Appendix B

Revision Problems

1. Define the term *tautology.* Determine whether the following propositions are tautologies.

$$(1) \quad p \wedge (p \to q) \to q; \qquad (2) \quad (p \to q) \wedge q \to p.$$

Class test, March 1999

2. Suppose that p and q are propositions so that $p \vee q \to p$ is FALSE. Find the *truth value* of the following propositions :

$$(1) \quad (p \to q) \to (q \to p); \qquad (2) \quad p \wedge (p \to q) \to q.$$

Class test, March 2000

3. Write down the *truth table* for $p \to q$, and then determine whether the propositions

$$(p \to q) \to r \quad \text{and} \quad p \to (q \to r)$$

are *logicaly equivalent.*

Exam, November 2001

4. Determine whether the propositions

$$(p \wedge q) \to r \quad \text{and} \quad (p \to r) \vee (q \to r)$$

are *logically equivalent*.

Exam, June 2003

5. Define the logical operator $\rightarrow$ and then show that the propositions

$$(p \vee q) \rightarrow r \quad \text{and} \quad (p \rightarrow r) \wedge (q \rightarrow r)$$

are *logically equivalent*

(a) using truth tables;

(b) using logical equivalences.

Class test, March 2004

6. Define the term *tautology*, and then determine whether the proposition

$$p \wedge q \rightarrow (p \rightarrow q)$$

is a tautology.

Exam, June 2004

7. The proposition "p nor q", denoted by $p \downarrow q$, is the proposition that is TRUE when both p and q are FALSE, and is FALSE otherwise.

(a) Construct the *truth table* for the logical operator $\downarrow$.

(b) Show that

$$p \downarrow p \iff \neg p.$$

(c) Show (by using a truth table or otherwise) that the propositions

$$(p \downarrow p) \downarrow (q \downarrow q) \quad \text{and} \quad p \wedge q$$

are *logically equivalent*.

Class test, August 2004

8.

(a) Construct the *truth table* for the logical operator $\rightarrow$.

(b) Determine whether the propositions

$$p \wedge q \leftrightarrow q \quad \text{and} \quad q \rightarrow p$$

are *logically equivalent.*

Supp Exam, February 2005

9.

(a) Suppose the variable x represents people, and

$$F(x): \ x \text{ is friendly}, \quad T(x): \ x \text{ is tall}, \quad A(x): \ x \text{ is angry}.$$

Write each of the following statements using the above predicates and any needed quantifiers :

i. Some people are not angry.

ii. All tall people are friendly.

iii. No friendly people are angry.

(b) Write each of the following in good English. DO NOT use variables in your answers.

$$(1) \quad A(Bill); \qquad (2) \quad \neg\exists x A(x) \wedge T(x); \qquad (3) \quad \neg\forall x F(x).$$

Class test, March 2000

10. Consider the predicates :

$$L(x,y); \ x < y, \quad Q(x,y): \ x = y, \quad E(x): \ x \text{ is even}, \quad G(x): \ x > 0,$$

$$\text{and} \quad I(x): \ x \text{ is an integer},$$

where the variables x and y represent real numbers. Write the following statements using the above predicates and any needed quantifiers :

(a) Every integer is even.

(b) If $x < y$, then x is not equal to y.

(c) There is no largest number.

(d) Some real numbers are not positive.

(e) No even integers are odd.

Exam, June 2000

11. Let $B(x)$, $G(x)$, $S(x)$, and $L(x, y)$ be the open sentences "x is a boy", "x is a girl", "x likes soccer", and "x likes y", respectively.

    (a) Use quantifiers and these predicates to express

    i. "Every boy likes some girl".

    ii. "All boys like all girls who like soccer".

    (b) In plain English negate the sentences (i) and (ii).

    (c) Use quantifiers and predicates to express the negated sentences. Simplify each expression so that no quantifier or implication remains negated.

12. Explain the terms *converse* and *contrapositive* of a conditional $p \to q$. Consider the proposition "Alice will win the game only if she plays by the rules."

    (a) Restate this proposition in good English in *three* different equivalent ways.

    (b) State the *converse* of this proposition.

    (c) State the *contrapositive* of this proposition.

    (d) Suppose that *Alice plays by the rules but loses.* Determine *with justification* whether the original proposition is TRUE or FALSE.

Supp Exam, February 2002

13. Consider the predicate

$$P(x, y) \, : \, x + 2y = xy$$

where the variables x and y represent real numbers. Determine *with justification* the truth values of the following propositions :

(a) $P(1, -1)$;

(b) $\forall x \exists y \, P(x, y)$;

(c) $\forall y \exists x \, P(x, y)$;

(d) $\exists x \exists y \, P(x, y)$;

(e) $\neg \forall x \exists y \, \neg P(x, y)$.

Class test, March 2004

14. Consider the predicate

$$P(x, y) \, : \, x^2 + 4y^2 = 4xy$$

where the variables x and y represent real numbers. Determine *with justification* the truth values of the propositions :

$$(1) \quad \forall x \exists y \, P(x, y) \quad \text{and} \quad (2) \quad \neg \exists y \forall y \, \neg P(x, y).$$

Exam, June 2004

15. Consider the predicate

$$T(x, y) \, : x \text{ is taking } y$$

where the variable x represents *students* and the variable y represents *courses*. Use quantifiers to express each of the following statements :

(a) No student is taking all courses.

(b) There is a course that no students are taking.

(c) Some students are taking no courses.

(d) Every student is being taken by at least one student.

Class test, August 2004

16. Consider the predicate

$$A(x) \, : x \text{ is aggressive}, \quad S(x) \, : x \text{ is short}, \quad T(x) \, : x \text{ is talkative}$$

where the variable x represents *people*. Write the following statements using the above predicate and any needed quantifiers :

(a) Some short people are not talkative.

(b) All talkative people are aggressive.

(c) No aggressive people are short.

Exam, November 2004

17. Investigate for *injectivity, surjectivity,* and *bijectivity* the function

$$f : \mathbb{R} \to \mathbb{Z}, \quad x \mapsto \lfloor x+2 \rfloor - \lfloor x \rfloor + \frac{x}{2}.$$

Class test, March 1999

18. TRUE or FALSE ? Motivate your answers.

(a) If A, B are sets, then $A \setminus B = A \setminus (A \cap B)$.

(b) If A, B, C are sets and $A \cup C = B \cup C$, then $A = B$.

(c) The function $f : \mathbb{Z} \to \mathbb{Z}, \quad f(x) = 2\lfloor \frac{x}{2} \rfloor$ is one-to-one.

(d) The function $g : \mathbb{N} \to \mathbb{N}, \quad g(n) = n!$ is *not* onto.

Exam, June 1999

19. Let $A = \{1, 2, 3, 4, 5\}$.

(a) Write *all* the subsets of A which contain the set $\{2, 5\}$.

(b) Write *all* the subsets B of A such that $B \cap \{2, 5\} = \{5\}$.

Class test, May 2000

20. Prove that for any sets A and B

$$(A \cap B) \cup (A \cap B^c) = A.$$

Class test, March 2001

21. Prove or disprove : If A, B, C are sets, then

$$(A \cup B) \cap C = A \cup (B \cap C).$$

Class test, August 2004

22. Let $A = \mathbb{R} \setminus \{1\}$ and consider the function

$$f : A \to A, \qquad x \mapsto \frac{x+1}{x-1} \cdot$$

(a) Determine $f(-2)$, $f(-1)$, $f(0)$, and $f\left(\frac{1}{3}\right)$.

(b) Investigate the function for *injectivity* and *surjectivity*.

(c) If the function is bijective, find its *inverse*.

(d) Find $f \circ f$ and $f \circ f \circ f$.

Class test, March 2001

23. TRUE or FALSE ? Motivate your answers.

(a) If A, B are sets, then $A \setminus (A \setminus B) = A \cap B$.

(b) If A, B are finite sets, then $|A \cup B| = |A| + |B|$.

(c) If A is a finite set, then any function $f : A \to A$ that is one-to-one is also onto.

(d) The function $g : \mathbb{N} \to \mathbb{N}, \quad n \mapsto 2n + 1$ is one-to-one but *not* onto.

Class test, March 2000

24. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 2x & \text{if } x < 0. \end{cases}$$

Sketch the graph of f and then investigate the function for *injectivity* and *surjectivity*. If the function is invertible, find its *inverse*.

Class test, March 2000

25. Let $A = \{1, 2, 3, 4\}$ and $B = \{\alpha, \beta\}$.

(a) Give an example of a function $f : A \to B$ that is a surjection but *not* an injection, and then explain why it meets these conditions.

(b) State the *domain, codomain*, and *range* of the function you gave in part (a). Is your function *invertible* ?

(c) Construct two different invertible functions from the set A onto the set (Cartesian product) $B \times B$. How many such functions are there ?

Class test, March 2003

26. TRUE or FALSE ? Justify your answers.

(a) The number 0 is an element of $\emptyset$.

(b) $\emptyset = \{\emptyset\}$.

(c) $\emptyset \in \{\emptyset\}$.

(d) If A, B, C are sets and $A \setminus C = B \setminus C$, then $A = B$.

(e) For all sets A and B, if $A \cap B = \emptyset$ then $A \times B = \emptyset$.

(f) If $f : A \to B$ and $g : B \to C$ are one-to-one, then $g \circ f : A \to C$ is also one-to-one.

(g) $\{a, b\} = \{b, a\}$.

(h) $\{a\} \subseteq \{\{a\}\}$.

(i) The power set of $\{\emptyset\}$ is $\{\{\emptyset\}\}$.

(j) The function $g : \{a, b, c\} \to \{1, 2, 3\}, \quad a \mapsto 2, b \mapsto 1, c \mapsto 3$ is *invertible*.

(k) There exists a function $f : \{1, 2, 3\} \to \{N, O, P, E\}$ which is *onto*.

(l) If $A, B, C \in 2^X$, then

$$A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C).$$

27. TRUE or FALSE ? Justify your answers.

(a) $\emptyset \in \{\emptyset\}$ and $\emptyset \subseteq \{\emptyset\}$.

(b) For A, B, C sets, $A \setminus (B \cap B) = (A \setminus B) \cup (A \setminus C)$.

(c) For A, B sets, if $A \setminus B = \emptyset$, then $A = B$.

(d) The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \dfrac{x-1}{x}$ is *invertible*.

(e) If $f : A \to B$ and $g : B \to C$ are both one-to-one functions, then the composite function $g \circ f : A \to C$ is also one-to-one.

Class test, March 2004

28. TRUE or FALSE ? Justify your answers.

(a) $\emptyset \in \{\emptyset\}$.

(b) For A, B sets, if $A \cup B = A \cap B$, then $A = B$.

(c) The function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto (x-1)^3$ is one-to-one but *not* onto.

Exam, June 2004

29. Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Recall the definitions of *the floor function* and *the ceiling function*, and then show that

(a) $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ (or, equivalently, $0 \leq x - \lfloor x \rfloor < 1$).

(b) $\lceil x \rceil - 1 < x \leq \lceil x \rceil$ (or, equivalently, $0 \leq \lceil x \rceil - x < 1$).

(c) $\lceil x \rceil + \lfloor -x \rfloor = 0$.

(d) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$.

(e) $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

30. Consider the function

$$f : \mathbb{R} \to \mathbb{Z}, \quad x \mapsto \lfloor 1-x \rfloor := \max\{n \in \mathbb{Z} \,|\, n \leq 1-x\}.$$

(a) Determine $f(-1)$, $f(0)$, $f\left(\frac{1}{2}\right)$, and $f(1)$.

(b) Sketch the *graph* of f.

(c) Explain what is meant by saying that a function is *injective* (or one-to-one), *surjective* (or onto). Hence determine *with justification* whether function f is injective or/and surjective.

Class test, August 2004

31. Let $a, b, c, d \in \mathbb{R}$. Prove that

(a) $a^2 + b^2 \geq 2ab$ with equality if and only if $a = b$.

(b) If $a + b + c \geq 0$, then

$$a^3 + b^3 + c^3 \geq 3abc$$

with equality if and only if $a = b = c$ or $a + b + c = 0$.

(c) If $a, b, c, d \geq 0$, then

$$a^4 + b^4 + c^4 + d^4 \geq 4abcd$$

with equality if and only if $a = b = c = d$.

[Hint : (b) Expand the product

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

(c) Write

$$\frac{a^4+b^4+c^4+d^4}{4} = \frac{\frac{a^4+b^4}{2} + \frac{c^4+d^4}{2}}{2}.]$$

32. Let $a, b, c, a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$. Prove that :

    (a) (**Mean inequalities**) If $0 < a \leq b \leq c$, then

    $$a \leq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \leq \sqrt[3]{abc} \leq \frac{a+b+c}{3} \leq \sqrt{\frac{a^2+b^2+c^2}{3}} \leq c$$

    with equality if and only if $a = b = c$.

    (b) (**Cauchy-Schwarz inequality**)

    $$(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

    with equality if and only if $a_1 = rb_1$, $a_2 = rb_2$ and $a_3 = rb_3$ $(r \in \mathbb{R})$.

    (c) (**Chebyshev inequality**) If $a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2 \leq b_3$, then

    $$(a_1 + a_2 + a_3)(b_1 + b_2 + b_3) \leq 3(a_1b_1 + a_2b_2 + a_3b_3)$$

    with equality if and only if $a_1 = a_2 = a_3$ and $b_1 = b_2 = b_3$.

33. Let $x, y \in \mathbb{R}$. Recall the definition of *the absolute value function* and then show that

    (a) $|x| \geq 0$; $|x| = 0 \iff x = 0$.

    (b) $|xy| = |x||y|$.

    (c) $|x + y| \leq |x| + |y|$.

    (d) $|\,|x| - |y|\,| \leq |x - y|$.

34. Let $a, b, c \in \mathbb{R}$ such that $1 \leq a$ and $b \leq c$. Prove that

    $$(1 + a)(b + c) \leq 2(b + ac)$$

    with equality if and only if $a = 1$ or $b = c$.

Class test, March 2004

35. Let $\alpha, \beta \in S_5$ be given by

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \beta = (1,2,4)(3,5).$$

(a) Write the permutation α in *cycle form.*

(b) Calculate

$$\alpha\beta, \quad \beta\alpha, \quad \alpha^2, \quad \beta^{-1}, \quad \text{and} \quad \beta^{-1}\alpha^{-1}.$$

(c) Determine the *signature* of

$$\alpha, \quad \alpha^2, \quad \text{and} \quad (\alpha\beta)^{-1}.$$

Class test, May 2001

36. Write all permutations on three elements in *cycle form*, and then find *all* permutations $\alpha \in S_3$ such that $\alpha^2 = (1,2,3)$.

Class test, May 2001

37. Define the term *permutation on n elements* and then explain what is meant by the *cycle notation.* Consider the permutation $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{bmatrix}$.

(a) Write $\alpha, \alpha^2,$ and α^{-1} in cycle form.

(b) Determine the permutation σ so that $\alpha\sigma = \alpha^{-1}$.

Class test, March 2003

38.

(a) Define the term *permutation on n elements* and then explain what is meant by the *cycle notation.*

(b) Consider the permutation $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{bmatrix}$.

i. Write $\alpha, \alpha^2,$ and α^{-1} in cycle form.

ii. Determine the permutation π so that $\alpha\pi = \alpha^{-1}$.

Class test, March 2004

39. Consider the permutations (on five elements)

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{bmatrix}.$$

Write each of the permutations

$$\alpha, \beta, \alpha\beta, \alpha^2, \quad \text{and} \quad \beta^{-1}$$

in *cycle form.*

Exam, June 2004

40. Calculate the sum

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)}$$

for $n = 1, 2, 3, 4,$ and 5. Guess a general formula for S_n, and then prove by *mathematical induction* this formula.

Class test, March 1999

41. Calculate the sum

$$S_n = \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots + \frac{2}{3^n}$$

for $n = 1, 2, 3,$ and 4. Guess a general formula for S_n, and then prove this formula by *mathematical induction.*

Exam, June 1999

42. For which natural numbers n is $n! > 2^n$? Prove your answer using *mathematical induction.*

Class test, March 2000

43. Calculate the sum

$$S_n = \frac{1}{n^2} + \frac{3}{n^2} + \frac{5}{n^2} + \cdots + \frac{2n-1}{n^2}$$

for $n = 1, 2, 3,$ and 4. Guess a general formula for S_n, and then prove this formula by mathematical induction.

Exam, June 2000

44.

(a) Write the sum

$$S = \frac{1}{3} - \frac{2}{5} + \frac{3}{7} - \frac{4}{9} \cdots + \frac{99}{199}$$

in *sigma notation*; that is in the form $\sum_{i=i_0}^{N} a_i$.

(b) Find

$$T = \sum_{i=1}^{199} (2i+1).$$

Class test, May 2001

45. Use *mathematical induction* to prove that

$$n(n+1)(n+2) \quad \text{is divisible by } 6$$

for every positive integer n.

Class test, May 2001

46. Calculate the sum

$$S_n = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1) \cdot (2n+1)}$$

for $n = 1, 2, 3, 4$ and 5. Guess a general formula for S_n, and then prove this formula by mathematical induction.

Class test, May 2001

47. Prove by *mathematical induction* that

$$5^n - 1 \vdots 4$$

for any natural number n.

Class test, March 2003

48. Prove by *mathematical induction* that

$$3^{2n-1} + 4^{2n-1} \vdots 7$$

for any positive integer number n.

49. Consider the sum

$$S_n = 4 \cdot 7 + 7 \cdot 10 + 10 \cdot 13 + 13 \cdot 16 + \cdots \quad \text{to } n \text{ terms.}$$

(a) What is the n^{th} term of S_n ?

(b) Calculate the sum.

(c) Use *mathematical induction* to verify your result.

Class test, March 2003

50. Prove by *mathematical induction* that

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 2\left(1 - \frac{1}{2^{n+1}}\right)$$

for all natural numbers n.

Exam, June 2003

51. Let $a, b \in \mathbb{R}$.

(a) Prove that

$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

with equality if and only if $a = b$.

(b) Use *mathematical induction* to prove that

$$\left(\frac{a+b}{2}\right)^n \leq \frac{a^n+b^n}{2}$$

for all natural numbers n.

Exam, June 2003

52.

(a) Explain the notation $n!$ (read : "n factorial"), and then compare the numbers 3^6 and $6!$.

(b) Prove by *mathematical induction* that

$$3^n < n! \quad \text{for all integers } n \geq 7.$$

Class test, May 2004

53. Explain the notation $a \,\vdots\, b$ (read : "a is divisible by b"), and then prove by *mathematical induction* that

$$n^3 - n \,\vdots\, 6 \quad \text{for all integers } n \geq 1.$$

Exam, June 2004

54. Consider the sum

$$S_n = 3\cdot 5 + 5\cdot 7 + 7\cdot 9 + 9\cdot 11 + \cdots \quad \text{to } n \text{ terms.}$$

(a) What is the n^{th} term of S_n ?

(b) Calculate the sum. (Hint : You may use the sums $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^2$.)

(c) Use *mathematical induction* to verify your result.

Class test, May 2004

55.

(a) Explain the notation $\sum_{i=p}^{q} a_i$, and then calculate (the sum)

$$S_n = \sum_{i=1}^{n} (2i - 1).$$

(Hint: You may use the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \cdot$)

(b) Use *mathematical induction* to verify your result.

Exam, June 2004

56. Consider the sum

$$S_n = 4 + 7 + 10 + 13 + \cdots \quad \text{to } n \text{ terms.}$$

(a) What is the n^{th} term of S_n ?

(b) Calculate the sum. (Hint : You may use the sum $\sum_{i=1}^{n} i$.)

(c) Use *mathematical induction* to verify your result.

Class test, October 2004

57. Prove by *mathematical induction* that

$$3^{2n+1} + 4^{2n+1} \quad \text{is divisible by } 7$$

for all natural numbers n.

Class test, October 2004

58. Consider the sum

$$S_n = \sum_{k=1}^{n} (3k + 2).$$

(a) Calculate the sum. (HINT : You may use the formula $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.)

(b) Use *mathematical induction* to verify your result.

Exam, November 2004

59. Using only the digits $1, 2, 3, 4, 5, 6,$ and 7, how many five-digit numbers can be formed that satisfy the following conditions ?

(a) no additional conditions ;

(b) at least one 7 ;

(c) no repeated digits ;

(d) at least one 1 and at least one 7.

Class test, May 1999

60. How many solutions are there to the equation

$$x + y + z = 100$$

if x, y, z are natural numbers ?

Class test, May 1999

61. Consider a group of 12 people consisting of 7 men and 5 women. How many 5-person teams can be chosen

(a) that contain 3 men and 2 women ?

(b) that contain at least one man ?

(c) that contain at most one man ?

(d) if a certain pair of people insist on been selected together or not at all ?

(e) if a certain pair of people refuse to be selected together ?

Exam, June 1999

62. How many positive integers not exceeding $1\,000$ are *not* divisible by either 8 or 12 ?

63.

(a) How many functions are there from a set with 3 elements to a set with 8 elements ?

(b) How many one-to-one functions are there from a set with 3 elements to a set with 8 elements ?

(c) How many onto functions are there from a set with 3 elements to a set with 8 elements ?

(d) What is the coefficient of x^3y^8 in $(x+y)^{11}$ and $(2x+3y)^{11}$?

Class test, May 2000

64. How many ways are there to assign six jobs to four employees so that every employee is assigned at least one job ?

Class test, May 2000

65.

(a) Evaluate $\binom{4}{2}$ by listing all *sets* of size 2 whose elements belong to the set $\{a, b, c, d\}$.

(b) Evaluate $\left(\!\binom{2}{4}\!\right)$ by listing all *multisets* of size 4 whose elements belong to the set $\{0, 1\}$.

(c) List all 2-*permutations with repetition* of the set $\{0, 1, a, b\}$.

Exam, June 2001

66. Determine the number of bit strings of length 10 that have

(a) exactly three 0s.

(b) the same number of 0s and 1s.

(c) at least seven 1s.

Exam, June 2001

67.

(a) Consider the set $S = \{0, 1, a, b\}$. List *all*

- 2-permutations
- 3-combinations
- 2-permutations with repetition
- 3-combinations with repetition.

of the set S.

(b) A person giving a party wants to set out 10 assorted cans of soft drink for his guests. He shops at a store that sells 5 different types of soft drinks. Use *multisets* to count how many selections of 10 soft drinks he can make.

Exam, June 2003

68. Explain what is meant by a *bit string*, and then determine the number of bit strings of length eight that

(a) have exactly three 0s;

(b) start and end with an 1;

(c) have at least six 1s.

Exam, June 2004

69.

(a) How many integers from 1 to 1 000 are *divisible* by 3 ?

(b) How many integers from 1 to 1 000 are *divisible* by 3 *and* 7 ?

(c) How many integers from 1 to $1\,000$ are *divisible* by 3 *or* 7 ?

(d) How many three-digit integers (i.e. integers from 100 through 999) are *divisible* by 6 *and* 9 ?

Class test, October 2004

70. How many different committees of *four* can be selected from a group of *twelve* people if

(a) a certain pair of people insist on serving together or not at all ?

(b) a certain pair of people refuse to serve together ?

Class test, October 2004

71. How many different committees of *six* can be selected from a group of *four* men and *four* women

(a) that consist of three men and three women ?

(b) that consist of *at least* one woman ?

(c) that consist of *at most* one man ?

Supp Exam, February 2005

72. Find :

(a) the coefficient of x^5y^6 in the expansion of $(2x-y)^{11}$.

(b) the middle term in the expansion of $(1-a)^{14}$.

(c) the largest coefficient in the expansion of $(x+2)^7$.

(d) $\binom{20}{1}+\binom{20}{2}+\binom{20}{3}+\cdots+\binom{20}{19}$.

Class test, May 1999

73. TRUE or FALSE ?

(a) The coefficient of x^2y^9 in the expansion of $(x+y)^{11}$ is 55.

(b) The *middle term* in the expansion of $\left(a - \frac{1}{a}\right)^{12}$ is -264.

(c) The largest coefficient in the expansion of $\left(x + \frac{1}{x}\right)^{7}$ is 35.

Exam, June 2001

74. Write down the *binomial formula*, and then use this formula to

(a) expand $(x-y)^5$;

(b) find the *middle term* in the expansion of $(2-a)^{10}$;

(c) calculate the sum

$$\binom{50}{0} + \binom{50}{1} + \binom{50}{2} + \cdots + \binom{50}{50}.$$

Exam, June 2004

75. Describe the sequence $(a_n)_{n\geq 0}$: $0, 1, 0, 1, 0, 1, \ldots$ recursively (include initial conditions) and then find an explicit formula for the sequence.

Class test, May 1999

76. Consider the sequence (of decimal fractions)

$$a_1 = 0.1, \quad a_2 = 0.11, \quad a_3 = 0.111, \quad a_4 = 0.1111, \quad \ldots$$

(a) Observe that

$$a_1 = \frac{1}{10}; \quad a_2 = \frac{1}{10} + \frac{1}{100}; \quad a_3 = \frac{1}{10} + \frac{1}{100} + \frac{1}{1000}.$$

Hence express a_n as a sum (of n terms).

(b) Show that

$$a_n = \frac{1}{9}\left(1 - \frac{1}{10^n}\right).$$

(c) Describe the sequence $(a_n)_{n\geq 1}$ *recursively*. (Include initial conditions.)

Class test, October 2004

77. Describe the sequence

$$1, \ 11, \ 111, \ 1111, \ 11111, \ \cdots$$

recursively. (Include initial conditions.)

Supp Exam, February 2005

78. Solve the following *recurrence relation* :

$$a_n = a_{n-2} + n; \quad a_0 = 1, \, a_1 = 2.$$

Exam, June 1999

79. Let d_n denote the number of people infected by a disease and suppose that *the change in the number infected in any period is proportional to the change in the number infected in the previous period.* Show that there exists some constant k such that

$$d_{n+2} - d_{n+1} = k(d_{n+1} - d_n).$$

Obtain an expression for d_n (in terms of d_0 and d_1) for $k = 2$.

Exam, June 1999

80.

(a) Find a *recurrence relation* of the number of ways to climb n stairs if stairs can be climbed two or three at a time.

(b) What are the initial conditions ?

(c) How many ways are there to climb eight stairs ?

Class test, May 2000

81. Find the solution of the *recurrence relation*

$$a_n = \frac{1}{2} a_{n-2}; \quad a_0 = a_1 = 1$$

and then determine a_{20}.

Class test, May 2000

82. Solve the *recurrence relation* :

$$a_n = a_{n-2} + n; \quad a_0 = 1, a_1 = \frac{7}{4}.$$

Exam, June 2000

83. Solve the following *recurrence relation* :

$$F_n = F_{n-1} + F_{n-2}; \quad F_0 = 1, F_1 = 0.$$

Exam, June 2001

84.

(a) Find a *recurrence relation* for the number of bit strings of length n that contain three consecutive 0s.

(b) What are the initial conditions ?

(c) How many bit strings of length seven contain three consecutive 0 ?

Exam, June 2001

85. Solve the following *recurrence relation* :

$$a_n = 2a_{n-1} - a_{n-2} + 2^n; \quad a_0 = 4, a_1 = 9.$$

86.

(a) Find a *recurrence relation* for the number of bit strings of length n that do *not* contain the pattern 11.

(b) What are the initial conditions ?

(c) How many bit strings of length eight do not contain the pattern 11 ?

Exam, June 2003

87. Let R_n denote the maximum number of regions into which n straight lines can cut a plane.

(a) Find the values of R_0, R_1, R_2, and R_3.

(b) Show that R_n satisfies the *recurrence relation*

$$R_n = R_{n-1} + n.$$

Give reasons for your answer.

(c) Solve this recurrence relation.

88. Solve the *recurrence relation*

$$a_n = -2a_{n-1} - a_{n-2} + n; \quad a_0 = \frac{1}{4} \text{ and } a_1 = \frac{1}{2}\cdot$$

(Hint : Try a particular solution of the form $An + B$.)

Exam, June 2004

89. Solve the *recurrence relation*

$$a_n = a_{n-1} + 10^{-n}, \quad a_1 = 10^{-1}.$$

(Hint : Try for a particular solution of the form $K \cdot 10^{-n}$.)

Class test, October 2004

90. Use *Gaussian elimination* to solve the linear system :

$$\begin{aligned} x - 2y + 3z &= 7 \\ 2x - 3y - z &= 6 \\ x - 3y + 10z &= 15 \end{aligned}$$

and then give a geometric interpretation.

Exam, June 1999

91. Use *Gaussian elimination* to solve the linear system :

$$\begin{aligned} x + 5y &= 12 \\ 3x - 7y &= 14 \\ 2x - 4y &= 10 \end{aligned}$$

and then give a geometric interpretation.

Exam, June 2000

92. Use *Gaussian elimination* to solve the following linear system :

$$\begin{aligned} 2x + 2y + z &= 3 \\ x + y + 3z &= -1 \\ 3x + 3y + 4z &= 2. \end{aligned}$$

Exam, November 2001

93. Explain what is meant by *linear system* and then use *Gaussian elimination* to solve the linear system :

$$\begin{aligned} 3x + y + z &= 0 \\ x - 2y + 3z &= 7 \\ 2x + y - z &= -1 \\ -x - 4y + 5z &= 11. \end{aligned}$$

Class test, May 2003

94. Use *Gaussian elimination* to solve the following linear system :

$$\begin{aligned} x + y + z &= 2 \\ 2x + 3y + z &= -1 \\ 3x + y + 2z &= 7 \\ x - 2y + z &= 8. \end{aligned}$$

Exam, June 2003

95. Use *Gaussian elimination* to solve the linear system

$$\begin{aligned} 2x - y + 3z &= 0 \\ x + 4y - z &= 1 \\ x + 13y - 6z &= 3. \end{aligned}$$

Class test, May 2004

96. Consider the linear system

$$\begin{aligned} x - 2y + 3z &= 0 \\ 3x + y - z &= 1 \\ x + 5y - 7z &= k. \end{aligned}$$

Determine for what values of the parameter $k \in \mathbb{R}$ the given system is *consistent*. If so, solve the system (by using *Gaussian elimination*).

Exam, June 2004

97. Consider the linear system

$$\begin{aligned} x - y + 4z &= -2 \\ 2x - y - 5z &= 0 \\ x - 2y + 7z &= k. \end{aligned}$$

(a) Determine for what values of the parameter $k \in \mathbb{R}$ the given system has *at least* one solution.

(b) Solve the system for those values of k determined in (a).

Supp Exam, February 2005

98. Find the inverse of the following matrix :

$$A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Use A^{-1} to solve the equation $Ax = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

Exam, June 1999

99. Evaluate the determinant

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{vmatrix}.$$

Exam, June 1999

100. Consider the matrix

$$A(x) = \begin{bmatrix} 0 & 1 & x \\ 1 & x & x \\ x & x & x \end{bmatrix}.$$

(a) Find all values of $x \in \mathbb{R}$ such that $A(x)$ is *invertible.*

(b) Compute the *inverse* of the matrix $A(2)$.

Class test, May 2000

101. Given the matrix $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, find all matrices A such that

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

How many solutions A does this problem have ?

Exam, June 2000

102. Find $\lambda \in \mathbb{R}$ such that the matrix

$$A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

is *not* invertible (i.e. $\det(A) = 0$).

Exam, June 2000

103. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}.$$

(a) Find the *inverse* A^{-1}.

(b) Show that your answer in (a) is correct without repeating the same calculation.

(c) Use A^{-1} to solve the linear system :

$$\begin{aligned} x + 2y + 3z &= 0 \\ 4y + 5z &= -3 \\ 6z &= 6. \end{aligned}$$

Class test, August 2001

104. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find all values $\lambda \in \mathbb{R}$ such that the matrix $A - \lambda I_3$ is *not* invertible.

Class test, August 2001

105. Evaluate the determinant :

$$\begin{vmatrix} 0 & -1 & 7 & 8 \\ 1 & 0 & -1 & 9 \\ 2 & 3 & 0 & -1 \\ 4 & 5 & 6 & 0 \end{vmatrix}.$$

Class test, August 2001

106. Consider the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -4 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Calculate A^2, $2E - A$, $\det(A)$, and $\det(2E - A)$.

(b) Is matrix A *invertible* ? Explain.

(c) Is matrix $2E - A$ *invertible* ? Explain.

(d) Find the *inverse* matrix $(2E - A)^{-1}$.

Exam, November 2001

107. Give the definition of the *determinant* of an $n \times n$ matrix and then evaluate the determinants

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{vmatrix}.$$

Class test, May 2003

108. Consider the matrices

$$A = \begin{bmatrix} 0 & 1 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}.$$

(a) Compute the matrices A^2, A^3, CB, and $A + BC$.

(b) Find the *inverse* $(A + BC)^{-1}$.

(c) Show that your answer in (b) is correct without repeating the same calculation.

(d) Let

$$E(t) := I + tA + \frac{t^2}{2}A^2 \qquad (t \in \mathbb{R})$$

where I is the identity matrix. Determine (by direct computation) whether the following equality holds for every $t, s \in \mathbb{R}$

$$E(t+s) = E(t)E(s).$$

Class test, May 2003

109. Show (by direct computation) that

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = x + y + z + 1.$$

Give a geometric interpretation.

Exam, June 2003

110. Consider the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -17 & -1 \end{bmatrix}.$$

(a) Find $A^2, A + BC$, and CB.

(b) Explain the notation A^T (read : "A transpose"), and then compute the matrix $AA^T - A^TA$.

(c) Is matrix A *invertible* ? If so, find its inverse A^{-1}.

Class test, May 2004

111. Use *Gaussian elimination* and *Laplace expansion* to evaluate – in two different ways – the determinant

$$\begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 2 & 4 & 0 & -1 \\ 3 & 5 & 6 & 0 \end{vmatrix}.$$

Class test, May 2004

112. Explain what is meant by saying that two (square) matrices *commute*, and then find all 2×2 matrices X which commute with $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

Exam, June 2004

113. Use *Gaussian elimination* and *Laplace expansion* to evaluate – in two different ways – the determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 5 & 0 \\ 3 & 7 & 8 & 9 \\ 4 & 0 & 10 & 6 \end{vmatrix}.$$

Exam, June 2004

114.

(a) Find the *inverse* of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \\ 3 & 0 & 1 \end{bmatrix}.$$

(b) Use the matrix A^{-1} to solve (for X) the equation

$$AX = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

(The unknown X is a 3×1 matrix.)

(c) Use *Cramer's rule* to solve the linear system

$$\begin{aligned} x + 2y &= 3 \\ 3y + 2z &= 1 \\ 3x + z &= 2. \end{aligned}$$

Class test, May 2004

115. Consider the matrices

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(a) Calculate

$$I + A^2, \quad (I + A)^2, \quad I + A^{-1} \quad \text{and} \quad (I + A)^{-1}.$$

(b) Determine the *inverse* of the matrix

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Exam, November 2004

116. Consider the matrices

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Calculate

(a) $A^{-1} + B^{-1}$.

(b) $(A + B)^{-1}$.

(c) $AB + BA$.

(d) $A^2 + B^2$.

(e) $(A + B)^2$.

Supp Exam, February 2005

117. Calculate the determinant

$$\begin{vmatrix} 0 & -1 & 2 & 3 \\ 1 & 0 & -4 & 0 \\ -2 & 4 & 0 & -5 \\ -3 & 0 & 5 & 0 \end{vmatrix}.$$

Class test, November 2004

118. Evaluate the determinant

$$\begin{vmatrix} 0 & -1 & \alpha & 2 \\ 1 & 0 & -1 & 0 \\ -\alpha & 1 & 0 & -1 \\ -2 & 0 & 1 & 0 \end{vmatrix}.$$

Supp Exam, February 2005

119. Let $a, b \in \mathbb{R}$ such that $a + b \neq 0$. Use *Cramer's rule* to solve for x, y, and z (in terms of a and b)

$$\begin{aligned} ax + by &= 1 \\ ay + bz &= 1 \\ az + bx &= 1. \end{aligned}$$

Exam, November 2001

120. Let $u, v \in \mathbb{R} \setminus \{1\}$ such that $u \neq v$. Use *Cramer's rule* to solve for x, y, and z (in terms of u and v)

$$\begin{aligned} x + uy + vz &= 1 \\ y + uz + vx &= -1 \\ z + ux + vy &= 0. \end{aligned}$$

Supp Exam, February 2002

121. Use *Cramer's rule* to solve for $x, y,$ and z (in terms of $u, v,$ and w)

$$\begin{aligned} -x+y+z &= u \\ x-y+z &= v \\ x+y-z &= w. \end{aligned}$$

Class test, May 2003

122. Let $\theta \in \mathbb{R}$. Use *Cramer's rule* to solve for $x, y,$ and z (in terms of $u, v,$ and w)

$$\begin{aligned} x &= u \\ (\cos\theta)y-(\sin\theta)z &= v \\ (\sin\theta)+(\cos\theta)z &= w. \end{aligned}$$

Exam, June 2004

123. Let $u, v, w \in \mathbb{R}$ such that $uvw \neq -1$. Use *Cramer's rule* to solve for $x, y,$ and z (in terms of $u, v,$ and w)

$$\begin{aligned} x+uy &= 1 \\ y+vz &= 0 \\ z+wx &= 0. \end{aligned}$$

Exam, November 2004

124. Let $a, b, c \in \mathbb{R}$ such that $abc \neq -1$. Use *Cramer's rule* to solve for $x, y,$ and z (in terms of $a, b,$ and c)

$$\begin{aligned} x+by &= c \\ y+cz &= a \\ z+ax &= b. \end{aligned}$$

Supp Exam, February 2005

125. TRUE or FALSE ?

(a) The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ is *invertible.*

(b) The vectors $\begin{bmatrix} 4 \\ -6 \\ -10 \end{bmatrix}$ and $\begin{bmatrix} -6 \\ 9 \\ 15 \end{bmatrix}$ are *collinear.*

(c) The equations

$$\vec{r} = \begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \frac{x-1}{-6} = \frac{y}{-2} = \frac{z+1}{2}$$

represent the *same* line.

Class test, May 2003

126. Consider the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} -6 \\ 0 \\ 3 \end{bmatrix}.$$

(a) Use the *dot product* to find the angle between $\vec{u}$ and $\vec{v}$.

(b) Use the *cross product* to find a vector $\vec{w}$ orthogonal to both $\vec{u}$ and $\vec{v}$ and such that $\|\vec{w}\| = 3$.

Exam, June 1999

127. TRUE or FALSE ?

(a) The points $A = (1, 2, -1)$, $B = (4, 2, 0)$, $C = (-2, -2, -2)$ are *collinear.*

(b) The lines $x = 1 + 2t, y = 2t, z = 1 - t$ and $\frac{x+1}{2} = \frac{y+2}{2} = \frac{z}{1}$ are *parallel.*

(c) The distance from the point $(1, -1, 1)$ to the plane $x + y + z = 3$ is $\frac{\sqrt{3}}{3}$.

Exam, June 1999

128. Consider the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

(a) Use the *dot product* to find the angle between $\vec{u}$ and $\vec{v}$.

(b) Compute the *area* of the triangle determined by $\vec{u}$ and $\vec{v}$.

(c) Find all values of $k \in \mathbb{R}$ such that the vectors $k\vec{u} + (1-k)\vec{v}$ and $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ are *collinear*.

Exam, June 2000

129. Consider the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

(a) Calculate

$$\vec{v} - 2\vec{u}, \quad (\vec{v} + \vec{u}) \bullet (\vec{v} - \vec{u}), \quad \text{and} \quad \|\vec{u}\| + \|\vec{v}\| - \|\vec{u} + \vec{v}\|.$$

(b) Find the *angle* between $\vec{u}$ and $\vec{v}$.

Class test, October 2001

130. TRUE or FALSE ? Justify your answers.

(a) The points $A(1, 1)$, $B(-5, 4)$ and $C(7, -2)$ are *collinear*.

(b) The vectors $\vec{u} = \begin{bmatrix} 1 \\ \alpha \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ \alpha \end{bmatrix}$ are *orthogonal* for some value of α.

(c) Given the points $P(2,1)$, $Q(3,4)$ and $R(1,3)$, the *area* of the parallelogram $\square\, OPQR$ is 5.

Exam, November 2004

131. Consider the planes

$$(\alpha) \quad x - 2y - z = 0 \quad \text{and} \quad (\beta) \quad 2x - y + 3z = 5.$$

(a) Find *parametric equations* of the line $\mathcal{L}$ of intersection of these planes.

(b) Write *symmetric equations* for the line through the point $P(1,1,-2)$ and parallel to the line $\mathcal{L}$.

(c) Find the equation of the plane through the point $P(1,1,-2)$ and perpendicular to the line $\mathcal{L}$.

Class test, October 2001

132. Consider the points

$$A(2,0,0), \quad B(1,1,0), \quad \text{and} \quad C(0,0,3).$$

(a) Find (the components of) the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) Determine the *angle* (in degrees) between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(c) Compute the *area* of the triangle $\triangle\, ABC$.

Class test, October 2001

133. Consider the points

$$A(-1,0), \quad B(3,0), \quad \text{and} \quad C(0,2).$$

(a) Find (the coordinates of) the point D such that the quadrilateral $\square\, ABDC$ is a *parallelogram.*

(b) Find (the components of) the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{AC}$.

(c) Compute and compare

$$(\overrightarrow{AB} + \overrightarrow{BC}) \bullet \overrightarrow{AC} \quad \text{and} \quad \| \overrightarrow{AC} \|^2.$$

Explain.

(d) Determine the *angle* (in degrees) between the vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$.

Exam, November 2001

134. Consider the points

$$A(-2,0), \quad B(1,0), \quad \text{and} \quad C(-1,3).$$

(a) Find (the coordinates of) the point M such that

$$\frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{AC}\right) = \overrightarrow{AM}\ .$$

(b) Find (the components of) the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{AC}$.

(c) Compute

$$(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CA}$$

and then explain the result obtained.

(d) Determine the *angle* (in degrees) between the vectors $\overrightarrow{AB}$ and $\overrightarrow{CA}$.

Supp Exam, February 2002

135. Consider the point $P(1,0,2)$ and the vectors

$$\vec{u} = \vec{i} - \vec{j} \quad \text{and} \quad \vec{v} = 3\vec{i} + \vec{k}.$$

(a) Determine the *angle* (in degrees) between the vectors $\vec{u}$ and $\vec{v}$.

(b) Write *parametric equations* for the line $\mathcal{L}$ through P with direction $\vec{u}$.

(c) Write the equation of the plane π through P with normal direction $\vec{v}$.

(d) Find the *distance* from the origin to the line $\mathcal{L}$.

Class test, May 2003

136. Consider the points

$$A(-1,0), \quad B(3,0), \quad \text{and} \quad C(0,2).$$

(a) Find (the coordinates of) the point D such that the quadrilateral $\square\, ABDC$ is a *parallelogram*.

(b) Find (the components of) the vectors $\overrightarrow{AB}$, $\overrightarrow{AC} - \overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{AD}$.

(c) Determine the *angle* (in degrees) between the vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$.

Supp Exam, February 2002

137. Consider the point $P(1,-1,1)$, the line $\mathcal{L}$ with equations

$$\frac{x-1}{1} = \frac{y-2}{1} = \frac{z}{-2}$$

and the plane π with equation

$$x + y + z - 3 = 0.$$

(a) Write *parametric equations* for the line $\mathcal{L}$.

(b) Find *equations* for

i. the plane through P and parallel to π;

ii. the plane through P and the line $\mathcal{L}$.

(c) Determine the *angle* between these two planes.

(d) Compute the *distance* from the point P to

i. the line $\mathcal{L}$;

ii. the plane π.

Exam, June 2003

138. TRUE or FALSE ? Justify your answers.

(a) The points $P(-1,2,3), Q(0,1,3)$, and $R(-4,5,3)$ are *collinear*.

(b) The *angle* between the vectors $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$ is $180°$.

(c) Given the points $A(0,2), B(-1,0)$, and $C(3,-1)$, the *area* of the triangle $\triangle ABC$ is 9.

Exam, June 2004

139. Consider the points $A(1,-1,0)$ and $B(2,0,3)$, and the plane π with equation

$$x + y - 2z + 8 = 0.$$

(a) Write *parametric equations* for the line $\mathcal{L}$ determined by the points A and B.

(b) Find the point P of intersection of the line $\mathcal{L}$ with the given plane π.

(c) Determine the (orhogonal) projection B' of the point B onto the plane π. (Hint : The point B' is the intersection of a certain line through B with the given plane.)

(d) Write *symmetric equations* for the line $\mathcal{L}'$ determined by the points P and B'.

Exam, June 2004

140. Consider the points $A(1,1,-3)$ and $B(1,-1,5)$, and the plane π with equation

$$y - 4z + 4 = 0.$$

(a) Write

- *parametric equations*
- *symmetric equations*

for the line $\mathcal{L}$ determined by the points A and B.

(b) Find the unique point L of intersection of line $\mathcal{L}$ with plane π.

(c) Calculate the *distance* between L and the midpoint M of the segment $\overline{AB}$.

(d) Determine whether the point $P(4, 0, 1)$ lies on the plane π.

(e) Determine the distance from P to the line $\mathcal{L}$. (HINT : You may use the formula

$$\delta = \frac{\| \overrightarrow{AP} \times \overrightarrow{AB} \|}{\| \overrightarrow{AB} \|}$$

for the distance from point P to line $\overleftrightarrow{AB}$.)

Exam, November 2004

141.

(a) If $z = 2 + i$ and $w = 1 + 3i$, find :

i. $3z + \bar{w}$;

ii. $(1 + i)z + w^2$;

iii. $\arg(z \cdot w)$;

iv. $|z - 2w|$;

v. $\operatorname{Re}\left[(z-1)^{10} + (\bar{z}-1)^{10}\right]$;

vi. $\operatorname{Im}\left(\frac{w+\bar{w}}{2i}\right)$.

(b) Solve the equation (for $z \in \mathbb{C}$)

$$z^6 = 1$$

and represent the roots as points in the complex plane.

Exam, June 2000

142. Let $w = \sqrt{3} + i$.

(a) Compute $w^2, \bar{w}, w^{-1}$, and $|w|$.

(b) Write w in *polar* (and exponential) form.

(c) Solve the equation (for $z \in \mathbb{C}$)

$$z^3 = w$$

and represent the roots as points in the complex plane.

Class test, November 2001

143. Let $w = -1 - i$.

(a) Compute $w^2, \bar{w}, w^{-1}$, and $|w|$.

(b) Write w in *polar* (and exponential) form.

(c) Solve the equation (for $z \in \mathbb{C}$)

$$z^3 = -1$$

and represent the roots as points in the complex plane.

Supp Exam, February 2002

144. Let $w = \sqrt{3} - i$.

(a) Compute w^2, $\bar{w}$, and w^{-1}.

(b) Write w in *exponential form* and then compute

$$w^6 + \bar{w}^6.$$

(c) Solve the equation (for $z \in \mathbb{C}$)

$$z^6 - 2\sqrt{3}z^3 + 4 = 0.$$

Exam, June 2003

145. Considert the complex number $w = \sqrt{3} - i$.

(a) Compute $\bar{w}, |w|$, and w^{-1}.

(b) Write w in *polar* form.

(c) Evaluate the expression

$$E = w^{18} + \bar{w}^{18}.$$

(HINT : You may use De Moivre's formula.)

Exam, June 2004

146. Solve the equation (for $z \in \mathbb{C}$)

$$z^4 - i = 0$$

and represent the roots as points in the complex plane. (HINT : Write the roots in *exponential* form.)

Exam, June 2004

147.

(a) Write $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^{1999}$ in the form $a + ib$.

(b) Show how $\sin\theta$ can be written in terms of the complex exponential function and use this to obtain $\sin^4\theta$ in terms of cosines of multiples of θ.

(c) Find all solutions of

$$z^5 + 1 = 0$$

in the form $a + ib$ and show them on a diagram.

148. Show that

$$\sin 5\theta = 5\cos^4\theta \cdot \sin\theta - 10\cos^2\theta \cdot \sin^3\theta + \sin^5\theta.$$

149.

(a) Let $w = 3 - i$. Plot w, $-w$, $\bar{w}$ and $-\bar{w}$ in the complex plane.

i. What shape is outlined ?

ii. What is $|w|$?

(b) Sketch (in the complex plane) the sets of complex numbers z satisfying, respectively, the conditions :

i. $|z + i| = 1$.

ii. $|z - i| = |z|$.

Supp Exam, February 2005

150. Given that $-2 + 3i$ is a *root* of the equation

$$z^4 + 4z^3 + 9z^2 - 16z - 52 = 0$$

find *all* the other roots of this equation.

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