Chapter 10

Complex Numbers

Topics :

1. Number systems
2. Algebraic operations on complex numbers
3. De Moivre's formula
4. Applications

Beginning with the *natural* numbers such as 0, 1 and 2, we proceed to the *integers*, then to the *rational* numbers, then to the *real* numbers, and then to the *complex* numbers. Each stage is motivated by our desire to be able to solve a certain kind of equation. Real numbers were understood remarkably well by the ancient Greeks. Complex numbers were used freely many years before they could be treated rigorously; that was how the word "imaginary" acquired its technical meaning.

10.1 Number systems

The first *numbers* that we consider in arithmetic are the **natural numbers**, forming a sequence that begins with 0 and never ends. On the *set* of natural numbers

$$\mathbb{N} := \{0, 1, 2, 3, \dots\}$$

the operations of *addition* and *multiplication* can be defined, and we shall call the triple $(\mathbb{N}, +, \cdot)$ the **natural number system**.

The problem of solving such an equation as

$$x + 2 = 1$$

motivates the discovery of the **integers**, which include not only the natural numbers (the "non-negative integers") but also the *negative* integers. The sequence of integers, which has neither beginning nor end, is conveniently represented by *points* evenly spaced along a straight line (which we may think of as the x-axis of ordinary analytic geometry). In this representation, addition and subtraction appear as *translations* : the transformation $x \mapsto x + a$ shifts each point through a spaces to the right if a is positive, and through $-a$ spaces to the left if a is negative; that is, the operation of adding a is the translation that transforms 0 into a.

The *set* of integers

$$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$$

is considered together with operations of *addition* and *multiplication*, and we shall call the triple $(\mathbb{Z}, +, \cdot)$ the **system of integers**.

NOTE : These new operations on $\mathbb{Z}$ are not the *same* as the ones on $\mathbb{N}$, but they are defined such that when the integers are just natural numbers, the operations reduce to the operations of the natural numbers system. Clearly,

$$\mathbb{N} \subseteq \mathbb{Z}$$

and anything that can be done with natural numbers, can be done with integers. In this sense, *the system of integers extends the natural number system.*

The problem of solving such an equation as

$$2x = 1$$

motivates the discovery of the **rational numbers** $r = \frac{m}{n}$, where m is an integer and n is a natural number ; these include not only the integers $m = \frac{m}{1}$, but also *fractions* such as $\frac{1}{2}$ and $-\frac{4}{3}$.

NOTE : We usually write each fraction in its "lowest terms", so that the numerator and the denominator have no common factor.

The rational numbers cannot be written down successively in their natural order, because between any two of them there is another, and consequently and infinity of others. The corresponding points are *dense* on the x-axis, and at first sight seem to cover it completely. Multiplication and division appear as *dilations* : the transformation $x \mapsto rx$ is the dilation of ratio r and center O, where O is the origin; that is, multiplication by r is the dilation of center O that transforms 1 into r. Of course, r may be either positive or negative. In particular, multiplication by -1 is the *half-turn* about O.

The *set* of rational numbers

$$\mathbb{Q} := \left\{ \frac{m}{n} \,|\, m, n \in \mathbb{Z},\ n \neq 0 \right\}$$

is considered together with operations of *addition* and *multiplication*, and we shall call the triple $(\mathbb{Q}, +, \cdot)$ the **system of rational numbers**.

NOTE : Again, the operations on $\mathbb{Q}$ are denoted by the same symbols as the ones on $\mathbb{Z}$ and $\mathbb{N}$ and *the system of rational numbers extends the system of integers* in the same sense as before. So we have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}.$$

The problem of solving such an equation as

$$x^2 = 2$$

motivates the discovery of the **real numbers**, which include not only the rational numbers but also the *irrational* numbers (such as $\sqrt{2}$ and π), which cannot be expressed as fractions. Roughly speaking, rational numbers have a *decimal representation* that terminates in zeros or which has a repeating block of digits. The *set* $\mathbb{R}$ of real numbers is taken to be the set of *all* decimal expansions. Geometrically, this means that the *number line* has now become a *continuum.* [A real number may be defined to be the *limit* of a convergent sequence of rational numbers, or (more precisely) the set of all sequences "equivalent" (in a specified sense) to a given sequence; for example, the real number π is the limit of the sequence

$$3, \ 3.1, \ 3.14, \ 3.141, \ 3.1415, \ \ldots]$$

NOTE : The operations of addition and multiplication on $\mathbb{Q}$ can also be extended to the larger set $\mathbb{R}$ and we shall call the triple $(\mathbb{R}, +, \cdot)$ the **real number system**. We have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq R$$

with the operations of addition and multiplication being extended all the way up, retaining the symbols $+$ and $\cdot$ as we go.

The problem of solving such an equation as

$$x^2 + 1 = 0$$

motivates the discovery of the **complex numbers**, which include not only the real numbers but also such "imaginary" numbers as the *the square root of* -1.

Complex numbers

Since the real numbers occupy the whole x-axis, it is natural to try to represent the complex numbers by all *points* (or *vectors*) in the (x, y)-plane (called the **complex plane**); that is, to define them as *ordered pairs of real numbers with suitable rules for their addition and multiplication.*

In the complex plane (also called the **Argand diagram**), points are added like the corresponding vectors from the origin O :

$$(x, y) + (a, b) := (x + a, y + b). \tag{10.1}$$

In other words, to add (a, b) we apply the translation that takes $(0, 0)$ to (a, b).

Multiplication by an integer still appears as a dilation; for instance,

$$2(x, y) = (x, y) + (x, y) = (2x, 2y).$$

In particular, multiplication by -1 is the half-turn about O. What, then, is multiplication by the "square root of -1" ? This must be a transformation whose "square" is the half-turn about O. The obvious answer is a *quater-turn* (or *rotation* through an angle of $90°$) about O.

Then multiplication by an arbitrary complex number should be a transformation which leaves O invariant and includes both dilations and rotations as special cases. The obvious transformation of this kind is a *rotation-dilation* (the product of a rotation and a dilation about O). It turns out that the rule for multiplication is

$$(a, b) \cdot (x, y) := \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (ax - by, bx + ay). \tag{10.2}$$

NOTE : We shall use *juxtaposition* $(a, b)(x, y)$ to denote $(a, b) \cdot (x, y)$, just as we often do with real numbers.

The *set* of complex numbers is denoted by $\mathbb{C}$ and we shall call the triple $(\mathbb{C}, +, \cdot)$ the **complex number system**.

The mapping

$$\varphi : \mathbb{R} \to \mathbb{C}, \quad x \mapsto (x, 0)$$

is a one-to-one mapping that "preserves" addition and multiplication ; that is,

$$\varphi(x + y) = \varphi(x) + \varphi(y) \quad \text{and} \quad \varphi(xy) = \varphi(x)\varphi(y)$$

for all $x, y \in \mathbb{R}$. It follows that $\varphi(\mathbb{R}) \subseteq \mathbb{C}$ *is a faithful copy of the (number system)* $\mathbb{R}$. We therefore *identify* $\mathbb{R}$ with $\varphi(\mathbb{R}) \subseteq \mathbb{C}$ and write x for $(x, 0)$.

NOTE : We shall allow ourselves to think of $\mathbb{R}$ as a *subset* of $\mathbb{C}$, and shall call members of $\varphi(\mathbb{R})$ "real". In other words, via this identification, $\mathbb{C}$ becomes a (number system) *extension* of $\mathbb{R}$ and thus we have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}.$$

Introducing the special symbol

$$\boxed{i := (0, 1) \in \mathbb{C}}$$

we have

$$i^2 = (0, 1)(0, 1) = (-1, 0) = -1.$$

The number i is often called the **imaginary unit** of $\mathbb{C}$.

10.1.1 EXAMPLE. Find

$$i^3,\ i^4,\ i^5,\ i^{23}, \quad \text{and} \quad i^{2000}.$$

SOLUTION : We have

$$\begin{aligned} i^3 &= i^2 i = -i, \\ i^4 &= i^3 i = -i^2 = 1, \\ i^5 &= i^4 i = i, \\ i^{23} &= (i^4)^5 i^3 = -i, \\ i^{2000} &= (i^4)^{500} = 1. \end{aligned}$$

Every complex number $z = (x, y) \in \mathbb{C}$ admits a *unique representation*

$$z = (x, y) = (x, 0) + (0, y) = x + y(0, 1) = x + yi = x + iy;$$

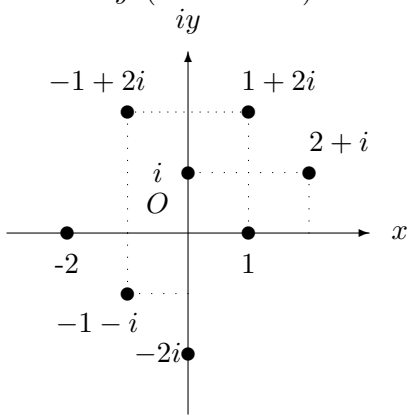
that is

$$\boxed{z = x + iy}$$

with $x, y \in \mathbb{R}$. This is the usual way to write complex numbers and will be called the **normal form**. The real numbers x and y are called the **real part** and the **imaginary part** of z, respectively, and we write

$$x = \mathrm{Re}\,(z) \quad \text{and} \quad y = \mathrm{Im}\,(z).$$

A complex number of the form iy (with $x = 0$) is called **imaginary**.



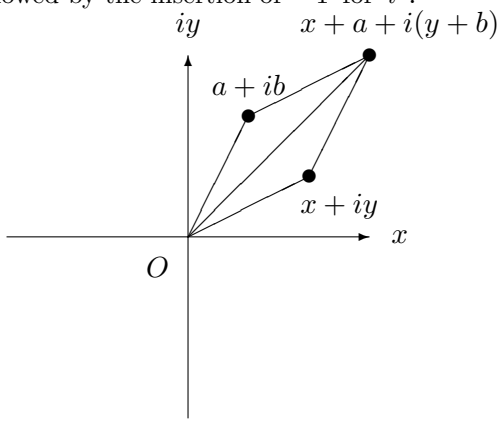
Complex numbers as points (in the plane).

NOTE : In this notation, the rules (1) and (2) become

$$(x+iy)+(a+ib) = (x+a)+i(y+b)$$

$$(a+ib)(x+iy) = ax-by+i(ay+bx)$$

which may be thought of as ordinary addition and multiplication, treating the symbol i as an indeterminate, followed by the insertion of -1 for i^2.



The sum of two complex numbers.

10.1.2 Example. Consider a nonzero complex number z. What is the geometric relationship between z and iz in the complex plane ?

Solution : If $z = a + ib$, then $iz = -b + ia$. We obtain the vector $\begin{bmatrix} -b \\ a \end{bmatrix}$ (representing iz) by rotating the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ (representing z) through an angle of $90°$ in the counterclockwise direction.

10.1.3 Example. If $z = 3 + 4i$ and $w = 1 + i$, find

$$\operatorname{Im}(z + 2w^2).$$

Solution : We have

$$z + 2w^2 = 3 + 4i + 2(1 + i)^2 = 3 + 4i + 4i = 3 + 8i$$

and hence

$$\operatorname{Im}(z + 2w^2) = 8.$$

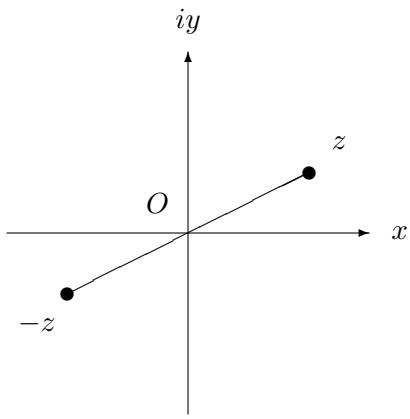
10.2 Algebraic operations on complex numbers

For a complex number $z = x + iy$ we define

$$\boxed{-z := -x - iy} \quad \text{(the \textbf{opposite} of } z\text{)}$$

and (for $z \neq 0$)

$$\boxed{z^{-1} := \frac{1}{x^2 + y^2}(x - iy).} \quad \text{(the \textbf{inverse} of } z\text{).}$$



The opposite of a complex number : $-z$.

If z_1, z_2 are two complex numbers, we write $z_1 - z_2$ instead of $z_1 + (-z_2)$ and $\frac{z_1}{z_2}$ instead of $z_1 z_2^{-1}$ (for $z_2 \neq 0$), just as we did with real numbers.

10.2.1 Example. Find the inverse of $2 + i$.

Solution : We have

$$(2+i)^{-1} = \frac{1}{5}(2-i).$$

We see that

$$(2+i)(2+i)^{-1} = \frac{1}{5}(2+i)(2-i) = \frac{5}{5} = 1.$$

10.2.2 Example. Write

$$E = \left(\frac{1+2i}{2+i}\right)^3$$

in the normal form.

Solution : We have

$$E = \left[(1+2i)(2+i)^{-1}\right]^3 = \left(\frac{1}{5}(1+2i)(2-i)\right)^3 = \left(\frac{1}{5}(4-3i)\right)^3 = \frac{1}{125}(-44+117i)\cdot$$

The following proposition summarizes the algebraic properties of the addition and multiplication of complex numbers.

10.2.3 Proposition. *If $z, z_1, z_2, z_3 \in \mathbb{C}$, then :*

(1) $z_1 + z_2 = z_2 + z_1$.

(2) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

(3) $z + 0 = 0 + z = z$.

(4) $z + (-z) = (-z) + z = 0$.

(5) $z_1 \cdot z_2 = z_2 \cdot z_1$.

(6) $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$.

(7) $z \cdot 1 = 1 \cdot z = z$.

(8) $z \cdot z^{-1} = z^{-1} \cdot z = 1 \quad (z \neq 0)$.

(9) $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$.

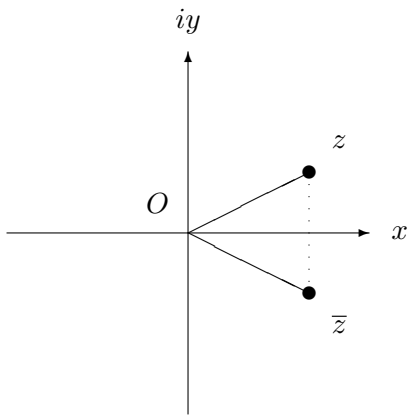
PROOF : Exercise.

NOTE : The properties listed above may be summarized by saying that the complex number system $(\mathbb{C}, +, \cdot)$ is a (commutative) *field*.

For a complex number $z = x + iy$ we define

$$\boxed{\bar{z} := x - iy} \quad \text{(the } \textbf{conjugate} \text{ of } z\text{)}.$$

Geometrically, the conjugate $\bar{z}$ is the *reflection* of z in the x-axis (the so-called **real axis**).



The conjugate of a complex number : $\overline{z}$.

10.2.4 Proposition. *If* $z, z_1, z_2 \in \mathbb{C}$, *then :*

(1) $\overline{z_1 + z_2} = \bar{z}_2 + \bar{z}_1$.

(2) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$.

(3) $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$.

(4) $\bar{\bar{z}} = z$.

(5) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$.

(6) $\mathrm{Re}\,(z) = \frac{1}{2}(z + \bar{z})$.

(7) $\mathrm{Im}\,(z) = \frac{1}{2i}(z - \bar{z})$.

(8) $z \in \mathbb{R} \iff z = \bar{z}$.

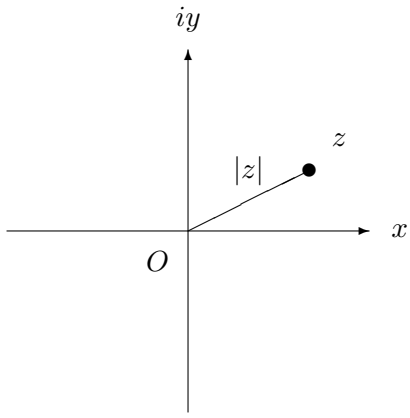
(9) $z \in i\,\mathbb{R} \iff z = -\bar{z}$.

Proof : Exercise.

If $z = x + iy \in \mathbb{C}$, then the real number

$$\boxed{|z| := \sqrt{x^2 + y^2}}$$

is called the **modulus** (or **absolute value**) of z. In other words, $|z|$ is nothing but the *distance* from the origin to the point (x, y) ; alternatively, $|z|$ is the *length* of the (geometric) vector $\begin{bmatrix} x \\ y \end{bmatrix}$.



The modulus of a complex number : $|z|$.

NOTE : If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $|z_1 - z_2|$ is the distance between the points (x_1, y_1) and (x_2, y_2).

10.2.5 PROPOSITION. *If $z, z_1, z_2 \in \mathbb{C}$, then :*

(1) $|z| \geq 0$, *and* $|z| = 0 \iff z = 0$.

(2) $z \cdot \bar{z} = |z|^2$.

(3) $|z_1 \cdot z_2| = |z_1||z_2|$.

(4) $|z| = |-z| = |\bar{z}|$.

(5) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \quad (z_2 \neq 0)$.

(6) $\mathrm{Re}\,(z) \leq |\mathrm{Re}\,(z)| \leq |z|$.

(7) $\mathrm{Im}\,(z) \leq |\mathrm{Im}\,(z)| \leq |z|$.

(8) $|\,|z_1| - |z_2|\,| \leq |z_1 - z_2|$.

PROOF : Exercise.

10.2.6 EXAMPLE. Let $z, w \in \mathbb{C}$. Then

$$|z + w| \leq |z| + |w|.$$

(This result is known as the **triangle inequality**).

SOLUTION : We have

$$\begin{aligned}
|z+w|^2 &= (z+w)\cdot(\overline{z+w}) \\
&= (z+w)\cdot(\bar{z}+\bar{w}) \\
&= z\cdot\bar{z} + z\cdot\bar{w} + \bar{z}\cdot w + w\cdot\bar{w} \\
&= |z|^2 + z\cdot\bar{w} + \bar{z}\cdot w + |w|^2 \\
&= |z|^2 + 2\mathrm{Re}\,(z\cdot\bar{w}) + |w|^2 \\
&\leq |z|^2 + 2|z\cdot\bar{w}| + |w|^2 \\
&= |z|^2 + 2|z|\,|\bar{w}| + |w|^2 \\
&= |z|^2 + 2|z|\,|w| + |w|^2 \\
&= (|z| + |w|)^2.
\end{aligned}$$

The result now follows.

10.3 De Moivre's formula

Sometimes it is useful to describe a complex number in *polar coordinates.*

If $z = x + iy \in \mathbb{C} \setminus \{0\}$, then we can write

$$\begin{aligned} z = x + iy &= \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} + i\frac{y}{\sqrt{x^2+y^2}}\right) \\ &= |z|\left(\frac{\mathrm{Re}\,(z)}{|z|} + i\frac{\mathrm{Im}\,(z)}{|z|}\right) \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

where $r = |z|$ and θ is an *angle* such that

$$\cos\theta = \frac{x}{r} \quad \text{and} \quad \sin\theta = \frac{y}{r}\cdot$$

NOTE : The existence of θ is assured since $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ and, in fact, there are many such θ. Geometrically, r is the distance (in the complex plane) between the origin and the point z, and θ measures the angle between the real axis and the vector z.

Any real number θ such that $z = |z|(\cos\theta + i\sin\theta)$ is said to be an **argument** of z. We denote by $\mathrm{Arg}\ z$ the set of all arguments of z ; that is,

$$\mathrm{Arg}\ z := \left\{\theta \in \mathbb{R} \mid \cos\theta = \frac{\mathrm{Re}\,(z)}{|z|} \ \text{ and } \ \sin\theta = \frac{\mathrm{Im}\,(z)}{|z|}\right\}.$$

The number $\arg z \in \mathrm{Arg}\ z$ such that $-\pi < \arg z \leq \pi$ is caled the **principal argument** of z. We have

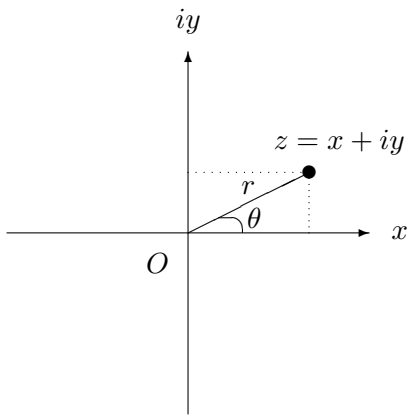
$$\mathrm{Arg}\ z = \{\arg z + 2k\pi \mid k \in \mathbb{Z}\}.$$

NOTE : Our *normalization* of θ to the interval $(-\pi, \pi]$ was arbitrary; in general, any half-open interval of length 2π is suitable.

The representation

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

is called the **polar form** of the complex number z. The real numbers $r = |z|$ and $\theta = \arg z$ are the **polar coordinates** of z.



The polar form of a complex number : $r(\cos\theta + i\sin\theta)$.

10.3.1 Example. Find the modulus and the (principal) argument of $z = -2 + 2i$.

Solution : We have

$$r = |z| = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}\,.$$

Representing z in the complex plane, we see that $\frac{3\pi}{4}$ is an argument of z (in fact, its principal argument).

10.3.2 Example. Determine $\operatorname{Arg}(-1)$ and $\operatorname{Arg}(1-i)$.

Solution : We have

$$\begin{aligned}
\operatorname{Arg}(-1) &= \left\{\theta \mid \cos\theta = \frac{-1}{1} \text{ and } \sin\theta = \frac{0}{1}\right\} \\
&= \{\theta \mid \cos\theta = -1 \text{ and } \sin\theta = 0\} \\
&= \{\pi + 2k\pi \mid k \in \mathbb{Z}\}
\end{aligned}$$

and

$$\begin{aligned}\operatorname{Arg}(1-i) &= \left\{\theta \mid \cos\theta = \frac{1}{\sqrt{2}} \text{ and } \sin\theta = -\frac{1}{\sqrt{2}}\right\} \\ &= \left\{-\frac{\pi}{4} + 2k\pi \mid k \in \mathbb{Z}\right\}.\end{aligned}$$

10.3.3 Example. Let $z \in \mathbb{C} \setminus \{0\}$. Then

$$\operatorname{Arg} \bar{z} = \operatorname{Arg}(z^{-1}).$$

Solution : Let $z = r(\cos\theta + i\sin\theta)$, where $\theta = \arg z$; then $\bar{z} = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + i\sin(-\theta))$ and hence

$$\operatorname{Arg} \bar{z} = \{-\theta + 2k\pi \mid k \in \mathbb{Z}\}.$$

On the other hand,

$$\begin{aligned}z^{-1} &= \frac{\bar{z}}{|z|^2} \\ &= \frac{r(\cos(-\theta) + i\sin(-\theta))}{r^2} \\ &= \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))\end{aligned}$$

and so

$$\operatorname{Arg}(z^{-1}) = \{-\theta + 2k\pi \mid k \in \mathbb{Z}\}.$$

10.3.4 Example. Write the complex numbers -3, i, and $-1+i$ in the polar form.

Solution : We have

$$\begin{aligned}-3 &= 3(\cos\pi + i\sin\pi), \\ i &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}, \\ -1+i &= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).\end{aligned}$$

The most important property of the polar form is given in the proposition below. It will allow us to have a very good geometric interpretation for the product of two complex numbers.

10.3.5 PROPOSITION. *Let* $z = \cos\alpha + i\sin\alpha$ *and* $w = \cos\beta + i\sin\beta$. *Then*

$$zw = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

SOLUTION : We have

$$\begin{aligned} zw &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha) \\ &= \cos(\alpha+\beta) + i\sin(\alpha+\beta). \end{aligned}$$

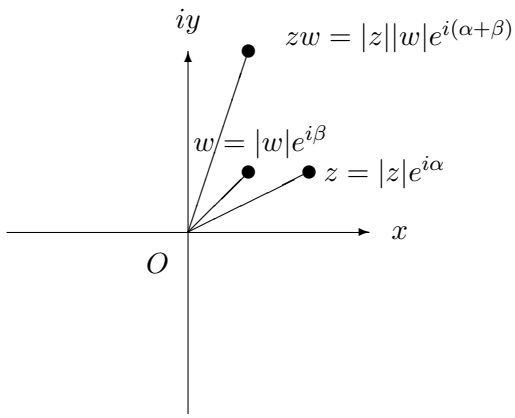
We observe that the modulus of zw is 1, and $\alpha+\beta$ is an argument of zw.□

In general, if $z = r(\cos\alpha + i\sin\beta)$ and $w = s(\cos\beta + i\sin\beta)$, then

$$zw = rs(\cos(\alpha+\beta) + i\sin(\alpha+\beta)).$$

From this we see that *when we multiply two complex numbers, we multiply the moduli and we add the arguments.* Thus

$$\boxed{|zw| = |z|\,|w| \quad \text{and} \quad \operatorname{Arg}(zw) = \operatorname{Arg} z + \operatorname{Arg} w.}$$



The product of two complex numbers.

10.3.6 Example. Describe the transformation $T : \mathbb{C} \to \mathbb{C}, \quad z \mapsto (3 + 4i)z$ geometrically.

Solution : We have

$$\begin{aligned} |T(z)| &= |3+4i|\,|z| = 5|z| \\ \operatorname{Arg}(T(z)) &= \operatorname{Arg}(3+4i) + \operatorname{Arg} z = \arctan\left(\frac{4}{3}\right) + \operatorname{Arg} z \approx 53^\circ + \operatorname{Arg} z\,. \end{aligned}$$

The transformation T is a *rotation-dilation* in the complex plane.

If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, we define the *power* z^n by

$$\boxed{z^n := \underbrace{z \cdot z \cdots z}_{n \text{ factors}}.}$$

We put $z^0 := 1$ and for a negative integer $m = -n$ with $n \in \mathbb{N}$, we define

$$\boxed{z^m = z^{-n} := \left(z^{-1}\right)^n\,.}$$

The following result is due to Abraham de Moivre (1667-1754).

10.3.7 Theorem. (De Moivre's Formula) *For* $\theta \in \mathbb{R}$ *and* $n \in \mathbb{Z}$

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Proof : We first use induction to prove that the result holds for all $n \in \mathbb{N}$.

If $n = 0$, then

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^0 = 1 = \cos 0 + i\sin 0 = \cos(n\theta) + i\sin(n\theta)\,.$$

Assume that the formula is true for some $n \in \mathbb{N}$; then

$$\begin{aligned} (\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^n \\ &= (\cos\theta + i\sin\theta)(\cos\, n\theta + i\sin\, n\theta) \\ &= \cos\theta\cos\, n\theta - \sin\theta\sin\, n\theta + i(\cos\theta\sin\, n\theta + \sin\theta\cos\, n\theta) \\ &= \cos\,(n+1)\theta + i\sin\,(n+1)\theta. \end{aligned}$$

It follows that the formula is true for all $n \in \mathbb{N}$.

If n is a negative integer, let $n = -m$ with $m \in \mathbb{N}$. Then

$$\begin{aligned}
(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m} \\
&= \left(\frac{1}{\cos\theta + i\sin\theta}\right)^m \\
&= (\cos(-\theta) + i\sin(-\theta))^m \\
&= \cos(-m\theta) + i\sin(-m\theta) \\
&= \cos\, n\theta + i\sin\, n\theta.
\end{aligned}$$

□

It is easy matter to check that the rules of exponents extend to complex numbers. Namely :

10.3.8 Proposition. *Let $z, w \in \mathbb{C}$ and $m, n \in \mathbb{N}$. Then :*

(1) $(zw)^n = z^n\, w^n$.

(2) $z^{mn} = (z^m)^n$.

(3) $z^m\, z^n = z^{m+n}$.

(4) $\frac{z^m}{z^n} = z^{m-n}$.

Proof : Exercise.

10.3.9 Example. Evaluate the following expression

$$E = (1+i)^{10} + (1-i)^{10}.$$

Solution : We have (for $n \in \mathbb{N}$) :

$$\begin{aligned}
E(n) &= (1+i)^n + (1-i)^n \\
&= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n + \left[\sqrt{2}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)\right]^n \\
&= 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) + 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right) \\
&= 2\cdot 2^{\frac{n}{2}}\cos\frac{n\pi}{4} = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}\cdot
\end{aligned}$$

In particular, for $n = 10$, we get :

$$E = E(10) = (1+i)^{10} + (1-i)^{10} = 2^6 \cos \frac{5\pi}{2} = 0.$$

10.4 Applications

$\boxed{\text{A}}$ (Complex-valued functions) A **complex-valued function** $t \mapsto z = f(t)$ is a function from $\mathbb{R}$ to $\mathbb{C}$: the *input* t is real, and the *output* z is complex.

10.4.1 Example. Here are two examples of complex-valued functions :

$$z = t + it^2 \quad \text{and} \quad z = \cos t + i \sin t.$$

For each t, the output z can be represented as a point in the *complex plane*. As we let t vary, we trace out a *trajectory* in the complex plane (a *parabola* and a *circle*, respectively).

Consider the complex-valued function

$$f : \mathbb{R} \to \mathbb{C}, \quad f(t) := \cos t + i \sin t.$$

This function has remarkable properties. For instance,

(1) $f(t) \cdot f(s) = f(t+s)$.

(2) $\frac{d}{dt} f(t) = i\, f(t)$.

(3) $f(0) = 1$.

It can be shown that *there exists a unique complex-valued function that satisfies conditions (2) and (3)*; this function also satisfies condition (1).

Note : The *exponential function*

$$\exp : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \exp(t) := e^{at} \quad (a \in \mathbb{R})$$

has the properties :

$$\frac{d}{dt} \exp(t) = a \exp(t) \quad \text{and} \quad \exp(0) = 1.$$

This motivates us to write

EULER'S FORMULA : *For any real number* θ

$$\boxed{e^{i\theta} := \cos\theta + i\sin\theta.}$$

NOTE: It has been known since the 18th century that the exponential function and the trigonometric functions are related. This remarkable relationship was discovered by LEONHARD EULER (1707-1783). The case $\theta = \pi$ leads to the intriguing formula $e^{i\pi} + 1 = 0$; this has been called the most beautiful formula in all mathematics.

Euler's formula can be used to write the polar form of a complex number more succintly :

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

This representation is known as the **exponential form** of the complex number z.

10.4.2 PROPOSITION. *If* $z = re^{i\theta} \in \mathbb{C}$, *then :*

$$z^n = r^n e^{in\theta}, \quad n \in \mathbb{Z}.$$

In particular, $z^{-1} = \frac{1}{r}e^{-i\theta}$ *and hence* $\overline{z} = re^{-i\theta}$.

PROOF : Exercise.

10.4.3 EXAMPLE. Write $z = i$ in exponential form.

SOLUTION : We have $r = 1$ and $\theta = \arg z = \frac{\pi}{2}\cdot$ Hence

$$z = e^{i\frac{\pi}{2}}.$$

10.4.4 EXAMPLE. Write $z = 2\,e^{i\pi}$ in normal form.

SOLUTION : Here we are given $r = 2$ and $\theta = \arg z = \pi$. Hence

$$z = 2(\cos\pi + i\sin\pi) = 2(-1 + i0) = -2.$$

10.4.5 Example. Find the real and imaginary parts of $(1+2i)\,e^{-it}$.

Solution : Let $z(t) = (1+2i)\,e^{-it}$. Then

$$z(t) = (1+2i)(\cos t + i\sin t) = (\cos t + 2\sin t) + i(2\cos t - \sin t).$$

Hence

$$\begin{aligned} \mathrm{Re}\,(z(t)) &= \cos t + 2\sin t \\ \mathrm{Im}\,(z(t)) &= 2\cos t - \sin t. \end{aligned}$$

If $\theta \in \mathbb{R}$, then

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta. \end{aligned}$$

Adding these and dividing by 2, we obtain :

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}\cdot}$$

Subtracting these and dividing by $2i$, we obtain :

$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\cdot}$$

Thus the familiar trigonometric functions can be expressed in terms of complex-valued functions. This finds application in a number of situations.

10.4.6 Example. Express $\cos 4\theta$ in terms of sines and cosines of θ.

Solution : We have

$$\begin{aligned} \cos 4\theta &= \mathrm{Re}\,(e^{i4\theta}) \\ &= \mathrm{Re}\,(\cos\theta + i\sin\theta)^4 \\ &= \mathrm{Re}\,\left(\cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4\right) \\ &= \mathrm{Re}\,\left((\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)\right) \\ &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta. \end{aligned}$$

10.4.7 EXAMPLE. Express $\sin^5\theta$ in terms of sines and cosines of multiples of θ.

SOLUTION : We have

$$\begin{aligned}
\sin^5\theta &= \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^5 \\
&= \frac{1}{(2i)^5}\left(e^{i5\theta}-5e^{i4\theta}e^{-i\theta}+10e^{i3\theta}e^{-i2\theta}-10e^{2\theta}e^{-i3\theta}+5e^{i\theta}e^{-i4\theta}-e^{-i5\theta}\right) \\
&= \frac{1}{32i}\left((e^{i5\theta}-e^{-i5\theta})-5(e^{i3\theta}-e^{-i3\theta})+10(e^{i\theta}-e^{-i\theta})\right) \\
&= \frac{1}{16}(\sin 5\theta-5\sin 3\theta+10\sin\theta).
\end{aligned}$$

10.4.8 EXAMPLE. Evaluate the sum

$$S=\sum_{k=0}^{n-1}\sin k\theta\,.$$

SOLUTION : The trick is to write each $\sin k\theta$ as $\operatorname{Im}(e^{ik\theta})$ and note that the sum is a geometric series with ratio $e^{i\theta}$. We have

$$S=\sum_{k=0}^{n-1}\sin k\theta=\operatorname{Im}\left(1+e^{i\theta}+e^{i2\theta}+\cdots+e^{i(n-1)\theta}\right)\,.$$

If the ratio $e^{i\theta}$ is 1, then the sum is simply n. We therefore assume that

$e^{i\theta} \neq 1$. Thus

$$\begin{aligned}
S = \sum_{k=0}^{n-1} \sin k\theta &= \operatorname{Im}\left(\frac{1-e^{in\theta}}{1-e^{i\theta}}\right) \\
&= \operatorname{Im}\left(\frac{1-e^{in\theta}}{1-e^{i\theta}} \cdot \frac{e^{-i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}}\right) \\
&= \operatorname{Im}\left(\frac{e^{i(n-\frac{1}{2})\theta} - e^{-i\frac{\theta}{2}}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}\right) \\
&= \operatorname{Im}\left(\frac{e^{i(n-\frac{1}{2})\theta} - e^{-i\frac{\theta}{2}}}{2i\sin\frac{\theta}{2}}\right) \\
&= \operatorname{Im}\left(-i\frac{e^{i(n-\frac{1}{2})\theta} - e^{-i\frac{\theta}{2}}}{2\sin\frac{\theta}{2}}\right) \\
&= \frac{\cos\frac{\theta}{2} - \cos(n-\frac{1}{2})\theta}{2\sin\frac{\theta}{2}} \\
&= \frac{\sin\frac{n\theta}{2}\sin\frac{(n-1)\theta}{2}}{\sin\frac{\theta}{2}}\,.
\end{aligned}$$

Since $e^{i\theta} \neq 1$, we know that $\sin\frac{\theta}{2} \neq 0$ and so calculations above are meaningful.

$\boxed{\text{B}}$ (Solutions of equations)

Equations of the form $x^n = w$

Equations of the form

$$x^n = w$$

where w is a fixed complex number, can be solved by writing x and w in polar (or exponential) form. If we let $x = r(\cos\theta + i\sin\theta) = re^{i\theta}$, then we obtain equations for r and θ which we can then solve.

10.4.9 Example. Solve

$$x^2 = -d^2\,, \quad d > 0.$$

SOLUTION : We write

$$x = r(\cos\theta + i\sin\theta) \quad \text{and} \quad -d^2 = d^2(\cos\pi + i\sin\pi)\,.$$

Then we have (using de Moivre's formula)

$$r^2(\cos 2\theta + i\sin 2\theta) = d^2(\cos\pi + i\sin\pi).$$

We obtain

$$r^2 = d^2 \quad \text{and} \quad 2\theta = \pi + 2k\pi\,, \quad k \in \mathbb{Z}$$

and hence

$$r = d \quad \text{and} \quad \theta = \frac{\pi}{2} + k\pi\,, \quad k \in \mathbb{Z}.$$

The set of solutions is

$$\begin{aligned}\left\{d(\cos\theta + i\sin\theta)\,|\,\theta = \frac{\pi}{2} + k\pi\,,\ k \in \mathbb{Z}\right\} &= \left\{d(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}),\ d(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2})\right\} \\ &= \{di,\ -di\}.\end{aligned}$$

10.4.10 EXAMPLE. Solve

$$x^4 = -1.$$

SOLUTION : We write

$$x = re^{i\theta} \quad \text{and} \quad -1 = e^{i\pi}.$$

Then we have

$$r^4 e^{i4\theta} = e^{i\pi}$$

and hence

$$r^4 = 1 \quad \text{and} \quad 4\theta = \pi + 2k\pi\,, \quad k \in \mathbb{Z}.$$

We get

$$r = 1 \quad \text{and} \quad \theta = \frac{\pi}{4} + k\frac{\pi}{2}\,, \quad k \in \mathbb{Z}.$$

The solution are (for $k = 0, 1, 2, 3$) :

$$\begin{aligned} x_0 &= e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)\,; \\ x_1 &= e^{i\frac{3\pi}{4}} = \frac{1}{\sqrt{2}}(-1+i)\,; \\ x_2 &= e^{i\frac{5\pi}{4}} = \frac{1}{\sqrt{2}}(-1-i)\,; \\ x_3 &= e^{i\frac{7\pi}{4}} = \frac{1}{\sqrt{2}}(1-i). \end{aligned}$$

For $k \geq 4$ we get only repeats of these four roots.

NOTE : The roots are complex-conjugate pairs.

10.4.11 EXAMPLE. Find all the solutions (roots) of

$$x^n = 1\,, \quad n \in \mathbb{N}.$$

SOLUTION : We write

$$x = re^{i\theta} \quad \text{and} \quad 1 = e^{i0}$$

and thus

$$x^n = r^n e^{in\theta} = e^{i0}.$$

It follows that

$$r^n = 1 \quad \text{and} \quad n\theta = 2k\pi\,, \quad k \in \mathbb{Z}.$$

Hence

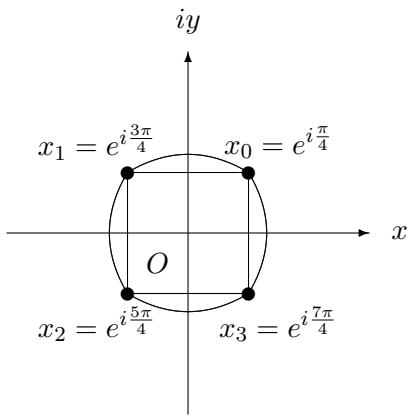
$$r = 1 \quad \text{and} \quad \theta = \frac{2k\pi}{n}\,, \quad k = 0, 1, 2, \ldots, n-1.$$

[For $k \geq n$ we get only repeats of these n roots.]

The solution are

$$x_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\,, \quad k = 0, 1, 2, \ldots, n-1.$$

NOTE : The roots are either real (for instance $x_0 = 1$) or in complex-conjugate pairs.



The roots of the equation $x^4 = -1$.

Quadratic equations

If $a, b,$ and c are real numbers, then the *quadratic equation*

$$ax^2 + bx + c = 0$$

has real solutions given by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $\Delta := b^2 - 4ac \geq 0$. However, if $\Delta < 0$ the solutions are complex.

Let $-\Delta := d^2$, $\quad d > 0$. We complete the square to obtain

$$a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] = 0$$

which leads to

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{d^2}{4a^2}\,.$$

Thus

$$x + \frac{b}{2a} = \pm\, i\,\frac{d}{2a}$$

and hence

$$\boxed{x_{1,2} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}\,.}$$

NOTE : (1) If $a, b, c \in \mathbb{R}$, the roots of the *polynomial* of degree 2 $p(x) = ax^2 + bx + c$ are either real or a complex-conjugate pair.

(2) The formula above remains valid if the coefficients a, b, c are complex numbers: we can still employ the method of the square to find the roots.

Polynomial equations of degree n

Let $p_n(x)$ be a *polynomial* of degree n (with complex coefficients). Then, after multiplying through by the reciprocal of the coefficient of x^n, we may assume that $p_n(x)$ has the form

$$p_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Polynomials whose highest power has coefficient 1 are called **monic**.

Perhaps the most remarkable property of the complex numbers is expressed in the *fundamental theorem of algebra*, first demonstrated by CARL F. GAUSS (1777-1855).

10.4.12 THEOREM. (FUNDAMENTAL THEOREM OF ALGEBRA) *Every polynomial with complex coefficients has at least one (complex) root.*

Suppose that $x = w$ is a root of $p_n(x)$. This means that

$$p_n(w) = w^n + a_{n-1}w^{n-1} + \cdots + a_1 w + a_0 = 0.$$

If $p_n(x)$ is divided by $x - w$, we obtain the identity

$$\frac{p_n(x)}{x - w} = q_{n-1}(x) + \frac{R}{x - w}$$

where R is a constant and $q_{n-1}(x)$ is a polynomial of degree $n - 1$. Hence,

$$p_n(x) = (x - w)q_{n-1}(x) + R.$$

But this is an *identity* in x, so that by setting $x = w$, we see that $R = 0$ if and only if $p_n(w) = 0$; that is, if and only if w is a root of $p_n(x)$. By

repeated use of this result, we obtain that *every polynomial of degree n with complex coefficients has precisely n roots (if they are properly counted with their multiplicities).* We can restate this result as follows :

10.4.13 PROPOSITION. *Any polynomial of degree n*

$$p_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

with complex coefficients can be written as a product of linear factors

$$p_n(x) = (x - w_1)(x - w_2)\cdots(x - w_n)$$

for some complex numbers $w_1, w_2, \ldots, w_n$. The numbers (roots) w_i need not be distinct.

A common situation arising particularly often in applications is the case in which all the coefficients of the polynomial $p_n(x)$ are real. The following result is easy to prove.

10.4.14 PROPOSITION. *Let $p_n(x)$ be a polynomial with real coefficients. If $w = a + ib$, $b \neq 0$ is a root of $p_n(x)$ then so is its conjugate $\overline{w} = a - ib$.*

PROOF : Exercise.

From these two results (PROPOSITION 12.4.13 and PROPOSITION 12.4.14) we obtain

10.4.15 PROPOSITION. *Any polynomial of degree n*

$$p_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

with real coefficients can be written as a product of linear and irreducible quadratic factors

$$p_n(x) = (x - r_1)(x - r_2)\cdots(x - r_k)(x^2 + \alpha_1x + \beta_1)\cdots(x^2 + \alpha_lx + \beta_l)$$

for some real numbers r_1, r_2, $\ldots$, r_k, α_1, β_1, $\ldots$, α_l, β_l. The numbers (roots) r_i as well as the numbers α_j, β_j need not be distinct.

10.4.16 Example. Find the roots of $p(x) = x^4 - 1$ and factor this polynomial.

Solution : Since $x^4 - 1$ is a difference of two squares, we have

$$p(x) = x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x^2 + 1) = (x - 1)(x + 1)(x - i)(x + i).$$

Thus the roots are

$$x_1 = 1,\ x_2 = -1,\ x_3 = i,\ x_4 = -i.$$

10.4.17 Example. Find a (monic) polynomial of degree 3 whose roots are $0,\ 1,\ 2$.

Solution : Since $(x - 0)$, $(x - 1)$ and $(x - 2)$ must be factors of any such polynomial, we have

$$p_3(x) = x(x - 1)(x - 2) = x^3 - 3x^2 + 2x.$$

10.4.18 Example. Find a (monic) polynomial of lowest degree with real coefficients having the roots $1,\ 1$ and $1 - i$.

Solution : Because the roots of a polynomial with real coefficients come in complex-conjugate pairs, the fact that $1 - i$ is a root implies that $1 + i$ is also a root. Hence,

$$(x - (1 - i))(x - (1 + i)) = x^2 - 2x + 2$$

is a factor of $p(x)$. Likewise, $(x - 1)^2$ ia also a factor. Therefore,

$$p(x) = (x^2 - 2x + 2)(x - 1)^2 = x^4 - 4x^3 + 7x^2 - 6x + 2$$

has the required roots. No lower degree polynomial could have four roots, so this is the monic polynomial of least degree with these roots.

10.4.19 Example. Solve the equation

$$x^3 + 3x - 4 = 0.$$

SOLUTION : Let $p(x) = x^3 + 3x - 4$. We first try, by inspection, to find a real root of $p(x)$. We spot $p(1) = 0$ and this means that $(x-1)$ is a factor. Dividing $(x-1)$ into $p(x)$ we obtain

$$p(x) = (x-1)(x^2 + x - 4).$$

Thus $p(x)$ is a product of a linear and an irreducible quadratic factor. The solutions of the equation $p(x) = 0$ are (the roots of $p(x)$) :

$$1 \quad \text{and} \quad -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\cdot$$

10.5 Exercises

Exercise 146 Find

$$i^{17},\ i^{23},\ i^{467}.$$

Exercise 147 If $z = 3 + 4i$ and $w = 1 + i$, find :

(a) $z + w$.

(b) z^2.

(c) $i\,w$.

(d) $(z - 3w)^{100}$.

(e) $\mathrm{Re}\,(z + 2\omega^2)$.

(f) $\mathrm{Im}\,(z - w)$.

Exercise 148 Find $\overline{z}$, $\mathrm{Re}\,(z)$, $\mathrm{Im}\,(z)$ and $|z|$ if

$$(a) \quad z = 7, \quad (b) \quad z = -2i, \quad (c) \quad z = (3+5i)^2, \quad (d) \quad z = \frac{2+3i}{4-5i}\cdot$$

Exercise 149 Show that the points (complex numbers) z_1, z_2, z_3 are collinear if and only if $\dfrac{z_2 - z_1}{z_3 - z_1} \in \mathbb{R}$.

Exercise 150 Prove that if $z, w \in \mathbb{C}$, then

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

This is known as the *parallelogram law*. Justify the name.

Exercise 151 Prove that for $z \in \mathbb{C}$

$$(1-z)(1+z+z^2+\cdots+z^{n-1}) = 1-z^n$$

and hence deduce that (for $z \neq 1$) :

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z}\,.$$

Exercise 152 Write in the polar (and exponential) form :

(a) $1+\sqrt{3}\,i$.

(b) $-1-i$.

(c) $-5+5i$.

(d) $\left(\dfrac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i}\right)^{10}$.

Exercise 153 Let $z = x+iy \in \mathbb{C}$ and define

$$e^z := e^x(\cos y + i\sin y).$$

Show that (for $z, w \in \mathbb{C}$ and $n \in \mathbb{N}$) :

(a) $e^{z+w} = e^z \cdot e^w$.

(b) $e^{nz} = (e^z)^n$.

(c) $|e^z| = e^x$.

(d) $\overline{e^z} = e^{\overline{z}}$.

(e) $e^z = e^w \iff z = w + 2k\pi, \quad k \in \mathbb{Z}$.

Solve the equation

$$e^z = 1+i.$$

Exercise 154 Evaluate

$$e^{-i}, \quad 2^i \quad \text{and} \quad \sqrt{i}.$$

Exercise 155 Show that :

(a) $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$.

(b) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

Exercise 156 Show that :

(a) $\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$.

(b) $\cos^4\theta = \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$.

Exercise 157 Prove that

$$\sum_{k=0}^{n-1} \cos\, k\theta = \frac{\sin\frac{n\theta}{2}\cos\frac{(n-1)\theta}{2}}{\sin\frac{\theta}{2}}\,.$$

Determine the values of θ for which this is valid and sum the (finite) series for these values of θ.

Exercise 158 Evaluate the following sums :

$$C = \sum_{k=0}^{n-1} 2^k \cos\, k\theta \quad \text{and} \quad S = \sum_{k=0}^{n-1} 2^k \sin\, k\theta.$$

Exercise 159 Solve the following equations and then represent the roots as points in the complex plane.

(a) $5x^2 + 2x + 10 = 0$.

(b) $x^2 + (2i - 3)x + 5 - i = 0$.

(c) $x^5 = 1$.

(d) $x^2 = i$.

(e) $x^3 = -1 + i$.

(f) $x^6 + \sqrt{2}x^3 + 1 = 0$.

(g) $x^{10} + 6ix^5 - 12 = 0$.

(h) $x^5 - 2x^4 - x^3 + 6x - 4 = 0$.

Exercise 160 Consider a polynomial of degree n

$$p_n(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

with complex coefficients, and let $w_1,\ w_2,\ \ldots,\ w_n$ denote its roots. Show that :

(a) $w_1 + w_2 + \cdots + w_n = -a_{n-1}$.

(b) $w_1 \cdot w_2 \cdots w_n = (-1)^n a_0$.