# Chapter 4

# Mathematical Induction

**Topics :**

1. Sequences of numbers
2. Summations
3. Mathematical induction

One of the tasks of mathematics is to discover and characterize patterns, such as those associated with processes that are repeated. The main mathematical structure used to study repeated processes is the *sequences*. An important mathematical tool used to verify conjectures about patterns is *mathematical induction*.

## 4.1 Sequences of numbers

Sequences are used to represent ordered collections, finite or infinite, of elements. To say that a collection of objects is *ordered* means that the collection has an identified first element, second element, third element, and so on. For the sake of simplicity, we may assume that the objects involved are all *numbers*. (Here, by number is meant any real number; that is, an element of the set $\mathbb{R}$.) A formal definition is given below.

**4.1.1** Definition. A **sequence** (of numbers) is a real-valued function whose domain is an infinite subset of $\mathbb{N}$ (usually either the set $\mathbb{N}$ or $\mathbb{Z}^+$).

Let $a : N \subseteq \mathbb{N} \to \mathbb{R}, \quad n \mapsto a(n)$ be a sequence. It is customary to use the notation $(a_n)_{n\in\mathbb{N}}$ (or $(a_n)_{n\geq n_0}$ or, simply, $(a_n)$ ) to denote such a sequence, where $a_n$ is called the $n^{\text{th}}$ **term** of the sequence.

**4.1.2** Example. The terms of the sequence $(a_n)_{n\geq 1}$, where $a_n = 3+(-1)^n$ are

$$3+(-1)^1,\ 3+(-1)^2,\ 3+(-1)^3,\ 3+(-1)^4,\ \dots ;$$

that is,

$$2,\ 4,\ 2,\ 4,\ \dots .$$

**4.1.3** Example. The terms of the sequence $\left(\frac{2n}{n+1}\right)_{n\geq 1}$ are

$$\frac{2\cdot 1}{1+1},\ \frac{2\cdot 2}{2+1},\ \frac{2\cdot 3}{1+3},\ \frac{2\cdot 4}{1+4},\ \dots ;$$

that is,

$$1,\ \frac{4}{3},\ \frac{3}{2},\ \frac{8}{5},\ \dots .$$

Sometimes the terms of a sequence are generated by some rule that does not explicitly identify the $n^{\text{th}}$ term of the sequence. In such cases, you may be required to discover a *pattern* in the sequence and to describe the $n^{\text{th}}$ term.

**4.1.4** Example. Find a sequence $(a_n)_{n\in\mathbb{N}}$ whose first five terms are

$$\frac{1}{1}, \frac{2}{3}, \frac{4}{5}, \frac{8}{7}, \frac{16}{9}, \dots$$

Solution : First, note that the numerators are successive powers of 2, and the denominators form the sequence of positive odd integers. By comparing $a_n$ with $n$, we have the following pattern

$$\frac{2^0}{1}, \frac{2^1}{3}, \frac{2^2}{5}, \frac{2^3}{7}, \frac{2^4}{9}, \dots, \frac{2^n}{2n+1}, \dots$$

**4.1.5** Example. Determine the $n^{\text{th}}$ term for a sequence whose first five terms are

$$-\frac{2}{1}, \frac{8}{2}, -\frac{26}{6}, \frac{80}{24}, -\frac{242}{120}, \dots$$

Solution : Note that the numerators are 1 less than $3^n$. Hence, we can reason that the numerators are given by the rule $3^n - 1$. Factoring the denominators produces

$$\begin{aligned} 1 &= 1 \\ 2 &= 1 \cdot 2 \\ 6 &= 1 \cdot 2 \cdot 3 \\ 24 &= 1 \cdot 2 \cdot 3 \cdot 4 \\ 120 &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 . \end{aligned}$$

This suggests that the denominators are represented by $n!$. Finally, because the signs alternate, we can write the $n^{\text{th}}$ term as

$$a_n = (-1)^n \left( \frac{3^n - 1}{n!} \right).$$

**Finite sequences**

An ordered collection of finitely many objects is usually referred to as a **list** (of terms) or a **string** (of symbols). We write lists (strings) by starting with an open paranthesis, followed by the elements of the list (string) separated by commas, and finishing with a close parenthesis. For example, $(1, \emptyset, \mathbb{Z})$ is a list whose first term is number 1, whose second term is the empty set, and whose third term is the set of integers.

NOTE : The order in which elements appear in a list (string) is significant. The string $(a, b, c)$ is *not* the same as the string $(b, c, a)$. Elements in a (list) string might be repeated.

The number of elements in a list (string) is called its **length**. For example, the string $(1, 0, 1, 0)$ is a string of length four (or a 4-string). Another term used for strings is *tuple*.

NOTE : The string $s = (s_1, s_2, \ldots, s_n)$ is also denoted simply by $s_1 s_2 \cdots s_n$. Strings constructed only with the symbols 0 and 1 are called **bit strings**. For example, 00, 01, 10, 11 are (all the) bit strings of length two. Bit strings are widely used in discrete mathematics as well as in computer science.

It is clear that lists (strings or tuples) are in fact *finite* sequences of objects. The formal definition is given below.

**4.1.6** DEFINITION. For each positive integer $n$, the set $[n] := \{1, 2, 3, \ldots n\}$ is called an **initial segment** of $\mathbb{Z}^+$. A **finite sequence** is a function whose domain is an initial segment of $\mathbb{Z}^+$.

## 4.2 Summations

Consider a sequence (of numbers) $(a_n)_{n \in \mathbb{N}}$. In order to express the sum of the terms

$$a_p, a_{p+1}, a_{p+2}, \ldots, a_q \quad (p \leq q)$$

it is often convenient to use the *summation notation*; we write

$$\sum_{i=p}^{q} a_i$$

to represent

$$a_p + a_{p+1} + a_{p+2} + \cdots + a_q.$$

NOTE : The variable $i$ is called the **index of summation**, and the choice of the letter $i$ is arbitrary; thus, for instance,

$$\sum_{i=p}^{q} a_i = \sum_{j=p}^{q} a_j = \sum_{k=p}^{q} a_k.$$

The uppercase Greek letter *sigma* ($\Sigma$) is used to denote summation.

Here, the index of summation runs through *all* integers starting with the **lower limit** $p$ and ending with the **upper limit** $q$. (There are $q - p + 1$ terms in the summation.)

**4.2.1** EXAMPLE. Find the value of the sum

$$S_1 = \sum_{i=0}^{4} (-2)^i.$$

SOLUTION : We have

$$\begin{aligned} S_1 = \sum_{i=0}^{4} (-2)^i &= (-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 \\ &= 1 + (-2) + 4 + (-8) + 16 \\ &= 11. \end{aligned}$$

**4.2.2** EXAMPLE. Compute the sum

$$S_2 = \sum_{j=0}^{4} (2j+1)^2.$$

SOLUTION : We have

$$\begin{aligned} S_2 = \sum_{j=0}^{4} (2j+1)^2 &= 1^2 + 3^2 + 5^2 + 7^2 + 9^2 \\ &= 1 + 9 + 25 + 49 + 81 \\ &= 165. \end{aligned}$$

**4.2.3** Example. Evaluate the following *double* sum

$$\sum_{i=1}^{3}\sum_{j=1}^{4} ij.$$

Solution : We have

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j=1}^{4} ij &= \sum_{i=1}^{3}(i+2i+3i+4i)\\ &= \sum_{i=1}^{3} 10i\\ &= 10+20+30\\ &= 60.\end{aligned}$$

**Some useful identities (formulas)**

**4.2.4** Example. Find explicit formulas for the sums

$$S_n^{(k)} = \sum_{i=1}^{n} i^k \quad \text{for } \ k = 1, 2.$$

Solution : We calculate first

$$S_n^{(1)} = \sum_{i=1}^{n} i = 1 + 2 + \cdots + n.$$

We write the sum in two ways

$$\begin{aligned}S_n^{(1)} &= 1 + 2 + \cdots + (n-1) + n\\ S_n^{(1)} &= n + (n-1) + \cdots + 2 + 1.\end{aligned}$$

On adding, we see that each pair of numbers in the same column yields the sum $n+1$ and since there are $n$ columns in all, it follows that

$$2S_n^{(1)} = n(n+1),$$

and hence

$$S_n^{(1)} = \frac{n(n+1)}{2}\,\cdot$$

We have obtained the formula (the sum of the first $n$ natural numbers) :

$$\boxed{1 + 2 + \cdots + n = \frac{n(n+1)}{2} \cdot}$$

Next, we calculate

$$S_n^{(2)} = \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2.$$

Consider the identity

$$(i+1)^3 = i^3 + 3i^2 + 3i + 1.$$

By making $i = n, n-1, n-2, \ldots, 2, 1$ we get

$$\begin{aligned}
(n+1)^3 &= n^3 + 3n^2 + 3n + 1 \\
n^3 &= (n-1)^3 + 3(n-1)^2 + 3(n-1) + 1 \\
(n-1)^3 &= (n-2)^3 + 3(n-2)^2 + 3(n-2) + 1 \\
&\vdots \\
3^3 &= 2^3 + 3 \cdot 2^2 + 3 \cdot 2 + 1 \\
2^3 &= 1^3 + 3 \cdot 1^2 + 3 \cdot 1 + 1.
\end{aligned}$$

Adding up, we have

$$(n+1)^3 + S_n^{(3)} - 1 = S_n^{(3)} + 3S_n^{(2)} + 3S_n^{(1)} + n$$

or

$$3S_n^{(2)} = (n+1)^3 - (n+1) - \frac{3n(n+1)}{2}$$

and hence

$$\begin{aligned}
S_n^{(2)} &= \frac{(n+1)\left[2(n+1)^2 - 2 - 3n\right]}{6} \\
&= \frac{(n+1)(2n^2+n)}{6} \\
&= \frac{n(n+1)(2n+1)}{6} \cdot
\end{aligned}$$

We have obtained the formula (the sum of the first $n$ squares) :

$$\boxed{1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \cdot}$$

NOTE : Similarly, using the identity

$$(i+1)^4 = i^4 + 4i^3 + 6i^2 + 4i + 1\,,$$

we can derive an explicit formula for $S_n^{(3)}$ ; that is, the formula (the sum of the first $n$ cubes) :

$$\boxed{1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} \cdot}$$

**4.2.5** EXAMPLE. Find an explicit formula for

$$A = a + (a+r) + (a+2r) + \cdots + (a+nr) = \sum_{i=0}^{n}(a+ir)$$

(the sum of the first $n+1$ terms of a **arithmetic progression** with initial term $a$ and *ratio* $r$ ).

SOLUTION : We have

$$\begin{aligned} A &= \sum_{i=0}^{n}(a+ir) \\ &= \sum_{i=0}^{n} a + r\sum_{i=0}^{n} i \\ &= (n+1)a + r\frac{n(n+1)}{2} \\ &= \frac{(n+1)(2a+rn)}{2} \\ &= \frac{n+1}{2}\left[a + (a+rn)\right]. \end{aligned}$$

Thus

$$\boxed{a + (a+r) + (a+2r) + \cdots + (a+nr) = \frac{n+1}{2}\left[a + (a+rn)\right].}$$

**4.2.6** Example. Find an explicit formula for

$$G = a + ar + ar^2 + \cdots + ar^n = \sum_{i=0}^{n} ar^i, \quad r \neq 1$$

(the sum of the first $n+1$ terms of a **geometric progression** with initial term $a$ and *ratio* $r$).

Solution : We have

$$\begin{aligned} rG &= r\sum_{i=0}^{n} ar^i \\ &= \sum_{i=0}^{n} ar^{i+1} \\ &= \sum_{k=1}^{n+1} ar^k \\ &= \sum_{k=0}^{n} ar^k + (ar^{n+1} - a) \\ &= G + (ar^{n+1} - a). \end{aligned}$$

Thus

$$rG = G + (ar^{n+1} - a),$$

and by solving for $G$ we get

$$\boxed{a + ar + ar^2 + \cdots + ar^n = \frac{a(r^{n+1} - 1)}{r - 1}, \quad r \neq 1.}$$

## 4.3 Mathematical induction

Mathematical induction is an important proof technique that can be used to prove statements of the form $\forall n\ P(n)$, where the universe of discourse (of the predicate $P(n)$) is the set of natural numbers. It is based on a principle, called the **principle of mathematical induction**.

**Principle of Mathematical Induction :** *Let $S$ be a subset of $\mathbb{N}$ such that*

- $0 \in S$ *;*
- *for all $n$, if $n \in S$ then $n+1 \in S$.*

*Then* $S = \mathbb{N}$.

To visualize the idea of mathematical induction, imagine a collection of dominoes positioned one behind the other in such a way that if any given domino falls backward, it makes the one behind him fall backward also. Then imagine that the first domino falls backward. What happens ? ... They all fall down !

NOTE : Strictly speaking, the validity of the principle of mathematical induction is an *axiom*. This is why it is referred to as the *principle* of mathematical induction rather than as a theorem.

Let $P(n)$ be a predicate whose universe of discourse is $\mathbb{N}$, and let $S$ be the *truth set* of $P(n)$, that is $S := \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$. Based on the principle of mathematical induction, a **proof by mathematical induction** that $P(n)$ is true for every natural number $n$ (that is, $S = \mathbb{N}$) consists of two steps :

1. *Basis step.* The property $P(0)$ is shown to be true.
2. *Inductive step.* The implication $P(n) \to P(n+1)$ is shown to be true for every $n \in \mathbb{N}$.

When we complete both steps of a proof by mathematical induction, we have proved that $P(n)$ is true for all natural numbers $n$; that is, we have shown that the proposition $\forall n\, P(n)$ is true.

NOTE : (1) This proof technique is based on the tautology

$$P(0) \wedge \forall n\, (P(n) \to P(n+1)) \to \forall n\, P(n).$$

(2) To prove the implication $P(n) \to P(n+1)$ is true for every $n \in \mathbb{N}$, we need to show that $P(n+1)$ *cannot* be false when $P(n)$ is true ; this can be accomplished by assuming that $P(n)$ is true and showing that *under this premise* $P(n+1)$ must also be true.

(3) The principle of mathematical induction is equally valid if, instead of starting with 0, we (1) start with a given natural number $a$, (2) show that $a \in S$, and (3) show that, if $a \in S$ and $n \geq a$, then $n+1 \in S$. When we do this we will know that every natural number greater than or equal to $a$ belongs to the set $S$.

**4.3.1** EXAMPLE. Use mathematical induction to prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for each positive integer $n$.

SOLUTION : Let $P(n)$ be the predicate

$$\text{“}\, 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \,\text{”}.$$

We shall prove by induction that the proposition $\forall n \; P(n)$ is true.

BASIS STEP : $P(1)$ is true, since $1 = \frac{1(1+1)}{2}$ ·

INDUCTIVE STEP : Assume that $P(n)$ is true. That is, assume that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \cdot$$

Under this assumption, we must show that $P(n+1)$ is true, namely, that

$$1 + 2 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2} \cdot$$

Adding $n+1$ to both sides of the equality in $P(n)$, it follows that

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + n + 1 \\ &= (n+1)\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2} \cdot \end{aligned}$$

The last equality shows that $P(n+1)$ is true.
This completes the proof by induction.

**4.3.2** Example. Use mathematical induction to prove that, for each natural number $n \geq 5$, $n^2 < 2^n$.

Solution : Let $P(n)$ denote the predicate "$n^2 < 2^n$", where the universe of discourse is the set $\{n \in \mathbb{N} \mid n \geq 5\}$. We shall prove by induction that the proposition $\forall n\, P(n)$ is true.

BASIS STEP : $P(5)$ is true, since $5^2 < 2^5$.

INDUCTIVE STEP : Assume that $P(n)$ is true. That is, assume that

$$n^2 < 2^n\,.$$

Under this assumption, we want to show that

$$(n+1)^2 < 2^{n+1}.$$

Now

$$(n+1)^2 = n^2 + 2n + 1 \quad \text{and} \quad 2^{n+1} = 2 \cdot 2^n$$

so we want to show that

$$n^2 + 2n + 1 < 2 \cdot 2^n.$$

Since $n^2 < 2^n$, $2n^2 < 2 \cdot 2^n$. Hence, it is sufficient to show that

$$n^2 + 2n + 1 < 2n^2.$$

But this inequality is equivalent to

$$1 < n(n-2)$$

which is obviously true, since $n \geq 5$. This completes the proof by induction.

**4.3.3** EXAMPLE. Prove by induction that, for each positive integer $n$, $7^n - 3^n$ is divisible by $4$.

SOLUTION : Let $P(n)$ denote the predicate "$7^n - 3^n$ is divisible by $4$", where the universe of discourse is the set of positive integers. We shall prove by induction that the proposition $\forall n\, P(n)$ is true.

BASIS STEP : $P(1)$ is true, since $7 - 3$ is divisible by $4$.

INDUCTIVE STEP : We assume that

$$7^n - 3^n \quad \text{is divisible by } 4$$

and want to show that

$$7^{n+1} - 3^{n+1} \quad \text{is divisible by } 4.$$

We write

$$\begin{aligned} 7^{n+1} - 3^{n+1} &= 7 \cdot 7^n - 3 \cdot 3^n \\ &= 7 \cdot 7^n - 7 \cdot 3^n + 7 \cdot 3^n - 3 \cdot 3^n \\ &= 7(7^n - 3^n) + 4 \cdot 3^n. \end{aligned}$$

Since $7(7^n - 3^n)$ and $4 \cdot 3^n$ are divisible by $4$ (why ?), $7^{n+1} - 3^{n+1}$ is divisible by $4$. This completes the proof by induction.

**4.3.4** EXAMPLE. Define a sequence $(a_n)_{n \geq 1}$ as follows :

$$a_1 = 2 \quad \text{and} \quad a_n = 5a_{n-1} \text{ for all } n \geq 1.$$

1. Write the first four terms of the sequence.

2. Use mathematical induction to show that the terms of the sequence satisfy the formula

$$a_n = 2 \cdot 5^{n-1} \quad \text{for all } n \geq 1.$$

Solution : We have

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 5a_1 = 5 \cdot 2 = 10 \\ a_3 &= 5a_2 = 5 \cdot 10 = 50 \\ a_4 &= 5a_3 = 5 \cdot 50 = 250. \end{aligned}$$

Let $P(n)$ denote the predicate " $a_n = 2 \cdot 5^{n-1}$ ", where the universe of discourse is the set of positive integers. We shall prove by induction that the proposition $\forall n \, P(n)$ is true.

BASIS STEP : $P(1)$ is true, since $a_1 = 2 \cdot 5^0 = 2$.

INDUCTIVE STEP : We assume that

$$a_n = 2 \cdot 5^{n-1}.$$

Under this assumption, we must show that

$$a_{n+1} = 2 \cdot 5^n.$$

We write

$$\begin{aligned} a_{n+1} &= 5a_n \\ &= 5 \cdot (2 \cdot 5^{n-1}) \\ &= 2 \cdot (5 \cdot 5^{n-1}) \\ &= 2 \cdot 5^n. \end{aligned}$$

This is what was to be shown. Since we have proved the basis and inductive steps, we conclude the formula holds for all terms of the sequence.

**4.3.5** EXAMPLE. Observe that

$$\begin{aligned}
1 &= 1, \\
1-4 &= -(1+2), \\
1-4+9 &= 1+2+3, \\
1-4+9-16 &= -(1+2+3+4), \\
1-4+9-16+25 &= 1+2+3+4+5.
\end{aligned}$$

Guess a general formula and prove it by induction.

SOLUTION : General formula is

$$1-4+9-16+\cdots+(-1)^{n-1}n^2 = (-1)^{n-1}(1+2+\cdots+n)$$

(in expanded form) or

$$\sum_{i=1}^{n}(-1)^{i-1}i^2 = (-1)^{n-1}\left(\sum_{i=1}^{n} i\right)$$

(in closed form). We shall prove (by mathematical induction) that this formula is true for all positive integers $n$.

BASIS STEP : The formula is true for $n = 1$ : $1 = (-1)^0 \cdot 1$.

INDUCTIVE STEP : Assume that the formula is true for some $n$; that is, assume that

$$1-4+9-16+\cdots+(-1)^{n-1}n^2 = (-1)^{n-1}(1+2+\cdots+n).$$

We write

$$\begin{aligned}
1-4+\cdots+(-1)^n(n+1)^2 &= \left(1-4+\cdots+(-1)^{n-1}n^2\right)+(-1)^n(n+1)^2 \\
&= (-1)^{n-1}(1+2+\cdots+n)+(-1)^n(n+1)^2 \\
&= (-1)^{n-1}\frac{n(n+1)}{2}+(-1)^n(n+1)^2 \\
&= (-1)^n\frac{n+1}{2}\left[-n+2(n+1)\right] \\
&= (-1)^n\frac{(n+1)(n+2)}{2} \\
&= (-1)^n\left(1+2+\cdots+n+(n+1)\right).
\end{aligned}$$

The last equality shows that the formula is true for $n+1$. This completes the proof and we are done.

### Second principle of mathematical induction

There is another form of mathematical induction that is often used in proofs. It is based on what is called the **second principle of mathematical induction**.

**Second Principle of Mathematical Induction :** *Let $S$ be a subset of $\mathbb{N}$ such that*

- $0 \in S$ *;*
- $\forall n,$ *if* $\{0, 1, \dots, n\} \subseteq S$ *then* $n+1 \in S$.

*Then* $S = \mathbb{N}$.

The corresponding proof by mathematical induction (of the proposition $\forall n\, P(n)$ ) consists of

1. *Basis step.* The proposition $P(0)$ is shown to be true.
2. *Inductive step.* It is shown that the implication $P(0) \wedge P(1) \wedge \cdots \wedge P(n) \to P(n+1)$ is true for every natural number $n$.

Note : (1) To prove that the implication

$$P(0) \wedge P(1) \wedge \cdots \wedge P(n) \to P(n+1)$$

is true for every $n \in \mathbb{N}$, we need to show that $P(n+1)$ *cannot* be false when $P(0)$, $P(1)$, $\dots$, $P(n)$ are all true; this can be accomplished by assuming that $P(0)$, $P(1)$, $\dots$, $P(n)$ are true and showing that *under these premises* $P(n+1)$ must also be true.

(2) Just as with the principle of mathematical induction, the second principle of mathematical induction is equally valid if, instead of starting with $0$, we (1) start with

a given $a$, (2) show that $a \in S$, and (3) show that, if $n \geq a$ and $\{a, a+1, \ldots, n\} \subseteq S$, then $n+1 \in S$. Again, when we do this, we will know that every natural number greater than or equal to $a$ belongs to the set $S$.

**4.3.6** EXAMPLE. Prove by induction that, if $n \in \mathbb{N}$ and $n \geq 4$, then $n$ can be written as a sum of numbers each of which is a $2$ or a $5$.

SOLUTION : Let $P(n)$ be the predicate "$n$ can be written as a sum of 2s and 5s", where the universe of discourse is the set $\{n \in \mathbb{N} \,|\, n \geq 4\}$. We shall prove by induction that the proposition $\forall n\, P(n)$ is true.

BASIS STEP : $P(4)$ is true, since $4 = 2 + 2$.

INDUCTIVE STEP : Assume that $n \geq 4$ and $4, 5, \ldots, n$ can all be written as a sum of 2s and 5s. Then $n-1$ can be written as a sum of 2s and 5s; that is,

$$n - 1 = a_1 + a_2 + \cdots + a_p, \quad a_i \in \{2, 5\}.$$

So,

$$n + 1 = a_1 + a_2 + \cdots + a_p + 2$$

and, therefore, $n+1$ can be written as a sum of $2$s and $5$s. This completes the proof by induction.

**4.3.7** EXAMPLE. Define a sequence $(b_n)_{n \geq 1}$ as follows :

$$b_1 = 0,\ b_2 = 2 \quad \text{and} \quad b_n = 3 \cdot b_{\lfloor k/2 \rfloor} + 2 \ \text{ for all } n \geq 3.$$

1. Write the first seven terms of the sequence.

2. Use mathematical induction to show that $b_n$ is even for all $n \geq 1$.

SOLUTION : We have

$$\begin{aligned}
b_1 &= 0\\
b_2 &= 2\\
b_3 &= 3\cdot b_{\lfloor 3/2\rfloor}+2 = 3\cdot b_1+2 = 3\cdot 0+2 = 2\\
b_4 &= 3\cdot b_{\lfloor 4/2\rfloor}+2 = 3\cdot b_2+2 = 3\cdot 2+2 = 8\\
b_5 &= 3\cdot b_{\lfloor 5/2\rfloor}+2 = 3\cdot b_2+2 = 3\cdot 2+2 = 8\\
b_6 &= 3\cdot b_{\lfloor 6/2\rfloor}+2 = 3\cdot b_3+2 = 3\cdot 2+2 = 8\\
b_6 &= 3\cdot b_{\lfloor 7/2\rfloor}+2 = 3\cdot b_3+2 = 3\cdot 2+2 = 8.
\end{aligned}$$

Let $P(n)$ be the property (predicate) "$b_n$ is even". We shall use mathematical induction (based on the second principle) to show that this property holds for all positive integers (i.e. the proposition $\forall n\, P(n)$ is true).

BASIS STEP : The property holds for $n = 1$ : $b_1 = 0$ is even.

INDUCTIVE STEP : Assume that the property holds for all $1 \leq k \leq n$. We need to show that the property then holds for $n+1$.

The number $b_{\lfloor (n+1)/2\rfloor}$ is even by assumption (inductive hypothesis), since

$$1 \leq \left\lfloor \frac{n+1}{2} \right\rfloor \leq n.$$

Thus $3 \cdot b_{\lfloor (n+1)/2\rfloor}$ is even (because $\mathit{odd} \cdot \mathit{even} = \mathit{even}$), and hence $3 \cdot b_{\lfloor (n+1)/2\rfloor} + 2$ is even (because $\mathit{even} + \mathit{even} = \mathit{even}$). Consequently, $b_{n+1}$ – which equals $b_{\lfloor (n+1)/2\rfloor} + 2$ – is even, as was to be shown.

We conclude that the statement is true.

We now ask a natural question : *"What is the relationship between the second principle of mathematical induction and the principle of mathematical induction ?"* It is clear that the second principle of mathematical induction logically implies the principle of mathematical induction. Indeed, if we are allowed to assume that $\{0, 1, \ldots, n\} \subseteq S$, then we are surely allowed to assume that $n \in S$.

In fact, it is also true that the principle of mathematical induction logically implies the second principle. So, *the two principles of mathematical induction are logically equivalent.*

NOTE : There is nothing wrong with guessing in mathematics, and presumably most mathematical results are first arrived at intuitively and only later established by proofs as theorems. There is a natural temptation, however, to make a plausible guess and then allow that guess to stand unproved.

## 4.4 Exercises

**Exercise 46** Find the value of each of the following sums :

(a) $\displaystyle\sum_{i=0}^{10} 3 \cdot 2^i$.

(b) $\displaystyle\sum_{i=0}^{10} 2 \cdot (-3)^i$.

(c) $\displaystyle\sum_{i=0}^{10} (3^i - 2^i)$.

(d) $\displaystyle\sum_{i=0}^{10} (2 \cdot 3^i + 3 \cdot 2^i)$.

(e) $\displaystyle\sum_{i=1}^{3}\sum_{j=1}^{4} (2i + 3j)$.

**Exercise 47** Use the formula for the sum of the first $n$ natural numbers and/or the formula for the sum of a geometric sequence to find the following sums :

(a) $3 + 4 + 5 + \cdots + 1\,000$.

(b) $5 + 10 + 15 + \cdots + 300$.

(c) $2 + 3 + \cdots + (k - 1)$.

(d) $1 + 2 + 2^2 + \cdots + 2^{25}$.

(e) $3 + 3^2 + 3^3 + \cdots + 3^n$.

(f) $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$.

(g) $1 - 2 + 2^2 - 2^3 + \cdots + (-1)^n 2^n$.

**Exercise 48**

(a) Let $(a_n)_{n\geq 1}$ be a sequence (of numbers). Verify that

$$\sum_{k=1}^{n}(a_k - a_{k+1}) = a_1 - a_{n+1}.$$

(b) Use identity

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

and part (a) to compute the sum

$$\sum_{k=1}^{n}\frac{1}{k(k+1)}\,.$$

(c) Use identity

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2}\frac{1}{k} - \frac{1}{k+1} + \frac{1}{2}\frac{1}{k+2}$$

to compute the sum

$$\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}\,.$$

(d) Evaluate the sum

$$\sum_{k=1}^{n}\frac{k}{(k+1)!}\,.$$

**Exercise 49** Prove by induction that (for each positive integer $n$)

(a) $5 + 7 + 9 + \cdots + (2n+3) = n(n+4)$.

(b) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}\,.$

(c) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}\,.$

(d) $1\cdot 2 + 2\cdot 3 + \cdots + n\cdot(n+1) = \dfrac{n(n+1)(n+2)}{3}\,.$

(e) $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \cdots + \dfrac{1}{(2n-1)\cdot(2n+1)} = \dfrac{n}{2n+1}\,.$

**Exercise 50** Prove by induction that

(a) $n^3 + 1 \geq n^2 + n\,, \quad n \in \mathbb{N}$.

(b) $n\,! > 2^n\,, \quad n \geq 4$.

(c) $1+\dfrac{1}{4}+\dfrac{1}{9}+\cdots+\dfrac{1}{n^2}<2-\dfrac{1}{n}\,,\quad n\geq 2.$

(d) $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}\,,\quad n\geq 2.$

**Exercise 51** Prove by induction that (for $n \geq 1$)

(a) $n(n+5)$ is divisible by 2.

(b) $n^3-n$ is divisible by 6.

(c) $4^n-1$ is divisible by 3.

(d) $2^{2n-1}+3^{2n-1}$ is divisible by 5.

(e) $n(n+1)(n+2)$ is divisible by 6.

**Exercise 52** A sequence $(a_n)_{n\geq 0}$ is defined by

$$a_0 = 3 \quad \text{and} \quad a_n = a_{n-1}^2 \quad \text{for all } n \geq 1.$$

Show that $a_n = 3^{2^n}$ for all $n \geq 0$.

**Exercise 53** A sequence $(b_n)_{n\geq 1}$ is defined by

$$b_1 = 1 \quad \text{and} \quad b_n = \sqrt{3b_{n-1}+1} \quad \text{for all } n \geq 2.$$

Show that $b_n < \frac{7}{2}$ for all $n \geq 1$.

**Exercise 54** A sequence $(c_n)_{n\geq 1}$ is defined by

$$c_1 = 1 \quad \text{and} \quad c_n = 2\cdot c_{\lfloor n/2\rfloor} \quad \text{for all } n \geq 2.$$

Show that $c_n \leq n$ for all $n \geq 1$.

**Exercise 55** A sequence $(d_n)_{n\geq 0}$ is defined by

$$d_0 = 12,\ d_1 = 29 \quad \text{and} \quad d_n = 5d_{n-1} - 6d_{n-2} \quad \text{for all } n \geq 2.$$

Show that $d_n = 5\cdot 3^n + 7\cdot 2^n$ for all $n \geq 0$.

**Exercise 56** Find the following sums :

(a) $3+6+9+\cdots+3n$.

(b) $1+3+5+\cdots+(2n-1)$.

(c) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}\,\cdot$

(d) $1 + 5 + 9 + 13 + \cdots$ to $n$ terms.

(e) $4 \cdot 7 + 7 \cdot 10 + 10 \cdot 13 + \cdots$ to $n$ terms.

Use mathematical induction to verify your answers.

**Exercise 57** Prove that :

(a) Every positive integer other than $1$ is either a prime number or the product of prime numbers.

(b) Every natural number $n \geq 14$ can be written as a sum of numbers, each of which is a $3$ or an $8$.

**Exercise 58** Prove that :

(a) For each *odd* natural number $n \geq 3$,

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{(-1)^n}{n}\right) = 1.$$

(b) For each *even* natural number,

$$\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{(-1)^n}{n}\right) = \frac{1}{2}\,\cdot$$

**Exercise 59** Let $a > -1$. Prove by induction that

$$(1+a)^n \geq 1 + na$$

for every $n \in \mathbb{N}$.

**Exercise 60** Let $a, b \geq 0$. Prove by induction that

$$\left(\frac{a+b}{2}\right)^n \leq \frac{a^n + b^n}{2}$$

for every $n \in \mathbb{N}$.