## Chapter 5

## Counting

## Topics :

1. Basic counting principles

## 2. Permutations and combinations

## 3. Binomial formula

Counting objects is necessary in order to solve many different types of problems. Furthermore, counting can provide mathematical insight. For example, counting can determine the proportion between the number of elements of a set that have a property and the number that do not; the study of discrete probability is founded on such proportions.

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### 5.1 Basic counting principles

A list is a finite sequence of objects. The order in which elements appear in a list is significant. For example, the list $(1,2,3)$ is not the same as the list $(3,2,1)$. Elements in a list might be repeated, as in $(1,1,2)$. The number of elements (or terms) in a list is called its length. For example, the list $(1,1,2,1)$ is a list of length four (or a 4 -list). A list of length two has a special name; it is called an ordered pair. A list of length zero is called the empty list.

Note: An $n$-list is also called an $n$-tuple or an $n$-string (see section 4.1). The 0 -string may be referred to as the null string.

Some counting problems are as simple as counting the elements of a list. For example, how many integers are there from 5 to 20 ? To answer this question, imagine going along the list of integers from 5 to 20 , counting each in turn. So the answer is 16 .

More generally, if $m$ and $n$ are integers and $m \leq n$, then there are $n-m+1$ integers from $m$ to $n$ inclusive.
5.1.1 Example. How many 3-digit integers (integers from 100 to 999 inclusive) are divisible by 5 ?

Solution : Imagine writing the 3 -digit integers in a row, noting those that are multiple of 5 :

$$
\begin{array}{|llllll|llllll}
\hline 100 & 101 & \cdots & 105 & 106 & \cdots & \boxed{110} & 111 & \cdots & 995 & 996 & \cdots
\end{array} 999 .
$$

Since $100=5 \cdot \mathbf{2 0}, 105=5 \cdot \mathbf{2 1}, \cdots, 995=5 \cdot \mathbf{1 9 9}$, it follows that there are as many 3 -digit integers that are multiples of 5 as there are integers from 20 to 199 inclusive. Hence, there are $199-20+1=180$ integers that are divisible by 5 .

Note: We have $180=\left\lceil\frac{999-100}{5}\right\rceil$ (see Exercise 29). So $\left\lceil\frac{n-m}{k}\right\rceil$ represents the number of integers from $m$ to $n$ inclusive which are divisible by $k ; k \leq m \leq n$.

We discuss now three basic counting principles, namely the addition rule, the multiplication rule, and the inclusion-exclusion principle.

## Addition rule

Suppose that a procedure can be performed by two different tasks. If there are $n_{1}$ ways to do the first task and $n_{2}$ ways to do the second task, and these tasks cannot be done at the same time, then there are $n_{1}+n_{2}$ ways to do the procedure. This basic principle is known as the addition rule.
5.1.2 Example. A representative to a university committee is to be chosen from the Mathematics Department or the Computer Science Department. How many different choices are there for this representative if there are 8 members of the Mathematics Department and 10 members of the Computer Science Department, and no faculty member belongs to both departments?

Solution : The task of choosing a representative from the Mathematics Department has 8 possible outcomes, and the task of choosing a representative from the Computer Science Department has 10 possible outcomes. By the addition rule, there are $8+10=18$ possible choices for the representative.

The addition rule can be stated in terms of sets.
The Addition Rule (for two sets) : If $A$ and $B$ are disjoint finite sets, then the number of elements in $A \cup B$ is the sum of the number of elements in $A$ and the number of elements in $B$; that is,

$$
|A \cup B|=|A|+|B| .
$$

To relate this to our statement of the addition rule, let $T_{1}$ be the task of choosing an element from $A$, and $T_{2}$ the task of choosing an element of $B$. There are $|A|$ ways to do $T_{1}$, and $|B|$ ways to do $T_{2}$. Since these tasks cannot be done at the same time, the number of ways to choose an element from one of the sets, which is the number of elements in the union, is $|A|+|B|$.

An important consequence of the addition rule is the fact that if the number of elements in a set $A$ and in a subset $B$ of $A$ are both known, then the number of elements that are in $A$ and not in $B$ can be computed.
5.1.3 Proposition. If $A$ is a finite set and $B \subseteq A$, then

$$
|A \backslash B|=|A|-|B| .
$$

Solution: If $B$ is a subset of $A$, then $B \cup(A \backslash B)=A$ and the two sets $B$ and $A \backslash B$ have no elements in common. Hence by the addition rule,

$$
|B|+|A \backslash B|=|A| .
$$

Substracting $|B|$ from both sides gives the relation

$$
|A \backslash B|=|A|-|B| .
$$

5.1.4 Example. How many 3 -digit integers are not divisible by 5 ?

Solution : There are $999-100+1=9003$-digit numbers. Among these numbers, 180 are divisible by 5 . Hence there are $900-180=720$ integers that are not divisible by 5 .

The addition rule can be extended to more than two sets.
The Addition Rule (for $n$ sets): If $A_{1}, A_{2}, \ldots, A_{n}$ are pairwise disjoint finite sets, then

$$
\left|A_{1} \cup A_{2} \cup \cdots \cup A_{n}\right|=\left|A_{1}\right|+\left|A_{2}\right|+\cdots+\left|A_{n}\right| .
$$

5.1.5 Example. Let $A=\{2,4,6\}, B=\{1,3,5,7\}$ and $C=\{8,9\}$. Find $A \cup B \cup C$ and verify the addition rule with $A, B$ and $C$.

Solution: We have $A \cup B \cup C=\{1,2,3,4,5,6,7,8,9\}$ and so

$$
|A \cup B \cup C|=9=3+4+2=|A|+|B|+|C| .
$$

## Multiplication rule

Suppose that a procedure can be performed by two different tasks. If there are $n_{1}$ ways to do the first task and $n_{2}$ ways to do the second task after the first task has been done, then there are $n_{1} \cdot n_{2}$ ways to do the procedure. This counting principle is known as the multiplication rule.
5.1.6 Example. The riders in a bycicle race are identified by a single letter and a single-digit number. If all riders are to be labeled differently, how many riders are permitted in the race ?

Solution : The procedure of labeling the riders consists of two tasks : placing a single letter on each rider and then placing a single-digit number on each rider. There are 26 ways of performing the first task and 10 ways of performing the second task, so the number of riders permitted in the race is $26 \cdot 10=260$.

The multiplication rule can also be stated in terms of sets.
The Multiplication Rule (for two sets) : If $A$ and $B$ are finite sets, then the number of elements of the Cartesian product $A \times B$ is the product of the number of elements in $A$ and the number of elements in $B$; that is,

$$
|A \times B|=|A| \cdot|B| .
$$

Again, one can relate this to our statement of the multiplication rule. Let $T_{1}$ be the task of choosing an element from $A$, and $T_{2}$ the task of choosing an element of $B$. There are $|A|$ ways to do $T_{1}$, and $|B|$ ways to do $T_{2}$. We note that the task of choosing an element in the Cartesian product $A \times B$ is done by choosing an element in $A$ and an element in $B$. The number of ways to do this, which is the number of elements of the Cartesian product, is $|A| \cdot|B|$.

The multiplication rule can be extended to more than two sets.

The Multiplication Rule (for $n$ sets): If $A_{1}, A_{2}, \ldots, A_{n}$ are finite sets, then

$$
\left|A_{1} \times A_{2} \times \cdots \times A_{n}\right|=\left|A_{1}\right| \cdot\left|A_{2}\right| \cdots\left|A_{n}\right| .
$$

5.1.7 Example. Let $A=\{1,2\}$ and $B=\{1,2, \alpha\}$. Find $A \times B$ and verify the product rule with $A$ and $B$.

Solution : We have $A \times B=\{(1,1),(1,2),(1, \alpha),(2,1),(2,2),(2, \alpha)\}$ and so

$$
|A \times B|=6=2 \cdot 3=|A| \cdot|B| .
$$

5.1.8 Example. Write all bit strings of length 4.

Solution : There are 16 bit strings of length 4:
0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111.
5.1.9 Example. How many different bit strings of length 8 are there?

Solution : Each of the eight bits can be chosen in two ways, and so the multiplication rule shows that there are $2^{8}=256$ different bit strings of length 8.
5.1.10 Example. How many functions are there from a set with $m$ elements to a set with $n$ elements ?

Solution : A function corresponds to a choice of one of the $n$ elements in the codomain for each of the $m$ elements in the domain. Hence, by the multiplication rule, there are $n \cdot n \cdots n=n^{m}$ functions from a set with $m$ elements to a set with $n$ elements.

Note : If $A$ and $B$ are finite sets such that $|A|=m$ and $|B|=n$, then

$$
\left|B^{A}\right|=n^{m}=|B|^{|A|} .
$$

5.1.11 Example. How many one-to-one functions are there from a set with $m$ elements to a set with $n$ elements ?

Solution : We note that for $m>n$ there are no one-to-one functions from a set with $m$ elements to a set with $n$ elements. Let $m \leq n$. Suppose the elements in the domain are $a_{1}, a_{2}, \ldots, a_{m}$. There are $n$ ways to choose the value of the function at $a_{1}$. Since the function is one-to-one, the value of the function at $a_{2}$ can be chosen in $n-1$ ways (since the value used for $a_{1}$ cannot be used again). In general, the value of the function at $a_{k}$ can be chosen in $n-k+1$ ways. By the multiplication rule, there are $n(n-1)(n-$ 2) $\cdots(n-m+1)$ one-to-one functions from a set with $m$ elements to a set with $n$ elements.

Note: If $A$ is a finite set such that $|A|=n$, then

$$
\left|S_{A}\right|=\left|S_{n}\right|=n(n-1) \cdots 2 \cdot 1=n!.
$$

So the symmetric group $S_{n}$ has $n$ ! elements.

## Inclusion-exclusion principle

Suppose that a procedure can be performed by two different tasks, which can be done at the same time. If there are $n_{1}$ ways to do the first task, $n_{2}$ ways to do the second task, and $n_{12}$ ways to do both tasks, then there are $n_{1}+n_{2}-n_{12}$ ways to do the procedure. This principle is known as the principle of inclusion-exclusion.
5.1.12 Example. How many integers from 1 to 1000 are multiples of 3 or multiples of 5 ?

Solution : The procedure of choosing such an integer can be performed by two different tasks : choosing a multiple of 3 and choosing a multiple of 5 . The first task has 333 possible outcomes, the second task has 200 possible outcomes, and there are 66 ways to do both tasks (since there are 66 multiples of 15 from 1 to 1000 ). By the principle of inclusion-exclusion,
there are $333+200-66=467$ integers from 1 through 1000 that are multiples of 3 or multiples of 5 .

Once again, we can state this counting principle in terms of sets.
The Inclusion-Exclusion Principle (for two sets): If $A$ and $B$ are finite sets, then

$$
|A \cup B|=|A|+|B|-|A \cap B| .
$$

Let $T_{1}$ be the task of choosing an element from $A$ and $T_{2}$ the task of choosing an element from $B$. There are $|A|$ ways to do $T_{1}$ and $|B|$ ways to do $T_{2}$. The number of ways to do either $T_{1}$ or $T_{2}$ is the sum $|A|+|B|$ minus the number of ways to do both $T_{1}$ and $T_{2}$. Since there are $|A \cup B|$ ways to do either $T_{1}$ or $T_{2}$, and $|A \cap B|$ ways to do both $T_{1}$ and $T_{2}$, we have

$$
|A \cup B|=|A|+|B|-|A \cap B| .
$$

5.1.13 Proposition. Let $A, B$, and $C$ be finite sets. Then

$$
|A \cup B \cup C|=|A|+|B|+|C|-|A \cap B|-|B \cap C|-|C \cap A|+|A \cap B \cap C| .
$$

Proof: We have

$$
\begin{aligned}
|A \cup B \cup C| & =\mid(A \cup B) \cup C) \mid \\
& =|A \cup B|+|C|-|(A \cup B) \cap C| \\
& =|A|+|B|-|A \cap B|+|C|-|(A \cap C) \cup(B \cap C)| \\
& =|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C| .
\end{aligned}
$$

5.1.14 Example. A car manufacturer finds that the most common production defects are faulty brakes and broken headlights. In testing a sample of 80 cars, the manufacturer recorded the following data : 20 cars have faulty brakes, 15 cars have broken headlights, and 10 cars have both defects. How many cars in the sample have at least one of these defects ?

Solution : Let $B$ be the set of cars in this sample with faulty brakes, and let $H$ be the set of cars in this sample with broken headlights. Using the principle of inclusion-exclusion, we have :

$$
|B \cup H|=|B|+|H|-|B \cap H|=20+15-10=25 .
$$

There are 25 cars in this sample with at least one of theses two defects.
The principle of inclusion-exclusion can be extended to more than three sets.

The Inclusion-Exclusion Principle (for $n$ sets): If $A_{1}, A_{2}, \ldots, A_{n}$ are finite sets, then

$$
\begin{aligned}
\left|A_{1} \cup A_{2} \cup \cdots \cup A_{n}\right|= & \sum_{i=1}^{n}\left|A_{i}\right|-\sum_{\substack{i, j=1 \\
i<j}}^{n}\left|A_{i} \cap A_{j}\right|+\sum_{\substack{i, j, k=1 \\
i<j<k}}^{n}\left|A_{i} \cap A_{j} \cap A_{k}\right|-\cdots \\
& +(-1)^{n-1}\left|A_{1} \cap A_{2} \cap \cdots \cap A_{n}\right| .
\end{aligned}
$$

5.1.15 Example. How many onto function are there from a set with $m$ elements to a set with $n$ elements?

Solution : We note that for $m<n$ there are no onto functions from a set with $m$ elements to a set with $n$ elements. Let $m \geq n$. Recall that the number of all functions between these given sets is $n^{m}$. Let's call, for convenience, the functions which are not onto "bad". If we can count the number of bad functions, then we are done because

$$
\# \text { onto functions }=n^{m}-\# \text { bad functions. }
$$

Suppose that the elements of the codomain are $b_{1}, b_{2}, \ldots, b_{n}$. Now a function might be bad because its range fails to contain $b_{1}$; or it might be bad because its range fails to contain $b_{2}$, and so on. Let $B_{i}$ denote the set of all functions that fail to contain the element $b_{i}$ in their range. Then the set $\mid B_{1} \cup B_{2} \cup$ $\cdots \cup B_{n} \mid$ contains precisely all the bad functions; what we want to do is to
calculate the size of this union. We have

$$
\begin{aligned}
\left|B_{i}\right| & =(n-1)^{m}, & 1 \leq i \leq n \\
\left|B_{i} \cap B_{j}\right| & =(n-2)^{m}, & 1 \leq i<j \leq n \\
\left|B_{i} \cap B_{j} \cap B_{k}\right| & =(n-3)^{m}, & 1 \leq i<j<k \leq n
\end{aligned}
$$

and so on. The question that remains is how many terms are on each row ? It turns out that the number of all such $l$-fold intersections is

$$
\binom{n}{l}:=\frac{n!}{l!\cdot(n-l)!}, \quad 1 \leq l \leq n .
$$

(See section 5.2 for more explanation and a proof of this statement.) So we have
$\left|B_{1} \cup \cdots \cup B_{n}\right|=\binom{n}{1}(n-1)^{m}-\binom{n}{2}(n-2)^{m}+\cdots+(-1)^{n+1}\binom{n}{n}(n-n)^{m}$
which can be rewritten in closed form as

$$
\left|B_{1} \cup B_{2} \cup \cdots \cup B_{n}\right|=\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}(n-i)^{m} .
$$

Recall that the set $B_{1} \cup B_{2} \cup \cdots \cup B_{n}$ counts the number of bad functions; we want the number of "good" functions (i.e. the onto functions). We get \# onto functions

$$
\begin{aligned}
& =n^{m}-\# \text { bad functions } \\
& =n^{m}-\left|B_{1} \cup B_{2} \cup \cdots \cup B_{n}\right| \\
& =n^{m}-\left[\binom{n}{1}(n-1)^{m}-\binom{n}{2}(n-2)^{m}+\cdots+(-1)^{n+1}\binom{n}{n}(n-n)^{m}\right] \\
& =n^{m}-\binom{n}{1}(n-1)^{m}+\binom{n}{2}(n-2)^{m}-\cdots-(-1)^{n+1}\binom{n}{n}(n-n)^{m} \\
& =\binom{n}{0} n^{m}-\binom{n}{1}(n-1)^{m}+\binom{n}{2}(n-2)^{m}-\cdots+(-1)^{n}\binom{n}{n}(n-n)^{m} \\
& =\sum_{i=0}^{n}(-1)^{i}\binom{n}{i}(n-i)^{m} .
\end{aligned}
$$

Let us collect what we learned about counting functions.
5.1.16 Proposition. Let $A$ and $B$ be finite sets with $|A|=m$ and $|B|=$ $n$.
(1) The number of functions from $A$ to $B$ is $n^{m}$.
(2) If $m \leq n$, the number of one-to-one functions $f: A \rightarrow B$ is

$$
(n)_{m}:=n(n-1)(n-2) \cdots(n-m+1) .
$$

If $m>n$, the number of such functions is zero.
(3) If $m \geq n$, the number of onto functions $f: A \rightarrow B$ is

$$
(n)^{m}:=\sum_{i=0}^{n-1}(-1)^{i}\binom{n}{i}(n-i)^{m} .
$$

If $m<n$, the number of such functions is zero.
Note : For $m=n$ we get

$$
n!=(n)_{n}=(n)^{n}=\sum_{i=0}^{n-1}(-1)^{i}\binom{n}{i}(n-i)^{n} .
$$

### 5.2 Permutations and combinations

We are interested in counting selections of objects of a finite set. We shall consider first the case when the objects are distinct; in other words, there is no repetition allowed.

There are two distinct ways of selecting $r$ (distinct) objects from a set of $n$ elements. In an ordered selection, it is not only what elements are chosen but also the order in which they are chosen that matter. Two ordered selections are said to be the same if the elements chosen are the same and also if the elements are chosen in the same order.

In an unordered selection, on the other hand, it is only the identity of the chosen elements that matters. Two unordered selections are said to be the same if they consists of the same elements, regardless of the order in which the elements are chosen.

## Permutations

5.2.1 Definition. Let $A$ be a set with $n$ elements and let $0 \leq r \leq n$. An ordered selection of $r$ elements of $A$ is called an r-permutation. An $n$-permutation of $A$ is simply called a permutation of $A$.

Note: An $r$-permutation of a set $A$ is simply an $r$-string (or $r$-list) of elements of $A$, when repetition is not allowed.
5.2.2 Example. Consider the set $A=\{a, b, c, d\}$. How many ordered selections of two elements can be made from the set $A$ ?

Solution : There are twelve 2-permutations of the set $A$ :

$$
a b, a c, a d, b a, b c, b d, c a, c b, c d, d a, d b, d c .
$$

The number of $r$-permutations of a set with $n$ elements is denoted by $P(n, r)$. We can find $P(n, r)$ using the multiplication rule.
5.2.3 Proposition. The number of $r$-permutations of a set with $n$ elements is

$$
P(n, r)=n(n-1)(n-2) \cdots(n-r+1), \quad 1 \leq r \leq n .
$$

Proof : The first element of the permutation can be chosen in $n$ ways. There are $n-1$ ways to choose the second element of the permutation, $n-2$ ways to choose the third element of the permutation, and so on, until there
are exactly $n-r+1$ ways to choose the $r^{t h}$ element. Consequently, by the multiplication rule, there are

$$
n(n-1)(n-2) \cdots(n-r+1)
$$

$r$-permutations of the set.

Note : An alternative notation for $n(n-1)(n-2) \cdots(n-r+1)$ is $(n)_{r}$. This notation is called falling factorial.

Equivalently, the number of all $r$-permutations of a set with $n$ elements is

$$
P(n, r)=\frac{n!}{(n-r)!}, \quad 0 \leq r \leq n
$$

In particular,

$$
P(n, 0)=\frac{n!}{n!}=1
$$

(the 0-permutation corresponds to the empty list (or null string)) and

$$
P(n, n)=(n)_{n}=n(n-1)(n-2) \cdots 2 \cdot 1=n!
$$

Note : By convention,

$$
0!:=1 \quad \text { and } \quad(n)_{0}:=1
$$

5.2.4 Example. How many ways can the letters in the word COMPUTER be arranged in a row?

Solution : All the eight letters in the word COMPUTER are distinct, so the number of ways to arrange the letters equals the number of permutations of a set with 8 elements, namely $8!=40320$.
5.2.5 Example. Suppose that there are 12 runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur ?

Solution: The number of different ways to award the medals is the number of 3-permutations of a set with 12 elements, namely $P(12,3)=12 \cdot 11 \cdot 10=$ 1320.
5.2.6 Example. How many different ways can three letters of the word MATHS be chosen and written in a row ? How many different ways can this be done if the first letter must be M ?

Solution : The answer to the first question equals the number of 3 -permutations of a set with 5 elements, namely $P(5,3)=5 \cdot 4 \cdot 3=60$. Since the first letter must be M , there are effectively only two letters to be chosen and placed in the other two positions. Hence the answer to the second question is the number of 2-permutations of a set with 4 elements, which is $P(4,2)=4 \cdot 3=12$.
5.2.7 Example. Find the number of permutations of the letters in COMPUTER, such that the letters in MUTE are together in any order.

Solution : COPRMUTE is one such permutation, as is COPRMTEU. In order to keep the letters in MUTE together, let $\Omega$ denote the set $\{M, U, T, E\}$. We consider $\Omega$ as a symbol in a permutation. The number of permutations of $C, O, P, R, \Omega$ is $5!=120$. In each of these 120 permutations, the presence of $\Omega$ will keep the letters in MUTE together. Now, for each of these permutations, there are $4!=24$ permutations of the letters in $\Omega$. Hence, the number of permutations of COMPUTER that contain the letters of MUTE together in any order is $120 \cdot 24=2880$.

## Combinations

5.2.8 Definition. Let $A$ be a set with $n$ elements and let $0 \leq r \leq n$. An unordered selection of $r$ elements of $A$ is called an $\mathbf{r}$-combination.

NOTE : An $r$-combination of a set with $n$ elements is simply a subset with $r$ elements of the given set.
5.2.9 Example. Find all the 2-combinations of the set $A=\{1,2,3,4\}$.

Solution : There are six 2-combinations of $A$, namely

$$
\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\} .
$$

The number of $r$-combinations of a set with $n$ elements is denoted by $C(n, r)$. We can determine the number of $r$-combinations of a set with $n$ elements using the formula for the number of $r$-permutations of a set.
5.2.10 Proposition. The number of $r$-combinations of a set with $n$ elements is

$$
C(n, r)=\frac{n!}{r!(n-r)!}, \quad 0 \leq r \leq n
$$

Proof : The $r$-permutations of the set can be obtained by forming the $C(n, r)$ combinations of the set, and then ordering the elements in each $r$ combination, which can be done in $P(r, r)$ ways. Hence,

$$
P(n, r)=C(n, r) \cdot P(r, r) .
$$

Thus

$$
C(n, r)=\frac{P(n, r)}{P(r, r)}=\frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}}=\frac{n!}{r!(n-r)!} .
$$

In particular,

$$
C(n, 0)=\frac{n!}{0!\cdot n!}=1
$$

(the 0 -combination corresponds to the empty set) and

$$
C(n, n)=\frac{n!}{n!\cdot 0!}=1 .
$$

5.2.11 Example. A group of twelve consists of five men and seven women. How many five-person teams can be chosen that consists of three men and two women?

Solution : We can think of forming a team as a two-step process : step 1 is to choose the men, and step 2 is to choose the women. There are $C(5,3)$
ways to choose the three men out of the five, and $C(7,2)$ ways to choose the two women out of the seven. Hence, by the product rule, the number of teams of five that contain three men and two women is

$$
C(5,3) \cdot C(7,2)=\frac{5!}{3!\cdot 2!} \cdot \frac{7!}{2!\cdot 5!}=210
$$

Note : There is another common notation for the number of $r$-combinations of a set with $n$ elements, namely

$$
\binom{n}{r} \quad(\text { read } \quad " n \text { choose } r ")
$$

This number is also called a binomial coefficient, since it occurs as a coefficient in the expansion of powers of binomial expressions such as $(a+b)^{n}$.
5.2.12 Example. Evaluate $\binom{5}{3}$.

Solution : We can simply list all the 3 -element subsets of the set $\{1,2,3,4,5\}$.
Here they are

$$
\begin{array}{lllll}
\{1,2,3\}, & \{1,2,4\}, & \{1,2,5\}, & \{1,3,4\}, & \{1,3,5\}, \\
\{1,4,5\}, & \{2,3,4\}, & \{2,3,5\}, & \{2,4,5\}, & \{3,4,5\}
\end{array}
$$

Alternatively, we have

$$
\binom{5}{3}=\frac{5!}{3!\cdot(5-3)!}=\frac{5!}{3!\cdot 2!}=\frac{3!\cdot 4 \cdot 5}{3!\cdot 2}=10
$$

Note : More generally,

$$
\binom{n}{3}=\frac{n(n-1)(n-2)}{6}, \quad n \geq 3
$$

Observe that

$$
\binom{5}{2}=\binom{5}{3}=10
$$

This equality is no coincidence. The following result holds.
5.2.13 Proposition. Let $n, r \in \mathbb{N}$ with $0 \leq r \leq n$. Then

$$
\binom{n}{r}=\binom{n}{n-r}
$$

Solution : We have

$$
\binom{n}{r}=\frac{n!}{r!\cdot(n-r)!}=\frac{n!}{(n-r)!\cdot[n-(n-r)]!}=\binom{n}{n-r} .
$$

Here is another way to think about this result. Imagine a class with $n$ children. The teacher has $r$ identical chocolate bars to give to exactly $r$ of the children. In how many ways can the chocolate bars be distributed ? The answer is $\binom{n}{r}$ because we are selecting a lucky set of $r$ children to get chocolate. But the pessimistic view is also interesting. We can think about selecting the unfortunate children who will not be receiving chocolate. There are $n-r$ children who do not get chocolate, and we can select that subset of the class in $\binom{n}{n-r}$ ways. Since the two countings are clearly the same, we must have $\binom{n}{r}=\binom{n}{n-r}$.
5.2.14 Example. How many bit strings of length eight have exactly three 1's ?

SOLUTION : To solve this problem, imagine eight empty positions into which the 0's and 1's of the bit string will be placed. Once a subset of three positions has been chosen from the eight to contain 1's, the remaining five positions must all contain 0 's. It follows that the number of ways to construct a bit string of length eight with exactly three 1's is the same as the number of subsets of three positions that can be chosen from the eight into which to place the 1 's. This number equals

$$
\binom{8}{3}=\frac{8!}{3!\cdot 5!}=56
$$

In many counting problems, objects may be used repeatedly. We are now interested in counting the ordered selections and unordered selections of objects of a finite set, when repetition is allowed.

## Permutations with repetition

5.2.15 Definition. Let $A$ be a set with $n$ elements and let $r \geq 0$. An r-permutation with repetition of $A$ is an ordered selection of $r$ elements of $A$, where each of the $r$ elements can be repeated. An $n$-permutation with repetition of $A$ is simply called a permutation with repetition of $A$.
5.2.16 Example. How many bit strings of length seven can be constructed from the set $\Sigma=\{0,1,2, \ldots, 9\}$ ? How many seven-digit telephone numbers can be constructed from the set $\Sigma$ if 0 or 1 is not allowed as a first digit in any telephone number ? Assume repeated digits can be used.

Solution : The number of strings of length seven (over $\Sigma$ ) is

$$
10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10=10^{7}=10000000
$$

If 0 or 1 cannot be used as a first digit in a telephone number, then the number of such seven-digit telephone numbers is

$$
8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10=8 \cdot 10^{6}=8000000
$$

The number of $r$-permutations with repetition of a set with $n$ elements can be obtained by using the multiplication rule.
5.2.17 Proposition. The number of $r$-permutations with repetition of $a$ set with $n$ elements is $n^{r}$.

Proof : First assume $r>0$. For each of the $r$ elements in the $r$-permutation with repetition, there are $n$ choices of elements, since repetition is allowed. By the multiplication rule, the number of $r$-permutations with repetition is $n \cdot n \cdots n=n^{r}$. For $r=0$, there is one 0-permutation, the empty permutation. Hence, the number of 0 -permutations is $n^{0}=1$.

## Combinations with repetition

5.2.18 Definition. Let $A$ be a set with $n$ elements and let $r \geq 0$. An $\mathbf{r}$ combination with repetition of $A$ is an unordered selection of $r$ elements of $A$, when repetition is allowed.

An $r$-combination with repetition is also called a multiset of size $r$.
5.2.19 Example. Write down all 3 -combinations with repetition of the set $\{1,2,3,4\}$. (Write such an unordered selection as $[a, b, c]$ )

Solution : We observe that because the order in which the elements are chosen does not matter, the elements of each selection may be written in increasing order. There are 20 such 3 -combinations with repetition :
$[1,1,1],[1,1,2],[1,1,3],[1,1,4],[1,2,2],[1,2,3],[1,2,4],[1,3,3],[1,3,4],[1,4,4]$,
$[2,2,2],[2,2,3],[2,2,4],[2,3,3],[2,3,4],[2,4,4],[3,3,3],[3,3,4],[3,4,4],[4,4,4]$.

Let $n, r \in \mathbb{N}$. The symbol $\left.\binom{n}{r}\right)$ denotes the number of mutisets of size $r$ whose elements belong to a set with $n$ elements.
Note : The notation $\left(\binom{n}{r}\right)$ is pronounced " $n$ multichoose $r$ ". The double parantheses remind us that we may include elements more than once.
5.2.20 Example. Let $n$ and $r$ positive integers. Evaluate

$$
\left(\binom{n}{1}\right) \quad \text { and } \quad\left(\binom{1}{r}\right) .
$$

Solution : The multisets of size 1 whose elements are selected from the set $\{1,2, \ldots, n\}$ are

$$
[1],[2], \ldots,[n]
$$

and so

$$
\left(\binom{n}{1}\right)=n .
$$

We need to count the multisets of size $r$ whose elements are selected from $\{1\}$. The only possibility is

$$
[1,1, \ldots, 1]
$$

and so

$$
\left(\binom{1}{r}\right)=1 .
$$

One can prove the following result.
5.2.21 Proposition. Let $n, r \in \mathbb{N}$. Then

$$
\left(\binom{n}{r}\right)=\binom{n+r-1}{r} .
$$

This equals the number of ways $r$ objects can be selected from $n$ categories of objects with repetition allowed.
5.2.22 Example. A person giving a party wants to set out 15 assorted cans of soft drinks for his guests. He shops at a store that sells five different types of soft drinks. How many different selections of 15 soft drinks can he make?

Solution : We think of the five different types of soft drinks as the $n$ categories and the 15 cans of soft drinks to be chosen as the $r$ objects. So $n=5$ and $r=15$. The total number of different selections of 15 cans of soft drinks of the five types is

$$
\left(\binom{5}{15}\right)=\binom{5+15-1}{15}=\binom{19}{5}=\frac{19 \cdot 18 \cdot 17 \cdot 16 \cdot 15!}{15!\cdot 4 \cdot 3 \cdot 2}=3876 .
$$

5.2.23 Example. If $n$ is a positive integer, how many triples of integers $(i, j, k)$ are there with $1 \leq i \leq j \leq k \leq n$ ?

Solution : We observe that there are exactly as many triples of integers $(i, j, k)$ with $1 \leq i \leq j \leq k \leq n$ as there are 3-combinations of integers from 1 through $n$ with repetition allowed, because the elements of any such

3 -combination can be written in increasing order in only one way. Hence, the number of such triples is

$$
\left(\binom{n}{3}\right)=\binom{n+3-1}{3}=\binom{n+2}{3}=\frac{(n+2)!}{3!\cdot(n-1)!}=\frac{n(n+1)(n+2)}{6} .
$$

5.2.24 Example. How many solutions are there to the equation

$$
x+y+z+t=10
$$

if $x, y, z, t$ are natural numbers ?
Solution: We observe that a solution corresponds to a way of selecting 10 objects from a set with 4 elements, so that $x$ objects of type $1, y$ objects of type $2, z$ objects of type 3 , and $t$ objects of type 4 are chosen. Hence, the number of solutions is equal to the number of 10 -combinations with repetition from a set with 4 elements, namely

$$
\left(\binom{4}{10}\right)=\binom{4+10-1}{10}=\binom{13}{10}=\frac{13!}{3!\cdot 10!}=\frac{13 \cdot 12 \cdot 11}{3 \cdot 2}=286 .
$$

### 5.3 Binomial formula

In algebra a sum of two terms, such as $a+b$, is called a binomial. The binomial formula gives an expression for the powers of a binomial $(a+b)^{n}$, for each positive integer $n$ and all real numbers $a$ and $b$.

If we compute the coefficients in the expansion $(a+b)^{n}$ for $n=1,2,3, \ldots$ we obtain :

$$
\begin{aligned}
(a+b)^{1} & =1 a+1 b \\
(a+b)^{2} & =1 a^{2}+2 a b+1 b^{2} \\
(a+b)^{3} & =1 a^{3}+3 a^{2} b+3 a b^{2}+b^{3} \\
(a+b)^{4} & =1 a^{4}+4 a^{3} b+6 a^{2} b^{2}+4 a b^{3}+1 b^{4} .
\end{aligned}
$$

If we arrange the coefficients, we obtain the following triangle, known as Pascal's triangle.


We observe that this can be written as
( ${ }_{0}^{1}$ ) $\quad\binom{1}{1}$
$\binom{2}{0}$
$\binom{2}{1}$
$\binom{2}{2}$
$\binom{3}{0} \quad\binom{3}{1}$
$\binom{3}{1} \quad\binom{3}{2}$
$\quad\binom{3}{3}$
$\binom{4}{0} \quad\binom{4}{1} \quad\binom{4}{2} \quad\binom{4}{3} \quad\binom{4}{4}$

Note : Pascal's triangle is a geometric version of a famous formula, called Pascal's formula. It relates the values of $\binom{n+1}{r}$ to the values of $\binom{n}{r-1}$ and $\binom{n}{r}$. Specifically, it says that

$$
\binom{n+1}{r}=\binom{n}{r-1}+\binom{n}{r}
$$

whenever $n$ and $r$ are positive integers with $r \leq n$. This formula makes it easy to compute higher combinations in terms of lower ones: if the values of $\binom{n}{r}$ are known for all $r$, then the values of $\binom{n+1}{r}$ can be computed for all $r$ such that $1 \leq r \leq n$.

The pattern that seems to be emerging is that :
5.3.1 Proposition. In the expansion of $(a+b)^{n}$, the coefficient of $a^{n-k} b^{k}$ is

$$
\binom{n}{k}=\binom{n}{n-k}
$$

Proof : We use a combinatorial argument.

$$
(a+b)^{n}=(a+b) \cdot(a+b) \cdots(a+b) \quad(n \text { factors })
$$

To multiply this out we choose a term $t_{i} \in\{a, b\}$ for each $i \in[n]=$ $\{1,2, \ldots n\}$ and form the product

$$
t_{1} \cdot t_{2} \cdots \cdot t_{n}
$$

and add up all such products. An example of such a product is

$$
a \cdot a \cdot b \cdot a \cdot b \cdot b \cdot a \cdot \cdots \cdot a
$$

which we can rearrange to form

$$
a^{n-k} b^{k}
$$

for some $k$. If $k$ is specified, the number of products which collapse to form $a^{n-k} b^{k}$ is the number of ways in which we can choose $n-k$ 's (and $k \quad b$ 's) from the $n$ factors. This is just $\binom{n}{n-k}=\binom{n}{k}$. So there are

$$
\binom{n}{k} \text { terms of the form } a^{n-k} b^{k}
$$

and this is the coefficient of $a^{n-k} b^{k}$.
We immediately obtain :

### 5.3.2 Proposition. (The Binomial Formula)

$$
\begin{aligned}
(a+b)^{n} & =\sum_{k=0}^{n}\binom{n}{k} a^{n-k} b^{k} \\
& =a^{n}+\binom{n}{1} a^{n-1} b+\binom{n}{2} a^{n-2} b^{2}+\cdots+\binom{n}{n-1} a b^{n-1}+b^{n}
\end{aligned}
$$

Proof :

$$
\begin{aligned}
(a+b)^{n} & =\text { the sum of all products of form } t_{1} \cdot t_{2} \cdots t_{n} \\
& =\sum_{k=0}^{n}(\text { the number of products with } k \quad b ' s) \cdot a^{n-k} b^{k} \\
& =\sum_{k=0}^{n}\binom{n}{k} a^{n-k} b^{k} .
\end{aligned}
$$

Note:

$$
(a+b)^{n}=\sum_{k=0}^{n}\binom{n}{k} a^{n-k} b^{k}=\sum_{k=0}^{n}\binom{n}{k} a^{k} b^{n-k}
$$

5.3.3 Example. Use the binomial formula to expand $(x+y)^{5}$.

Solution : We have

$$
\begin{gathered}
(x+y)^{5}=\sum_{k=0}^{5}\binom{5}{k} x^{5-k} y^{k}= \\
=\binom{5}{0} x^{5}+\binom{5}{1} x^{4} y+\binom{5}{2} x^{3} y^{2}+\binom{5}{3} x^{2} y^{3}+\binom{5}{4} x y^{4}+\binom{5}{5} y^{5}= \\
=x^{5}+5 x^{4} y+10 x^{3} y^{2}+10 x^{2} y^{3}+5 x y^{4}+y^{5}
\end{gathered}
$$

5.3.4 Example. What is the coefficient of $a^{7} b^{8}$ in the expansion of $a+$ b) ${ }^{15}$ ?

Solution : From the binomial formula it follows that this coefficient is

$$
\binom{15}{7}=\frac{15!}{7!\cdot 8!}=6435 .
$$

5.3.5 Example. Use the binomial formula to show that

$$
\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^{n}
$$

for all natural numbers $n$.
Solution : We have

$$
2^{n}=(1+1)^{n}=\sum_{k=0}^{n}\binom{n}{k} 1^{n-k} \cdot 1^{k}=\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n} .
$$

NOTE : $2^{n}$ represents the number of all the subsets of a set with $n$ elements.
5.3.6 Example. The following identity holds

$$
\binom{n}{3}=\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{n-1}{2} .
$$

Solution : We have (using Pascal's formula)

$$
\begin{aligned}
\binom{n}{3} & =\binom{n-1}{2}+\binom{n-1}{3} \\
& =\binom{n-1}{2}+\binom{n-2}{2}+\binom{n-2}{3} \\
& \vdots \\
& =\binom{n-1}{2}+\binom{n-2}{2}+\cdots+\binom{4}{2}+\binom{4}{3} \\
& =\binom{n-1}{2}+\binom{n-2}{2}+\cdots+\binom{4}{2}+\binom{3}{2}+\binom{3}{3} .
\end{aligned}
$$

### 5.4 Exercises

## Exercise 61

(a) How many integers from 1 through 1000 do not have any repeated digits?
(b) How many four-digit integers (integers from 1000 through 9999 ) are divisible by 5 ?
(c) How many integers from 1 through 100000 contain the digit 6 exactly once?

## Exercise 62

(a) A typical PIN (personal identification number) is a sequence of any four symbols chosen from 26 letters in the alphabet and ten digits, with repetition allowed. How many different PINs are possible?
(b) Now suppose that repetition is not allowed. How many different PINs are there?

Exercise 63 How many permutations of the seven letters $A, B, C, D, E, F, G$
(a) are there?
(b) have $E$ in the first position ?
(c) have $E$ in one of the first two positions ?
(d) do not have vowels on the ends ?
(e) have 2 vowels before the 5 consonants ?
(f) have $A$ immediately to the left of $E$ ?
(g) neither begin nor end with $A$ ?
(h) do not have the vowels next to each other ?

Exercise 64 A survey of 100 university students revealed the following data : 18 like to eat chicken, 40 like to eat beef, 20 like to eat lamb, 12 like to eat chicken and beef, 5 like to eat chicken and lamb, 4 like to eat beef and lamb, and 3 like to eat all three. We will classify the student who does not like to eat any of the three kinds of meat as a non-meat eater.
(a) How many students in our sample like to eat at least one of the three kinds of meat?
(b) How many non-meat eaters are in our sample ?
(c) How many students in our sample like to eat only lamb ?

Exercise 65 Suppose you have 30 books ( 15 novels, 10 history books, and 5 math books). Assume that all 30 books are different. In how many ways can you :
(a) put the 30 books in a row on the shelf?
(b) get a bunch of 4 books to give to a friend?
(c) get a bunch of 3 history books and 7 novels to give to a friend?
(d) put the 30 books in a row on a shelf if the novels are on the left, the math books are in the middle, and the history books are on the right?

## Exercise 66

(a) Determine the number of functions from a five-element set to an eightelement set.
(b) Determine the number of functions from a five-element set to an eightelement set that are not one-to-one.
(c) Determine the number of functions from a five-element set to a threeelement set that are onto.
(d) Determine the number of subsets of a set with 10 elements. (Hint: Count all the characteristic functions of the given set)

Exercise 67 Using only the digits $1,2,3,4,5,6$, and 7, how many five-digit numbers can be formed that satisfy the following conditions?
(a) no additional conditions.
(b) at least one 5 .
(c) at least one 5 and at least one 6 .
(d) no repeated digits.
(e) at least one 5 and no repeated digits.
(f) the first and the last digits the same.

Exercise 68 Consider the word ALGORITHM.
(a) How many ways can three of the letters of the word be selected an written in a row?
(b) How many ways can five of the letters of the word be selected and written in a row if the first letter must be $A$ ?

Exercise 69 How many different committees of five from a group of fourteen people can be selected if
(a) a certain pair of people insist on serving together or not at all?
(b) a certain pair of people refuse to serve together ?

Exercise 70 Consider a group of twelve consisting of five men and seven women.
(a) How many five-person teams can be chosen that consists of three men and two women?
(b) How many five-person teams contain at least one man?
(c) How many five-person teams contain at most one man?

## Exercise 71

(a) How many bit strings of length 16 contain exactly nine 1 's?
(b) How many bit strings of length 16 contain at least one 1 ?

Exercise 72 Nine points $A, B, C, D, E, F, G, H, I$ are arranged in a plane in such a way that no three lie on the same straight line.
(a) How many straight lines are determined by the nine points?
(b) How many of these straight lines do not pass through point $A$ ?
(c) How many triangles have three of the nine points as vertices?
(d) How many of these triangles do not have $A$ as a vertex?

Exercise 73 Use the binomial formula to expand:
(a) $(2-x)^{5}$.
(b) $(2 a-3 b)^{6}$.
(c) $(a+2)^{6}$.
(d) $\left(x-\frac{3}{x}\right)^{5}$.
(e) $\left(x^{2}+\frac{1}{x}\right)^{7}$.

Exercise 74 Find :
(a) the coefficient of $x^{5}$ in $(1+x)^{12}$.
(b) the coefficient of $x^{5} y^{6}$ in $(2 x-y)^{11}$.
(c) the coefficient of $x^{5}$ in $\left(2+x^{2}\right)^{12}$.
(d) the middle term of $(a-b)^{10}$.
(e) the number of terms in the expansion of $(5 a+8 b)^{15}$.
(f) the largest coefficient in the expansion of $(x+3)^{5}$.
(g) the last term of $(x-2 y)^{6}$.

Exercise 75 Use the binomial formula to prove that
(a) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^{n}\binom{n}{n}=0 ; \quad n \geq 1$.
(b) $\quad\binom{n}{0}+2\binom{n}{1}+2^{2}\binom{n}{2}+\cdots+2^{n}\binom{n}{n}=3^{n}, \quad n \geq 0$.

