbk^{n\times n}$, there is a curve

$$\sigma : (-\epsilon, \epsilon) \to \Bbbk^{n\times n}, \quad t \mapsto \mathbf{1} + tA$$

such that $\sigma(t) \in \mathsf{T}(n, \Bbbk)$ and $\dot{\sigma}(0) = A$. Hence *the Lie algebra* $\mathfrak{t}(n, \Bbbk)$ *of* $\mathsf{T}(n, \Bbbk)$ *consists of all* $n \times n$ *upper triangular matrices*, with the usual commutator as the Lie bracket. Thus

$$\mathfrak{t}(n, \Bbbk) = T_{\mathbf{1}}\,\mathsf{T}(n, \Bbbk) = \left\{ A \in \Bbbk^{n\times n} \,:\, a_{ij} = 0 \;\text{ for } i > j \right\}.$$

It follows that

$$\dim\, \mathsf{T}(n, \Bbbk) = \dim_{\mathbb{R}}\, \mathfrak{t}(n, \Bbbk) = \frac{n(n+1)}{2}\, \dim_{\mathbb{R}} \Bbbk.$$

An upper triangular matrix $A \in \Bbbk^{n\times n}$ is *strictly upper triangular* if all its diagonal entries are 0. Then *the Lie algebra of the unipotent group* $\mathsf{UT}(n, \Bbbk)$ *consists of all* $n \times n$ *strictly upper triangular matrices*, with the usual commutator as the Lie bracket. So

$$\mathfrak{ut}(n, \Bbbk) = T_{\mathbf{1}}\,\mathsf{UT}(n, \Bbbk) = \left\{ A \in \Bbbk^{n\times n} \,:\, a_{ij} = 0 \;\text{ for } i \geq j \right\}.$$

◇ **Exercise 105.** Find $\dim_{\mathbb{R}} \mathfrak{ut}(n, \Bbbk)$.

Problems (33–44)

(33) Let $\mathsf{G} \leq \mathsf{GL}(n, \mathbb{k})$ be a linear Lie group.

(a) Prove that if $A = \dot{\alpha}(0) \in T_{\mathbf{1}}\mathsf{G}$, then $\exp(A) \in \mathsf{G}$. (This means that the matrix exponential map $\exp : \mathbb{k}^{n\times n} \to \mathsf{GL}(n, \mathbb{k})$ maps the Lie algebra $\mathfrak{g} = T_{\mathbf{1}}\mathsf{G}$ into G.)

(b) Hence deduce that

$$T_{\mathbf{1}}\mathsf{G} = \{X \in \mathbb{k}^{n\times n} \ : \ \exp(tX) \in \mathsf{G} \ \text{ for all } t \in \mathbb{R}\}.$$

(34) Let G be a linear Lie group. Prove that the following statements are logically equivalent.

(a) Any two elements of G may be joined by a path in G.

(b) G is *not* the disjoint union of two non-empty open sets.

(c) G is generated by any neighborhood of $\mathbf{1}$.

(d) G is generated by $\exp(\mathfrak{g})$. (A subset of G *generates* G if every element of G is a finite product of elements of the subset and their inverses; in this case, it means that every element of G is of the form $\exp(X_1)\exp(X_2)\cdots\exp(X_k)$ for some $X_1, X_2, \ldots, X_k$ in the Lie algebra $\mathfrak{g}$ of G.)

(35) Let G be a linear Lie group with associated Lie algebra $\mathfrak{g}$. Prove that the group G is Abelian <u>if and only if</u> the Lie algebra $\mathfrak{g}$ is Abelian (i.e., $[x, y] = 0$ for all $x, y \in \mathfrak{g}$).

(36) Let $\mathfrak{g}$ be a (real) Lie algebra. A vector subspace $\mathfrak{k}$ of $\mathfrak{g}$ is called an **ideal** if

$$[x, y] \in \mathfrak{k} \quad \text{for all } x \in \mathfrak{g},\ y \in \mathfrak{k}.$$

(a) Verify that any ideal $\mathfrak{k}$ is a Lie subalgebra (of $\mathfrak{g}$).

(b) Show that the **center** of $\mathfrak{g}$

$$\mathfrak{z}(\mathfrak{g}) := \{x \in \mathfrak{g} \ : \ [x, y] = 0 \ \text{ for all } y \in \mathfrak{g}\}$$

is an ideal in $\mathfrak{g}$.

(c) Show that the vector subspace

$$[\mathfrak{g}, \mathfrak{g}] := \operatorname{span}\{[x, y] \ : \ x, y \in \mathfrak{g}\}$$

is an ideal in $\mathfrak{g}$. (It is called the **commutator subalgebra**.)

(d) Show that the set

$$\mathfrak{sl}(n, \mathbb{k}) := \{x \in \mathbb{k}^{n\times n} \ : \ \operatorname{tr} x = 0\}$$

is an ideal in $\mathbb{k}^{n\times n}$. (It is called the **special linear Lie algebra.)**

(37) Show that if $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism, then the kernel $\operatorname{Ker}\phi$ is an ideal of $\mathfrak{g}_1$, and the image $\operatorname{Im}\phi$ is a Lie subalgebra of $\mathfrak{g}_2$.

(38) Let $\mathsf{G} \leq \mathsf{GL}(n, \Bbbk)$ be a linear Lie group. Prove that if H is a *normal* subgroup of G, then $T_1 \mathsf{H}$ is an *ideal* of the Lie algebra $T_1 \mathsf{G}$.

(39) A matrix $A \in \mathbb{C}^{n\times n}$ is called **skew-Hermitian** if $A^* + A = 0$.

(a) Show that the diagonal terms of a skew-Hermitian matrix are purely imaginary and hence deduce that the set $\mathsf{Sk\text{-}Herm}(n)$ of all skew-Hermitian matrices in $\mathbb{C}^{n\times n}$ is *not* a vector space over $\mathbb{C}$.

(b) Prove that $\mathsf{Sk\text{-}Herm}(n)$ is a *real* vector space of dimension

$$n + 2\,\frac{n(n-1)}{2} = n^2.$$

(c) If σ is a curve through the identity in $\mathsf{U}(n)$, show that $\dot{\sigma}(0)$ is skew-Hermitian and hence

$$\dim \mathsf{U}(n) \leq n^2.$$

(40) Consider the set (of $n \times n$ **skew-symmetric** matrices)

$$\mathfrak{so}(n) = \{x \in \mathbb{R}^{n\times n} \ : \ x^\top + x = 0\}.$$

(It is called the special **orthogonal Lie algebra**.)

(a) Show that $\mathfrak{so}(n)$ is a Lie subalgebra of $\mathbb{R}^{n\times n}$.

(b) Show that the Lie algebra $\mathfrak{so}(3)$ contains no ideals other than itself and (the trivial ideal) $\{0\}$. (Such a Lie algebra is called **simple**.)
[Hint : Show that any non-trivial ideal must contain *all* the elements of the standard basis.]

(41) For each of the following linear Lie group G, determine its Lie algebra $\mathfrak{g}$ and hence its dimension.

(a) $\mathsf{G} = \left\{A \in \mathsf{GL}(2,\mathbb{R}) \ : \ A^\top Q\, A = Q\right\}$, where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) $\mathsf{G} = \left\{A \in \mathsf{GL}(2,\mathbb{R}) \ : \ A^\top Q\, A = Q\right\}$, where $Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(c) $\mathsf{G} = \mathsf{AGL}(3,\mathbb{R})$.

(d) $\mathsf{G} = \mathsf{H}_3$.

(e) $\mathsf{G} = \mathsf{G}_4 \leq \mathsf{UT}^u(4,\mathbb{R})$ from **Exercise 65**.

(f) $\mathsf{G} = \mathsf{E}(n)$.

(g) $\mathsf{G} = \mathsf{SE}(n)$.

(42) (a) Show that the Lie algebra of the symplectic group $\mathsf{Sp}(2n,\mathbb{R})$ is

$$\mathfrak{sp}(2n,\mathbb{R}) = \left\{A \in \mathbb{R}^{2n\times 2n} \ : \ A^\top \mathbb{J} + \mathbb{J}A = 0\right\}.$$

(b) If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathfrak{sl}(2n,\mathbb{R})$$

show that $A \in \mathfrak{sp}(2n, \mathbb{R})$ <u>if and only if</u>

$$d = -a^\top, \quad c = c^\top, \quad \text{and} \quad b = b^\top.$$

(c) Calculate the dimension of $\mathfrak{sp}(2n, \mathbb{R})$.

(43) Show that the Lie algebra of the Lorentz group Lor is

$$\mathfrak{lor} = \left\{A \in \mathbb{R}^{4\times 4} \ : \ SA + A^\top S = 0\right\} = \left\{ \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ a_3 & a_5 & a_6 & 0 \end{bmatrix} \ : \ a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R} \right\}.$$

(44) Consider the linear Lie group $\mathbb{k}^\times = \mathsf{GL}(1, \mathbb{k})$. (Its Lie algebra is clearly $\mathbb{k}$.)

(a) Show that the determinant function

$$\det : \mathsf{GL}(n, \mathbb{k}) \to \mathbb{k}^\times$$

is a *Lie homomorphism* (i.e., a continuous homomorphism of linear Lie groups that is also *differentiable*; cf. DEFINITION 70).

(b) Show that the induced *homomorphism* of Lie algebras (i.e., the derivative of det) is the trace function

$$\mathrm{tr} : \mathbb{k}^{n\times n} \to \mathbb{k}.$$

(c) Derive from (b) that (for $A, B \in \mathbb{k}^{n\times n}$)

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$