Chapter 1

Propositions and Predicates

Topics :

1. Propositions and connectives
2. Propositional equivalences
3. Predicates and quantifiers

Logic is the basis of all mathematical reasoning. The rules of logic specify the precise meaning of mathematical statements. Most of the definitions of formal logic have been developed so that they agree with the natural or intuitive logic used by people who have been educated to think clearly and use language carefully. The difference that exists between formal and intuitive logic are necessary to avoid ambiguity and obtain consistency.

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1.1 Propositions and connectives

In any *mathematical theory*, new terms are defined by using those that have been previously defined. However, this process has to start somewhere. A few initial terms necessarily remain *undefined.* In logic, the words **sentence, true,** and **false** are initial undefined terms.

Propositions

1.1.1 DEFINITION. A **proposition** (or **statement**) is a sentence that is either TRUE or FALSE (but *not* both).

1.1.2 EXAMPLES. All the following sentences are propositions.

(1) Pretoria is the capital of South Africa.

(2) $3 + 3 = 5$.

(3) If x is a real number, then $x^2 < 0$. (FALSE)

(4) All kings of the United States are bald. (TRUE)

(5) There is intelligent life outside our solar system.

1.1.3 EXAMPLES. The following are *not* propositions.

(6) Is this concept important ?

(7) Wow, what a day !

(8) $x + 1 = 7$.

(9) When the swallows return to Capistrano.

(10) This sentence is false.

A proposition ends in a period, not a question mark or an exclamation point. Thus (6) and (7) are *not* propositions. (8) is *not* a proposition, even though it has the proper *form*, until the variable x is replaced by meaningful terms. Because (9) *makes no sense*, it cannot be TRUE or FALSE.

Sentence (10) is deceiving; it looks like a proposition. If it is a proposition, then it is either TRUE or FALSE, but not both. Suppose it is TRUE. Then what it says is TRUE, and it is FALSE. But it cannot be both TRUE and FALSE. Hence (10) cannot be TRUE. Well, suppose (10) is FALSE. Then what it says is FALSE, and (10) is not FALSE, it is TRUE. Again, it cannot be both TRUE and FALSE. Therefore, (10) *cannot be classified* as either TRUE or FALSE. Hence it is not a proposition.

We will use lower case letters, such as $p, q, r, s, \ldots$ to denote propositions. Any proposition symbolized by a single letter is called a **primitive proposition**. If p is a (primitive) proposition we define its **truth value** by

$$\tau(p) := \begin{cases} 1 & \text{if } p \text{ is } \mathsf{TRUE} \\ 0 & \text{if } p \text{ is } \mathsf{FALSE}. \end{cases}$$

Thus the symbol 1 stands for TRUE and the symbol 0 for FALSE.

Logical operators

New propositions, called **compound propositions**, may be constructed from existing propositions using **logical operators** (or **connectives**). We define here five such connectives : $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$. Three more connectives – $\oplus$, $\downarrow$, and $|$ – are defined in the exercises.

1.1.4 Definition. Let p be a proposition. The proposition "not p", denoted by $\neg p$, is TRUE when p is FALSE and is FALSE when p is TRUE. The proposition $\neg p$ is called the **negation** of p.

We use a **truth table** to display the relationship between the truth values.

NOT	
p	$\neg p$
1	0
0	1

1.1.5 Example. Find the negation of the proposition "Today is Monday".

Solution : The negation is "Today is not Monday".

1.1.6 Definition. Let p and q be propositions. The proposition "p and q", denoted by $p \wedge q$, is TRUE when both p and q are TRUE, and is FALSE otherwise. The proposition $p \wedge q$ is called the **conjunction** of p and q.

We can use a truth table to illustrate how conjunction works.

AND		
p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

1.1.7 Example. Find the conjunction of the propositions p and q, where p is the proposition "Today is Sunday" and q is the proposition "The moon is made of cheese".

Solution : The conjunction of these propositions is the proposition "Today is Sunday and the moon is made of cheese". This proposition is always FALSE.

1.1.8 Definition. Let p and q be propositions. The proposition "p or q", denoted by $p \vee q$, is the proposition that is FALSE when p and q are FALSE, and is TRUE otherwise. The proposition $p \vee q$ is called the **disjunction** of p and q.

We can again use a truth table to illustrate how disjunction works.

OR		
p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

1.1.9 Example. Find the disjunction of the propositions p and q, where p is the proposition "Today is Sunday" and q is the proposition "The moon is made of cheese".

Solution : The disjunction of these propositions is the proposition "Today is Sunday or the moon is made of cheese". This proposition is TRUE only on Sundays.

1.1.10 Definition. Let p and q be propositions. The proposition "p implies q", denoted by $p \to q$, is FALSE when p is TRUE and q is FALSE, and is TRUE otherwise. The proposition $p \to q$ is called an **implication** (or **conditional**). In this implication p is called the **antecedent** (or **hypothesis**), whereas q is called the **consequent** (or **consequence**).

A truth table for the conditional is :

IMPLIES		
p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

1.1.11 EXAMPLE. Consider the propositions

(1) p : 7 is a positive integer.

(2) q : $2 < 3$.

(3) r : 5 is an even integer.

(4) s : No one will pass this course.

Then the conditional propositions $p \to q$, $r \to q$, $r \to s$ are all TRUE.

NOTE : The proposition $r \to s$ is TRUE since the hypothesis is FALSE. Even though this proposition is TRUE, it does not say that no one will pass this course.

It is worth noting certain synonyms for $p \to q$ which occur in mathematical literature. They are as follows :

- "if p, then q"
- "p, only if q"
- "p is sufficient for q"
- "q is implied by p"
- "q, if p"
- "q, whenever p"

- "q is necessary for p".

NOTE : (1) The mathematical concept of an implication is independent of a cause-and-effect relationship between antecedent and consequent.

(2) The if-then construction used in many programming languages is different from that in logic. Most programming languages contain statements such as "if p then S", where p is a proposition and S is a program segment (one or more statements to be executed). When execution of a program encounters such a statement, S is executed if p is TRUE, whereas S is not executed if p is FALSE.

There are some related implications that can be formed from $p \to q$. We call

- $q \to p$ the **converse**
- $\neg q \to \neg p$ the **contrapositive**
- $\neg p \to \neg q$ the **inverse**

of the implication $p \to q$.

1.1.12 EXAMPLE. Find the converse, the contrapositive, and the inverse of the implication "If today is Friday, then I will party tonight."

SOLUTION : The converse is "If I will party tonight, then today is Friday." The contrapositive of this implication is "If I will not party tonight, then today is not Friday." And the inverse is "If today is not Friday, then I will not party tonight."

Observe that the given implication and its contrapositive are either both TRUE or both FALSE.

NOTE : When one says "If it rains, then I will stay home" the meaning usually includes the inverse statement "If it does not rain, then I will not stay home". This is *not* the case in an implication. As an implication, this statement gives absolutely no information for the case when it does not rain.

1.1.13 DEFINITION. Let p and q be propositions. The proposition "p if and only if q", denoted by $p \leftrightarrow q$, is the proposition that is TRUE when p and q have the same truth values, and is FALSE otherwise. The proposition $p \leftrightarrow q$ is called an **equivalence** (or **biconditional**).

The truth table for the biconditional is :

IF AND ONLY IF		
p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

The biconditional is the conjunction of two conditional propositions (see the appropriate logical equivalence from the list on page 11). This leads to the terminology "if and only if". Other ways of expressing this connective are

- "if p, then q and conversely"
- "p is necessary and sufficient for q".

1.1.14 EXAMPLE. Find the truth value of the biconditional "The moon is made of cheese if and only if 1=2".

SOLUTION : The proposition is TRUE, since it is composed of two propositions each of which is FALSE.

NOTE : (Remark on punctuation) For symbolized statements, punctuation is accomplished by using parentheses. To lessen the use of parentheses, we agree that in the absence of parentheses the logical operator NOT takes precedence over AND and OR, and the logical operators AND and OR take precedence over the conditional (IMPLIES) and biconditional (IF AND ONLY IF). Thus

- $\neg p \wedge q$ stands for $(\neg p) \wedge q$, not for $\neg(p \wedge q)$.

- $\neg p \to q$ stands for $(\neg p) \to q$, not for $\neg(p \to q)$.
- $\neg p \vee \neg q$ stands for $(\neg p) \vee (\neg q)$, not for $\neg(p \vee (\neg q))$.
- $p \wedge q \to r$ stands for $(p \wedge q) \to r$, not for $p \wedge (q \to r)$.
- $\neg p \vee q \to r \wedge q$ stands for $((\neg p) \vee q) \to (r \wedge q)$.
- $\neg p \leftrightarrow p \to q \wedge r$ stands for $(\neg p) \leftrightarrow (p \to (q \wedge r))$.

A basic principle of (propositional) logic is that *the truth values of a proposition composed of other propositions and the connectives* $\neg, \wedge, \vee, \to,$ *and* $\leftrightarrow$ *are determined by the truth values of the constituent propositions and by the way in which those propositions are combined with the connectives.*

1.1.15 Example. Construct the truth table for the proposition $(\neg p \to q) \wedge r$.

Solution : The truth table is

p	q	r	$\neg p \to q$	$(\neg p \to q) \wedge r$
1	1	1	1	1
1	1	0	1	0
1	0	1	1	1
1	0	0	1	0
0	1	1	1	1
0	1	0	1	0
0	0	1	0	0
0	0	0	0	0

It is often desirable that we translate English (or any other human language) sentences into expressions involving propositional variables and logical connectives in order to remove the ambiguity.

Note : This may involve making certain reasonable assumptions based on the intended meaning of the sentence.

Once we have translated sentences from English into logical expressions we can analyze them (i.e. determine their truth values), as well as manipulate and reason about them.

1.1.16 EXAMPLE. Let p, q, and r be the propositions

p : You get an A on the final exam.

q : You do every prescribed exercise.

r : You get an A in this class.

Write the following propositions using p, q, and r and logical operators.

(1) You get an A on the final, but you don't do every prescribed exercise; nevertheless, you get an A in this class.

(2) Getting an A on the final and doing every prescribed exercise is sufficient for getting an A in this class.

SOLUTION : The answer is : $p \wedge \neg q \wedge r$ and $p \wedge q \rightarrow r$.

1.2 Propositional equivalences

Methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments.

1.2.1 DEFINITION. A compound proposition that is always TRUE, no matter what the truth values of the propositions that occur in it, is called a **tautology**.

1.2.2 EXAMPLES. All the following propositions are tautologies.

(1) $p \vee \neg p$.

(2) $p \rightarrow p \vee q$.

(3) $p \wedge (p \to q) \to q$.

In each case, the truth value of the compound proposition is 1 for every one of the lines in the truth table. For example, the truth table for the proposition $p \wedge (p \to q) \to q$ is :

p	q	$p \wedge (p \to q)$	$p \wedge (p \to q) \to q$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	1

1.2.3 Definition. A compound proposition that is always FALSE, no matter what the truth values of the propositions that occur in it, is called a **contradiction**.

1.2.4 Examples. All the following propositions are contradictions.

(1) $p \wedge \neg p$.

(2) $p \wedge \neg(p \vee q)$.

(3) $(p \to q) \wedge (p \wedge \neg q)$.

1.2.5 Definition. The propositions p and q are called **logically equivalent**, denoted $p \iff q$, provided $p \leftrightarrow q$ is a tautology.

In other words, two compound propositions are logically equivalent provided that they have the same truth values for all possible assignments of truth values to the constituent propositions. Thus *all tautologies are logically equivalent, as are all contradictions.*

Note : Although one proposition may be easier to read than the other, insofar as the truth values of any propositions which depend on one or the other of the two propositions, it makes no difference which of the two is used. Logical equivalence may allow us to replace a complex proposition with a much simpler one.

1.2.6 Example. Show that the propositions $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Solution : We construct the truth table for these propositions.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
1	1	1	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	0	1
0	0	0	1	1	1	1

Since the truth values of $\neg(p \wedge q)$ and $\neg p \vee \neg q$ agree, these propositions are logically equivalent.

Note : It is important to understand the difference between $\leftrightarrow$ and $\iff$. $p \leftrightarrow q$ is a statement which may or may not be TRUE. In other words, the truth value of $\tau(p \leftrightarrow q)$ may be 1 or 0. But $p \iff q$ is a statement about the statement (proposition) $p \leftrightarrow q$: it says that "$p \leftrightarrow q$ is always TRUE".

1.2.7 Definition. Let p and q be propositions. We say that p **logically implies** q, denoted $p \Rightarrow q$, provided $p \to q$ is a tautology.

In other words, q is true whenever p is TRUE. For example, we have seen that

$$p \wedge (p \to q) \to q \quad \text{is a tautology}$$

and so

$$p \wedge (p \to q) \Rightarrow q.$$

Note : (1) Again, we emphasize the difference between $\to$ and $\Rightarrow$. $p \to q$ is is just a proposition, whereas $p \Rightarrow q$ is a statement about the statement (proposition) $p \to q$: it says that "$p \to q$ is always TRUE".

(2) Also, we note that, in order to show that $p \Rightarrow q$ it is sufficient to show that $p \to q$ is never FALSE. For example, to show that

$$p \wedge q \Rightarrow p$$

we need only show that we cannot have $\tau(p \wedge q) = 1$ and $\tau(p) = 0$. This is immediate since if $\tau(p \wedge q) = 1$ then $\tau(p) = 1$.

We list some important logical equivalences and logical implications. In this, we let $\mathbf{T}$ denote any tautology and we let $\mathbf{C}$ denote any contradiction.

(1) $p \wedge \mathbf{T} \iff p$ (identity) ;

(2) $p \vee \mathbf{C} \iff p$ (identity) ;

(3) $p \wedge \mathbf{C} \iff \mathbf{C}$ (domination) ;

(4) $p \vee \mathbf{T} \iff \mathbf{T}$ (domination) ;

(5) $\neg\neg p \iff p$ (double negation) ;

(6) $p \wedge p \iff p$ (idempotency) ;

(7) $p \vee p \iff p$ (idempotency) ;

(8) $p \wedge q \iff q \wedge p$ (commutativity) ;

(9) $p \vee q \iff q \vee p$ (commutativity) ;

(10) $p \wedge (q \wedge r) \iff (p \wedge q) \wedge r$ (associativity) ;

(11) $p \vee (q \vee r) \iff (p \vee q) \vee r$ (associativity) ;

(12) $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$ (distributivity) ;

(13) $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$ (distributivity) ;

(14) $\neg(p \wedge q) \iff \neg p \vee \neg q$ (De Morgan's law) ;

(15) $\neg(p \vee q) \iff \neg p \wedge \neg q$ (De Morgan's law) ;

(16) $p \to q \iff \neg p \vee q$ (OR form of a conditional) ;

(17) $\neg(p \to q) \iff p \wedge \neg q$ (negation of a conditional) ;

(18) $p \to q \iff \neg q \to \neg p$ (contraposition) ;

(19) $p \leftrightarrow q \iff (p \to q) \wedge (q \to p)$ (biconditional) ;

(20) $p \Rightarrow p \vee q$ (addition) ;

(21) $p \wedge q \Rightarrow p$ (simplification) ;

(22) $p \wedge (p \to q) \Rightarrow q$ (detachment) ;

(23) $\neg p \wedge (p \vee q) \Rightarrow q$ (disjunction) ;

(24) $(p \to q) \wedge (q \to r) \Rightarrow p \to r$ (syllogism).

The logical equivalences above, as well as any others (like $p \wedge \neg p \iff \mathbf{C}$ or $p \vee \neg p \iff \mathbf{T}$), can be used to construct additional logical equivalences. The reason for this is that *a proposition in a compound proposition can be replaced by one that is logically equivalent to it without changing the truth value of the compound proposition.*

1.2.8 Example. Show that the propositions $p \wedge q \to r$ and $p \to (q \to r)$ are logically equivalent.

Solution : To show these compound propositions are logically equivalent, we will developing a sequence of logical equivalences starting with $p \wedge q \to r$ and ending with $p \to (q \to r)$. We have

$$p \wedge q \to r \iff \neg(p \wedge q) \vee r \iff (\neg p \vee \neg q) \vee r \iff \\ \neg p \vee (\neg q \vee r) \iff \neg p \vee (q \to r) \iff p \to (q \to r).$$

1.2.9 Example. Show that the proposition $\neg(p \to q) \to p$ is a tautology.

Solution : To show this proposition is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to $\mathbf{T}$. We have

$$\neg(p \to q) \to p \iff (p \wedge \neg q) \to p \iff \neg(p \wedge \neg q) \vee p \iff \\ (\neg p \vee q) \vee p \iff (\neg p \vee p) \vee q \iff \mathbf{T} \vee q \iff \mathbf{T}.$$

1.3 Predicates and quantifiers

Propositions are sentences that are either TRUE or FALSE (but *not* both). The sentence “He is a student at Rhodes University” is not a proposition because it may be either true or false depending on the value of the pronoun *he*. Similarly, the sentence “ $x + y > 0$ ” is not a proposition because its truth value depends on the values of the *variables* x and y.

NOTE : In grammar, the word *predicate* refers to the part of a sentence that gives information about the subject. In the sentence “James is a student at Rhodes University”, the word *James* is the subject and the phrase “is a student at Rhodes University” is the predicate.

In logic, predicates can be obtain by removing some or all nouns from a statement. For instance, the sentences “ x is a student at Rhodes University” and “ x is a student at y ” are predicates ; here x and y are *predicate variables* that take values in appropriate sets.

Predicates

We make the following definition.

1.3.1 DEFINITION. A sentence that contains a finite number of variables and becomes a proposition when specific values are substituted for the variables is called a **predicate** (or **open sentence**).

Predicates are sometimes referred to as **propositional functions**.

1.3.2 EXAMPLES. The following are all predicates.

(1) $x + 3 = 4$.

(2) The sum of the first n odd integers is n^2.

(3) $x > 2$.

(4) If $x < y$, then $x^2 < y^2$.

(5) $x + y = z$.

(6) x is the capital of France.

We generally denote predicates by capital letters such as $P, Q, R, \ldots$, and following the name of the predicate we list in parentheses the variables which are used by the predicate. Thus in the example above we might describe the predicates as $P(x)$, $Q(n)$, $R(x)$, $S(x,y)$, $T(x,y,z)$, $U(x)$.

NOTE : The predicate $S(x,y)$: "If $x < y$, then $x^2 < y^2$" is actually a *compound predicate*, consisting of predicates which are combined by the same kind of logical connectives that are used in propositional logic. This means that we can write $S(x,y)$ as $A(x,y) \to B(x,y)$, where $A(x,y)$ is the predicate "$x < y$" and $B(x,y)$ is the predicate "$x^2 < y^2$".

1.3.3 EXAMPLE. Consider the predicate

$$P(x,y): \quad \text{“} y = x + 3 \text{”}.$$

What are the truth values of the propositions $P(1,2)$ and $P(0,3)$?

SOLUTION : $P(1,2)$ is the proposition "$2 = 1 + 3$" which is false. The statement $P(0,3)$ is the proposition "$3 = 0 + 3$" which is true.

When all the variables in a predicate are assigned values, the resulting sentence is a proposition. The set of all values that may be substituted in place of a variable constitutes the **universe of discourse**. If $P(x)$ is a predicate, where the universe of discourse is $\mathcal{U}$, the **truth set** of $P(x)$ is the set of all elements in $\mathcal{U}$ that make $P(x)$ true when substituted for x.

However, there is another important way to change predicates into propositions, namely **quantification**.

Quantifiers

1.3.4 DEFINITION. The proposition "$P(x)$ is true for all values of x belonging to the universe of discourse $\mathcal{U}$", denoted $\forall x\, P(x) \quad (x \in \mathcal{U})$, is called the **universal quantification** of the predicate $P(x)$.

The proposition $\forall x\, P(x)$ (or $\forall x \in \mathcal{U},\ P(x)$) is also expressed as

- "For all x (in $\mathcal{U}$) $P(x)$".
- "For every x (in $\mathcal{U}$) $P(x)$".

1.3.5 EXAMPLE. Express the statement "Every student in this class has seen a computer" as a universal quantification.

SOLUTION : Let $Q(x)$ be the predicate "x has seen a computer". Then the statement "Every student in this class has seen a computer" can be written as $\forall x\, Q(x)$, where the universe of discourse consists of all students in this class. Also, this statement can be expressed as $\forall x\ (P(x) \to Q(x))$, where $P(x)$ is the predicate "x is in this class" and the universe of discourse consists of all students.

NOTE : There is often more than one good way to express a quantification.

1.3.6 EXAMPLE. Let $P(x)$ be the predicate "$x^2 > x$". What is the truth value of the universal quantification $\forall x\, P(x)$, where the universe of discourse is the set of real numbers ?

SOLUTION : $P(x)$ is not true for all real numbers x; for instance, $P(\frac{1}{2})$ is false. Thus, the proposition $\forall x\, P(x)$ is false.

When all the elements of the universe of discourse can be listed, say $x_1, x_2, \dots, x_n$, it follows that the universal quantification $\forall x\, P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n).$$

1.3.7 EXAMPLE. What is the truth value of the proposition "$\forall x\ P(x)$", where $P(x)$ is the predicate "$x^{10} < 50\,000$", and the universe of discourse consists of the positive integers not exceeding 3 ?

SOLUTION : The proposition $\forall x\, P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3).$$

Since the proposition $P(3)$: "$3^{10} < 50\,000$" is false, it follows that the universal quantification $\forall x\ P(x)$ is false.

Statements such as "there exists an x in U such that $P(x)$" are also common.

1.3.8 DEFINITION. The proposition "There exists an x in the universe of discourse $\mathcal{U}$ such that $P(x)$ is TRUE", denoted $\exists x\ P(x) \quad (x \in \mathcal{U})$, is called the **existential quantification** of the predicate $P(x)$.

The proposition $\exists x\ P(x)$ (or $\exists x \in \mathcal{U},\ P(x)$) is also expressed as

- "There exists at least one x (in $\mathcal{U}$) such that $P(x)$".
- "For some x (in $\mathcal{U}$) $P(x)$".

1.3.9 EXAMPLE. Let $Q(n)$ be the predicate "$n^2 + n + 1$ is a prime number". What is the truth value of the existential quantification $\exists n\ Q(n)$, where the universe of discourse is the set of positive integers ?

SOLUTION : Since the proposition $Q(1)$: "3 is a prime number" is TRUE, it follows that the existential quantification $\exists n\ Q(n)$ is TRUE.

When all the elements in the universe of discourse can be listed, say $x_1, x_2, \ldots, x_n$, the existential quantification $\exists x\ P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n).$$

1.3.10 EXAMPLE. What is the truth value of the proposition $\exists x\ P(x)$, where $P(x)$ is the predicate "$x^3 = 15x + 4$", and the universe of discourse consists of all positive integers not exeeding 4 ?

SOLUTION : The proposition $\exists x\ P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Since the proposition $P(4)$: "$4^3 = 60 + 4$" is TRUE, it follows that the existential quantification $\exists x\ P(x)$ is TRUE.

1.3.11 Example. Rewrite the formal statement

$$\forall x \in \mathbb{R},\ x^2 \geq 0$$

in several equivalent but more informal ways. Do not use the symbol (universal quantifier) $\forall$.

Solution :

All real numbers have nonnegative squares.

Every real number has a nonnegative square.

Any real number has a nonnegative suare.

x has a nonnegative square, for each real number x.

The square of any real number is nonnegative.

1.3.12 Example. Rewrite the statement "No dogs have wings" formally. Use quantifiers and variables.

Solution :

$\forall$ dogs d, d does not have wings.

$\forall d \in D$, d does not have wings (where D is the set of dogs).

$\forall d\ P(d)$, where D is the set of dogs and $P(d)$ is the predicate " d does not have wings".

$\forall d\ \neg Q(d)$, where D is the set of dogs and $Q(d)$ is the predicate " d has wings".

Many mathematical statements involve multiple quantifications of predicates involving more than one variable. It is important to note that *the order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.*

1.3.13 EXAMPLE. Express the statement "If somebody is female and is a parent, then this person is someone's mother" formally. Use quantifiers and variables.

SOLUTION : Consider the predicates

$F(x)$: " x is female".

$P(x)$: " x is parent".

$M(x, y)$: " x is the mother of y ".

Here, the universe of discourse can be taken to be the set of all people. We can write the statement symbolically as

$$\forall x\,(F(x) \wedge P(x) \to \exists y\, M(x, y)).$$

1.3.14 EXAMPLE. Let $P(x, y)$ be the predicate " $x + y = 3$ ". What are the truth values of the quantifications

$$\forall x \forall y\ P(x, y), \quad \forall x \exists y\ P(x, y), \quad \exists x \forall y\ P(x, y) \quad \text{and} \quad \exists x \exists y\ P(x, y)\,?$$

SOLUTION : The quantification $\forall x \forall y\ P(x, y)$ denotes the proposition "For every pair x, y $P(x, y)$ is TRUE". Clearly, this proposition is FALSE.

The quantification $\forall x \exists y\ P(x, y)$ denotes the proposition "For every x there is an y such that $P(x, y)$ is TRUE". Given a real number x, there is a real number y such that $x + y = 3$, namely $y = 3 - x$. Hence, the proposition $\forall x \exists y\ P(x, y)$ is TRUE.

We can see that $\exists x \forall y\ P(x, y)$ is FALSE and $\exists x \exists y\ P(x, y)$ is TRUE.

NOTE : The order in which quantifiers appear makes a difference. For instance, the propositions $\exists x \forall y\ P(x, y)$ and $\forall y \exists x\ P(x, y)$ are *not* logically equivalent. However,

$$\exists x \forall y\ P(x, y) \ \Rightarrow\ \forall y \exists x\ P(x, y).$$

We will often want to consider the negation of a quantification. In this regard, the following logical equivalences are useful.

(1) $\neg\forall x\ P(x) \iff \exists x\ \neg P(x)$;

(2) $\neg\exists x\ P(x) \iff \forall x\ \neg P(x)$;

(3) $\neg\forall x\forall y\ P(x,y) \iff \exists x\exists y\ \neg P(x,y)$;

(4) $\neg\forall x\exists y\ P(x,y) \iff \exists x\forall y\ \neg P(x,y)$;

(5) $\neg\exists x\forall y\ P(x,y) \iff \forall x\exists y\ \neg P(x,y)$;

(6) $\neg\exists x\exists y\ P(x,y) \iff \forall x\forall y\ \neg P(x,y)$.

1.3.15 Example. Express the negations of the following propositions using quantifiers. Also, express these negations in English.

(1) "Every student in this class likes mathematics".

(2) "There is a student in this class who has been in at least one room of every building on campus".

Solution : (1) Let $L(x)$ be the predicate "x likes mathematics", where the universe of discourse is the set of students in this class. The original statement is $\forall x\, L(x)$ and its negation is $\exists x\, \neg L(x)$. In English, it reads "Some student in this class does not like mathematics".

(2) Consider the predicates $P(z,y)$: "room z is in building y" and $Q(x,z)$: "student x has been in room z". Then the original statement is

$$\exists x\, \forall y\, \exists z\, (P(z,y) \wedge Q(x,z))\, .$$

To form the negation, we change all the quantifiers and put the negation on the inside, then apply De Morgan's law. The negation is therefore

$$\forall x\, \exists y\, \forall z\, (\neg P(z,y) \vee \neg Q(x,z))\, ,$$

which is also equivalent to

$$\forall x\, \exists y\, \forall z\, (P(z,y) \to \neg Q(x,z)).$$

In English, this could be read "For every student there is a building on the campus such that for every room in that building, the student has not been in that room".

1.3.16 Example. Rewrite the statement "No politicians are honest" formally. Then write formal and informal negations.

Solution :

formal version : $\forall$ politicians x, x is not honest.

formal negation : $\exists$ a politician x such that x is honest.

informal negation : "Some politicians are honest".

1.3.17 Example. Write informal negations for each of the following statements:

(1) "All computer programs are finite".

(2) "Some computer hackers are over 40".

(3) "Every polynomial function is continuous".

Solution : The informal negations are :

(1) "Some computer programs are not finite".

(2) "No computer hackers are over 40" (or "All computer hackers are 40 or under").

(3) "There is a non-continuous polynomial function".

1.4 Exercises

Exercise 1 TRUE or FALSE? The negation of "If Tom is Ann's father, then Jim is her uncle and Sue is her aunt" is "If Tom is Ann's father, then either Jim is not her uncle or Sue is not her aunt".

Exercise 2 Assume that "Joe is a girl" is a FALSE proposition and that "Mary is ten years old" is a TRUE proposition. Which of the following are TRUE ?

(a) Joe is a girl and Mary is ten years old.

(b) Joe is a girl or Mary is ten years old.

(c) If Joe is a girl, then Mary is ten years old.

(d) If Mary is ten years old, then Joe is a girl.

(e) Joe is a girl if and only if Mary is ten years old.

Exercise 3 Suppose that p and q are propositions so that $p \to q$ is FALSE. Find the truth value of the following propositions.

(a) $\neg p \to q$.

(b) $p \vee q$.

(c) $q \to p$.

Exercise 4 Construct truth tables for each of the following propositions.

(a) $p \wedge \neg p$.

(b) $p \vee \neg p$.

(c) $p \wedge q \to p$.

(d) $p \to p \vee q$.

(e) $p \vee q \to p \wedge q$.

(f) $(p \to q) \to (q \to p)$.

(g) $p \wedge (p \to q) \to q$.

(h) $(p \to q) \vee (\neg p \to r)$.

(i) $(p \vee q) \wedge r \to \neg p \vee q$.

(j) $\neg p \wedge \neg q \vee (r \to q)$.

Exercise 5 Let p and q be propositions. The proposition "p exclusive or q", denoted by $p \oplus q$, is the proposition that is TRUE when exactly one of p and q is TRUE, and is FALSE otherwise. Construct the truth table for this logical operator (XOR), and then show that

$$\tau(p \oplus q) = \tau((p \wedge \neg q) \vee (\neg p \wedge q)).$$

Exercise 6 Verify that each of the following propositions is a tautology.

(a) $p \wedge q \to p$.

(b) $p \to p \vee q$.

(c) $\neg p \to (p \to q)$.

(d) $p \wedge q \to (p \to q)$.

(e) $p \wedge (p \to q) \to q$.

(f) $(p \to q) \to (p \vee r \to q \vee r)$.

Exercise 7 Verify each of the following logical equivalences.

(a) $p \wedge r \to q \iff p \to (r \to q)$.

(b) $p \vee q \to r \iff (p \to r) \wedge (q \to r)$.

(c) $\neg(p \oplus q) \iff p \leftrightarrow q$.

(d) $\neg(p \leftrightarrow q) \iff \neg p \leftrightarrow q$.

Exercise 8 Show that $(p \to q) \to r$ and $p \to (q \to r)$ are *not* logically equivalent.

Exercise 9 Which of the following expressions are logically equivalent to $p \to q$?

(a) $p \to q$.

(b) $p \vee q$.

(c) $\neg q \to \neg p$.

(d) $\neg q \to p$.

(e) $\neg p \wedge q$.

(f) $p \wedge \neg q$.

(g) $\neg p \to q$.

(h) $p \wedge q$.

(i) $p \vee \neg q$.

(j) $\neg p \to \neg q$.

(k) $\neg p \vee q$.

(l) $q \to p$.

Exercise 10 Suppose that "John is smart", "John or Mary is ten years old", and "If Mary is ten years old, then John is not smart" are each TRUE propositions. Which of the following propositions are TRUE ?

(a) John is not smart.

(b) Mary is ten years old.

(c) John is ten years old.

(d) Either John or Mary is not ten years old.

Exercise 11 Let p and q be propositions. The proposition "p NOR q", denoted by $p \downarrow q$, is TRUE when both p and q are FALSE, and it is FALSE otherwise.

(a) Construct the truth table for the logical operator $\downarrow$ (also known as the **Peirce arrow**).

(b) Show that :

i. $p \downarrow q \iff \neg(p \vee q)$.

ii. $p \downarrow p \iff \neg p$.

iii. $(p \downarrow q) \downarrow (p \downarrow q) \iff p \vee q$.

iv. $(p \downarrow p) \downarrow (q \downarrow q) \iff p \wedge q$.

Exercise 12 Let p and q be propositions. The proposition "p NAND q", denoted by $p \mid q$, is TRUE when either p and q, or both, are FALSE, and is FALSE when both p and q are TRUE.

(a) Construct the truth table for the logical operator $\mid$ (also known as the **Scheffer stroke**).

(b) Show that :

i. $p \mid q \iff \neg(p \wedge q)$.

ii. $p \mid p \iff \neg p$.

iii. $(p \mid p) \mid (q \mid q) \iff p \vee q$.

iv. $(p \mid q) \mid (p \mid q) \iff p \wedge q$.

Exercise 13 Let $F(x, y)$ be the predicate "x can fool y", where the universe of discourse is the set of all people in the world. Use quantifiers to express each of the following statements.

(a) Everybody can fool Bob.

(b) Kate can fool everybody.

(c) Everybody can fool somebody.

(d) There is no one who can fool everybody.

(e) Everyone can be fooled by somebody.

(f) No one can fool Fred and Jerry.

(g) No one can fool himself or herself.

Exercise 14 Let $P(x, y)$ be the predicate "$x + y = x - y$". If the universe of discourse is the set of integers, what are the truth values of the following propositions?

(a) $P(1, 1)$.

(b) $P(2, 0)$.

(c) $\forall y\ P(1, y)$.

(d) $\exists x\ P(x, 2)$.

(e) $\exists x \exists y\ P(x, y)$.

(f) $\forall x \exists y\ P(x, y)$.

(g) $\exists y \forall x\ P(x, y)$.

(h) $\forall x \forall y\ P(x, y)$.

(i) $\forall y \exists x\ P(x, y)$.

Exercise 15 Write the negation for each of the following propositions. Determine whether the resulting proposition is TRUE or FALSE; let $\mathcal{U} = \mathbb{R}$.

(a) $\forall x\,(x^2 + 2x - 3 = 0)$.

(b) $\exists x\,(x^2 - 2x + 5 \leq 0)$.

(c) $\forall x\,\exists r\,(xr = 1)$.

(d) $\forall x \exists m\ (x^2 < m)$.

(e) $\exists m \forall x\ (x^2 < m)$.

(f) $\exists m \forall x\ \left(\frac{x}{|x|+1} < m\right)$.

(g) $\forall x\,\forall y\,(x^2 + y^2 \geq xy)$.