## Chapter 6

## Recursion

## Topics :

## 1. Recursively defined sequences

2. Modelling with recurence relations
3. Linear recurrence relations

Sometimes an object is difficult to define explicitly; it may be easy to define it in terms of itself. This procedure is called recursion. Recursive definitions are at the hart of discrete mathematics. Furthermore, recursion is one of the central ideas of computer science.

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### 6.1 Recursively defined sequences

Recall that a sequence (of numbers), denoted $\left(a_{n}\right)_{n \geq 0}$ (or simply $\left(a_{n}\right)$ ), is just a function from $\mathbb{N}$ to $\mathbb{R}$; that is, an element of $\mathbb{R}^{\mathbb{N}}$.

A common way to define a sequence is to give an explicit formula for its $n^{\text {th }}$ term. For example, a sequence $\left(a_{n}\right)$ can be specified by writing

$$
a_{n}=2^{n} \quad \text { for } n=0,1,2 \ldots
$$

The advantage of defining a sequence by such an explicit formula is that each term of the sequence is uniquely determined and any term can be computed in a fixed, finite number of steps.

Another way to define a sequence is to use recursion. This requires giving both an equation, called a recurrence relation, that relates later terms in the sequence to earlier terms and a specification, called initial conditions, of the values of the first few terms of the sequence. For example, the sequence of powers of 2 can also be defined recursively as follows :

$$
a_{n}=2 a_{n-1} \quad \text { for } n \geq 1 \text { and } a_{0}=1 .
$$

Sometimes it is very difficult or impossible to find an explicit formula for a sequence, but it is possible to define the sequence using recursion.

Note : Defining a sequence recursively is similar to proving statements by mathematical induction. The recurence relation is like the inductive step and the initial conditions are like the basis step. Indeed, the fact that sequences can be defined recursively is equivalent to the fact that mathematical induction works as a method of proof.
6.1.1 Definition. A recurrence relation for a sequence $\left(a_{n}\right)$ is a formula that expresses $a_{n}$ in terms of one or more of the previous terms of the sequence, namely $a_{n-1}, a_{n-2}, \ldots$. One or more of these terms may be given specific values; the given values are called initial conditions.
6.1.2 Example. Consider the factorial sequence ( $n$ !). We can see that

$$
0!=1, \quad 1!=1, \quad 2!=2 \cdot 1, \quad 3!=3 \cdot 2 \cdot 1=3 \cdot 2!, \quad 4!=4 \cdot 3!, \quad \ldots
$$

In general, $n!=n \cdot(n-1)!$. If we let $\varphi_{n}:=n$ !, then we have $\varphi_{n}=n \varphi_{n-1}$. So

$$
\varphi_{n}=n \varphi_{n-1} \quad \text { for } n=1,2,3, \ldots
$$

is a recurrence relation defining $n!$. However, $a_{n}=n a_{n-1}$ does not define a unique sequence; for example, the sequence $\pi, \pi, 2 \pi, 6 \pi, \ldots$ also satisfies this recurrence relation. If we require that

$$
\varphi_{n}=n \varphi_{n-1} \quad \text { for } n \geq 1 \quad \text { and } \quad \varphi_{0}=1
$$

we obtain $n$ ! uniquely. This illustrates the need for initial conditions if we want a recurrence relation to define a sequence uniquely. The initial condition $\varphi_{0}=1$ for the factorial sequence is $0!=1$.

Note: A sequence defined recursively need not start with a subscript of zero.
6.1.3 Example. Show that the sequence $\left(s_{n}\right)_{n \geq 1}$ with $s_{n}=\frac{(-1)^{n+1}}{n!}$ satisfies the recurrence relation

$$
s_{n+1}=-\frac{1}{n+1} s_{n} \quad \text { for } n \geq 1
$$

Solution : We have

$$
\begin{aligned}
-\frac{1}{n+1} s_{n} & =-\frac{1}{n+1} \frac{(-1)^{n+1}}{n!} \\
& =\frac{(-1)^{n+2}}{(n+1)!} \\
& =s_{n+1}
\end{aligned}
$$

Given a sequence that satisfies a certain recurrence relation and initial conditions, it is often helpful to know an explicit formula for the sequence; such an explicit formula is called a solution to the recurrence relation.

The most basic method for finding an explicit fromula for a recursively defined sequence is iteration. Iteration works as follows : given a sequence $\left(a_{n}\right)$ defined by a recurrence relation and initial conditions, we start from the initial conditions and calculate succesive terms of the sequence until we can see a pattern developing. At that point we guess an explicit formula.
6.1.4 Example. Let $\left(a_{n}\right)$ be the sequence defined recursively as follows :

$$
a_{n}=a_{n-1}+2 ; \quad a_{0}=1 .
$$

Use iteration to guess an explicit formula for $a_{n}$.
Solution : We have

$$
\begin{array}{ll}
a_{0}=1 & =1+0 \cdot 2 ; \\
a_{1}=a_{0}+2=1+2 & =1+1 \cdot 2 ; \\
a_{2}=a_{1}+2=(1+1 \cdot 2)+2 & =1+2 \cdot 2 ; \\
a_{3}=a_{2}+2=(1+2 \cdot 2)+2 & =1+3 \cdot 2 ; \\
a_{4}=a_{3}+2=(1+3 \cdot 2)+2 & =1+4 \cdot 2 .
\end{array}
$$

Guess : $a_{n}=1+2 n$.
Note: A sequence $\left(a_{n}\right)_{n \geq 0}$ is called an arithmetic sequence (or progression) if and only if there is a constant $r$ such that

$$
a_{n}=a_{n-1}+r \text { for all } n \geq 1 .
$$

Or, equivalently,

$$
a_{n}=a_{0}+r \cdot n .
$$

6.1.5 Example. Consider the sequence $\left(b_{n}\right)$ defined recursively as follows :

$$
b_{n}=3 b_{n-1} ; \quad b_{0}=2 .
$$

Use iteration to guess an explicit formula for this sequence.

Solution : We have

$$
\begin{aligned}
b_{0} & =2=1 \cdot 2 ; \\
b_{1} & =3 \cdot b_{0}=3 \cdot 2 ; \\
b_{2} & =3 \cdot b_{1}=3^{2} \cdot 2 ; \\
b_{3} & =3 \cdot b_{2}=3^{3} \cdot 2 .
\end{aligned}
$$

Guess: $b_{n}=2 \cdot 3^{n}$.
Note: A sequence $\left(b_{n}\right)_{n \geq 0}$ is called an geometric sequence (or progression) if and only if there is a constant $q \neq 0$ such that

$$
b_{n}=q \cdot b_{n-1} \quad \text { for all } n \geq 1
$$

Or, equivalently,

$$
b_{n}=b_{0} \cdot q^{n}
$$

6.1.6 Example. Find an explicit formula for the sequence $\left(H_{n}\right)_{n \geq 1}$ defined recursively by

$$
H_{n}=2 H_{n-1}+1 \quad \text { for } n \geq 2 \text { and } \quad H_{1}=1 .
$$

(This sequence may be referred to as the Tower of Hanoi sequence; see the entertaining discussion of the Tower of Hanoi puzzle further in this section).

Solution: By iteration

$$
\begin{array}{lll}
H_{1}=1 & & \\
H_{2} & =2 H_{1}+1=2 \cdot 1+1 & =2^{1}+1 \\
H_{3} & =2 H_{2}+1=2(2+1)+1 & =2^{2}+2+1 \\
H_{4} & =2 H_{3}+1=2\left(2^{2}+2+1\right) & =2^{3}+2^{2}+2+1 .
\end{array}
$$

Guess : $H_{n}=2^{n-1}+2^{n-2}+\cdots+2^{2}+2+1$. By the formula for the sum of a geometric sequence,

$$
2^{n-1}+2^{n-2}+\cdots+2^{2}+2+1=\frac{2^{n}-1}{2-1}=2^{n}-1
$$

Hence, the explicit formula seems to be

$$
H_{n}=2^{n}-1 \text { for all integers } n \geq 1 .
$$

The process of solving a recurrence relation by iteration can involve complicated calculations. It is all too easy to make a mistake and come up with the wrong formula. That is why it is important to confirm your calculations by checking the correctness of your formula. The most common way to do this is to use mathematical induction.
6.1.7 Example. Use mathematical induction to show that the formula we have obtained for the Tower of Hanoi sequence is correct.

Solution : Let $P(n)$ be the predicate

$$
" H_{n}=2^{n}-1 . "
$$

We shall prove by induction that the proposition $\forall n P(n)$ is true.
BASIS STEP : $\quad P(1)$ is true, since $H_{1}=2^{1}-1$.
INDUCTIVE STEP : Assume that $P(n)$ is true. That is, assume that

$$
H_{n}=2^{n}-1 .
$$

Under this assumption, we must show that $P(n+1)$ is true, namely that

$$
H_{n+1}=2^{n+1}-1 .
$$

We have

$$
H_{n+1}=2 H_{n}+1=2\left(2^{n}-1\right)+1=2^{n+2}-2+1=2^{n+1}-1 .
$$

This shows that $P(n+1)$ is true, and therefore we are done.
The following example shows how the process of trying to verify a formula by mathematical induction may reveal a mistake.
6.1.8 Example. Consider the sequence $\left(c_{n}\right)$ defined by

$$
c_{n}=2 c_{n-1}+n \quad \text { for all integers } n \geq 1 ; \quad c_{0}=1 .
$$

Suppose that your calculations suggests that

$$
c_{n}=2^{n}+n \text { for all integers } n \geq 0 .
$$

Is this formula correct?
Solution : Let us start to prove the proposition $\forall n\left(c_{n}=2^{n}+n\right)$ and see what develops. The proposed formula passes the basis step with no trouble, since $2^{0}+0=1$. In the inductive step we suppose that

$$
c_{n}=2^{n}+n \text { for some integer } n
$$

and we must show that

$$
c_{n+1}=2^{n+1}+(n+1) .
$$

We have

$$
c_{n+1}=2 c_{n}+(n+1)=2\left(2^{n}+n\right)+(n+1)=2^{n+1}+3 n+1 .
$$

To finish the verification, we need to show that

$$
2^{n+1}+3 n+1=2^{n+1}+(n+1),
$$

which is equivalent to $2 n=0$. But this is false, since $n$ may be any natural number. Hence, the proposed formula is wrong. (The right formula is $c_{n}=$ $3 \cdot 2^{n}-(n+2)$.)

Once we have found a proposed formula to be false, we should look back at our calculations to see where we have made a mistake, correct it, and try again.

### 6.2 Modelling with recurrence relations

Recurrence relations can be used to model a variety of problems such as finding compound interest or counting various objects with certain properties (e.g. bacteria, sequences, bit strings, etc.).
6.2.1 Example. (Compound interest) On your twenty-first birthday you get a letter informing you that on the day you were born an eccentric rich aunt deposited $R 1000$ in a bank account earning $5.5 \%$ interest compounded annually and she now intends to turn the account over to you provided you can figure out how much it is worth. What is the amount currently in the account?

Solution : We observe that the amount in the account at the end of any particular year equals the amount at the end of the previous year plus the interest earned on the account during the year. Now the interest earned during the year equals the interest rate, $5.5 \%=0.055$, times the amount in the account at the end of the previous year. For any positive integer $n$, let $A_{n}$ denote the amount in the account at the end of year $n$ and let $A_{0}=1000$ be the initial amount in the account. Then we get

$$
A_{n}=A_{n-1}+(0.055) \cdot A_{n-1}=(1.055) \cdot A_{n-1} .
$$

Thus a (complete) recurrence relation for the sequence $\left(A_{n}\right)$ is as follows

$$
A_{n}=(1.055) \cdot A_{n-1} ; \quad A_{0}=1000
$$

The value on your twenty-first birthday can be computed by repeated substitution : $A_{1}=1055.00, A_{2}=1113.02, A_{3}=1174.24, \ldots, A_{20}=2917.76, A_{21}=$ 3078.23.
6.2.2 Example. (Number of bacteria in a colony) Suppose that the number of bacteria in a colony triples every hour.

1. Set up a recurrence relation for the number of bacteria after $n$ hours have elapsed.
2. If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

Solution : (1) Since the number of bacteria triples every hour, the recurrence relation should say that the number of bacteria after $n$ hours is 3 times
the number of bacteria after $n-1$ hours. Letting $b_{n}$ denote the number of bacteria after $n$ hours, this statement translates into the recurrence relation $b_{n}=3 b_{n-1}$.
(2) The given statement is the initial condition $b_{0}=100$ (the number of bacteria at the beginning is the number of bacteria after no hours have elapsed). We can solve the recurrence relation easily by iteration :

$$
b_{n}=3 b_{n-1}=3^{2} b_{n-2}=\cdots=3^{n} b_{n-n}=3^{n} b_{0}
$$

Letting $n=10$ and knowing that $b_{0}=100$, we see that

$$
b_{10}=3^{10} \cdot 100=5,904,900
$$

### 6.2.3 EXAMPLE. (Number of sequences with a certain property)

1. Find a recurrence relation for the number of strictly increasing sequences of positive integers that have 1 as their first term and $n$ as their last term, where $n$ is a positive integer. (That is, finite sequences $a_{1}, a_{2}, \ldots, a_{k}$ where $a_{1}=1, a_{k}=n$, and $a_{i}<a_{i+1}$ for $i=1,2, \ldots, k-$ 1.)
2. What are the initial conditions ?
3. How many sequences of the type described in (a) are there, when $n$ is a positive integer with $n \geq 2 ?$

Solution : (1) Let $s_{n}$ be the number of such sequences. A string ending in $n$ must consist of a string ending in something less than $n$, followed by an $n$ as the last term. Therefore the recurrence relation is

$$
s_{n}=s_{n-1}+s_{n-2}+\cdots+s_{2}+s_{1}
$$

Here is another approach, with a more compact form of the answer. A sequence ending in $n$ is either a sequence ending in $n-1$ followed by $n$ (and there are
clearly $s_{n-1}$ of these), or else it does not contain $n-1$ as a term at all, in which case it is identical to a sequence ending in $n-1$ in which the $n-1$ has been replaced by $n$ (and there are clearly $s_{n-1}$ of these as well). Therefore,

$$
s_{n}=2 s_{n-1}
$$

(2) We need two initial conditions if we use the second formulation above, $s_{1}=1$ and $s_{2}=1$ (otherwise, our argument is invalid, because the first and the last terms are the same). There is one sequence ending in 1 , namely the sequence with just this 1 in it, and there is only the sequence (string) ( 1,2 ) ending in 2. If we use the first formulation above, then we can get by with just the initial condition $s_{1}=1$.
(3) Clearly, the solution to this recurrence relation and initial condition is $s_{n}=2^{n-2}$ for al $n \geq 2$.
6.2.4 EXAMPLE. (Number of bit strings with a certain property)

Find the number of bit strings of length 10 that do not contain the pattern 11.

Solution : To find the required number you could list all $2^{10}=1024$ strings of length 10 and cross off those that contain the pattern 11. However, this approach would be very time-consuming. A more efficient solution uses recursion. Suppose the number of bit strings of length $n$ that do not contain the pattern 11 is known. Any such bit string begins with either a 0 or a 1. If the string begins with a 0 , the remaining $n-1$ characters can be any sequence of 0 's and 1's except that the pattern 11 cannot appear. If the string begins with a 1 , then the second character must be a 0 , for otherwise the string would contain the pattern 11 ; the remaining $n-2$ characters can be any sequence of 0 's and 1's that does not contain the pattern 11. Let $s_{n}$ denote the number of bit strings of length $n$ that do not contain the pattern 11. Then we can write

$$
s_{n}=s_{n-1}+s_{n-2} .
$$

So, a (complete) recurrence relation for the sequence $\left(s_{n}\right)$ is

$$
s_{n}=s_{n-1}+s_{n-2} ; \quad s_{0}=1, s_{1}=2\left(\text { or } s_{1}=2, s-2=3\right) .
$$

It follows that $s_{2}=3, s_{3}=5, s_{4}=8, \ldots, s_{10}=144$. Hence, there are 144 bit strings of length 10 that do not contain the pattern 11 .

We give some more interesting examples.
6.2.5 EXAMPLE. (Fibonacci numbers) One of the earliest examples of recursively defined sequences arises in the writings of Leonardo of Pisa, commonly known as Fibonacci. In 1202 Fibonacci posed the following problem.

A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions :

1. Rabbit pairs are not fertile during their first month of life, but thereafter give birth to one new male/female pair at the end of every month;
2. No rabbits die.

How many rabbits will there be at the end of the year ?
Solution : One way to solve this problem is to plunge right into the middle of it using recursion. Suppose we know how many rabbit pairs there were at the ends of previous months. How many will there be at the end of the current month? The crucial observation is that the number of rabbit pairs born at the end of month $k$ is the same as the number of pairs alive at the end of month $k-2$. (Why ? Because it is exactly the rabbit pairs that were alive at the end of the month $k-2$ that were fertile during month $k$. The rabbits born at the end of month $k-1$ were not.) Now the number of rabbit pairs alive at the end of month $k$ equals the ones alive at the end of the month $k-1$ plus the pairs newly born at the end of the month. For any positive integer $n$, let $F_{n}$ denote the number of rabbit pairs alive at the end of month $n$ and
let $F_{0}=1$ be the initial number of rabbit pairs. Then we get the following recurrence relation

$$
F_{n}=F_{n-1}+F_{n-2} ; \quad F_{0}=1, \quad F_{1}=1 .
$$

Note : The terms of the sequence $\left(F_{n}\right)_{n \geq 0}$ are called Fibonacci numbers.
To answer Fibonacci's question, compute $F_{2}, F_{3}$, and so forth through $F_{12}$. We get $F_{2}=2, F_{3}=3, F_{4}=5, \ldots, F_{12}=233$. At the end of the twelfth month there are 233 rabbit pairs in all.

### 6.2.6 Example.

1. Find a recurrence relation for the number of bit strings of length $n$ that contain a pair of consecutive 0 s.
2. What are the initial conditions ?
3. How many bit strings of length seven contain two consecutive 0 s?

Solution: (1) Let $a_{n}$ be the number of bit strings of length $n$ containing a pair of consecutive 0 s . In order to construct a bit string of length $n$ containing a pair of consecutive 0 s we could start with 1 and follow with a string of length $n-1$ containing a pair of consecutive 0 s, or we could start with a 01 and follow with a string of length $n-2$ containing a pair of consecutive 0 s, or we could start with 00 and follow with any string of length $n-2$. These three cases are mutually exclusive and exhaust the possibilities for how the string might start. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 2$ :

$$
a_{n}=a_{n-1}+a_{n-2}+2^{n-2} .
$$

(2) There are no bit strings of length 0 or 1 containing a pair of consecutive 0 s, so the initial conditions are $a_{0}=a_{1}=0$ (or $a_{1}=0, a_{2}=1$ ).
(3) We will compute $a_{2}$ through $a_{7}$ using the recurrence relation :

$$
\begin{aligned}
& a_{2}=a_{1}+a_{0}+2^{0}=0+0+1=1 ; \\
& a_{3}=a_{2}+a_{1}+2^{1}=1+0+2=3 ; \\
& a_{4}=a_{3}+a_{2}+2^{2}=3+1+4=8 ; \\
& a_{5}=a_{4}+a_{3}+2^{3}=8+3+8=19 ; \\
& a_{6}=a_{5}+a_{4}+2^{4}=19+8+16=43 ; \\
& a_{7}=a_{6}+a_{5}+2^{5}=43+19+32=94 .
\end{aligned}
$$

Thus, there are 94 bit strings of length 7 containing two consecutive 0 s.

### 6.2.7 Example.

1. Find a recurrence relation for the number of ways to climb $n$ stairs if the person climbing the stairs can take one stair or two stairs at the time.
2. What are the initial conditions ?
3. How many ways can this person climb a flight of eigth stairs ?

Solution : (1) Let $S_{n}$ be the number of ways to climb $n$ stairs. In order to climb $n$ stairs, a person must either start with a step of one stair and then climb $n-1$ stairs (and this can be done in $S_{n-1}$ ways), or else start with a step of two stairs and then climb $n-2$ stairs (and this can be done in $S_{n-2}$ ways). From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 2$ :

$$
S_{n}=S_{n-1}+S_{n-2} .
$$

We note that the recurrence relation is the same as that for the Fibonacci sequence.
(2) The initial conditions are $S_{0}=1$ and $S_{1}=1$, since there is one way to
climb no stairs (do nothing) and clearly one way to climb one stair.
(3) We have

$$
S_{2}=2, S_{3}=3, S_{4}=5, S_{5}=8, S_{6}=13, S_{7}=21, S_{8}=34
$$

Thus, a person can climb a flight of 8 stairs in 34 ways under the restrictions in this problem.
6.2.8 Example. (The Tower of Hanoi puzzle) This puzzle was invented in 1883 by the French mathematician Edouard Lucas, who made up the legend to accompany it. According to "legend", a certain Hindu temple contains three thin diamond poles on one of which, at the time of creation, God placed 64 golden disks that decrease in size as they rise from the base. The priests of the temple work unceasingly to transfer all the disks one by one from the first pole to one of the others, but they must never place a larger disk on top of a smaller one. As soon as they have completed their task, "tower, temple, and Brahmins alike will crumble into dust, and with a thunderclap the world will vanish."
The question is : Assuming the priests work as efficiently as possible, how long will it be from the time of creation until the end of the world?

Solution: Suppose that we have found the most efficient way possible to transfer a tower of $k-1$ disks one by one from one pole to another, obeying the restriction that we never place a larger disk on top of a smaller one. What is the most efficient way to move a tower of $k$ disks from one pole to another ? Let $A$ be the initial pole, $B$ the intermediate one, and $C$ the target pole. The minimum sequence of moves includes three steps : step 1 - is to move the top $k-1$ disks from pole $A$ to pole $B$; step 2 - is to move the bottom disk from pole $A$ to pole $C$; step 3 - is to move the top $k-1$ disks from pole $B$ to pole $C$. For any positive integer $n$, let $H_{n}$ denote the minimum number of moves needed to move a tower of $n$ disks from one pole to another. One can show that

$$
H_{n}=H_{n-1}+1+H_{n-1}=2 H_{n-1}+1 .
$$

Hence, a (complete) recursive description of the sequence $\left(H_{n}\right)$ is as follows :

$$
H_{n}=2 H_{n-1}+1 ; \quad H_{1}=1
$$

Going back to the legend, suppose the priests work rapidly and move one disk every second. Then the time from the beginning of creation to the end of the world would be $H_{64}$ seconds. We can compute this number on a calculator or a computer ; the approximate result is
$1.844674 \times 10^{19}$ seconds $\approx 5.84542 \times 10^{11}$ years $\approx 584.5$ billion years.

### 6.3 Linear recurrence relations

We shall restrict our investigation to a special class of recurrence relations that can be explicitly solved in a systematic way. These are recurrence relations that express the terms of a sequence as a linear combination of previous terms.
6.3.1 Definition. A recurrence relation of the form

$$
\begin{gathered}
a_{n}=\beta_{1} a_{n-1}+\beta_{2} a_{n-2}+\cdots+\beta_{k} a_{n-k}+\varphi(n) ; \\
a_{0}=\gamma_{0}, a_{1}=\gamma_{1}, \ldots, a_{k-1}=\gamma_{k-1},
\end{gathered}
$$

where $\beta_{1}, \beta_{2}, \ldots, \beta_{k}$ are real numbers, and $\beta_{k} \neq 0$, is called a linear (non-homogeneous) recurrence relation (of degree $k$ ) with constant coefficients. If $\varphi(n) \equiv 0$, the recurrence relation is called homogeneous.

Note : (1) The recurrence relation in the definition is linear since the RHS is a sum of multiples of the previous terms of the sequence.
(2) The recurrence relation is homogeneous if no terms occur (in the RHS) that are not multiples of the $a_{i}$ 's.
(3) The coefficients $\beta_{1}, \beta_{2}, \ldots, \beta_{k}$ of the terms of the sequence are all constants.
(4) The degree is $k$ because $a_{n}$ is expressed in terms of the previous $k$ terms of the sequence.

### 6.3.2 Examples.

1. The recurrence relation $t_{n}=t_{n-1} \cdot t_{n-2}$ is not linear.
2. The recurrence relation $\varphi_{n}=n \varphi_{n-1}$ does not have constant coefficients.
3. The recurrence relation $F_{n}=F_{n-1}+F_{n-2}$ is a linear homogeneous recurrence relation of degree two.
4. The recurrence relation $H_{n}=2 H_{n-1}+1$ is a linear (non-homogeneous) recurrence relation of degree one.
5. The recurrence relation $A_{n}=(1.055) A_{n-1}$ is a linear homogeneous recurrence relation of degree one.

## The homogeneous case

We first tackle the homogeneous case, so we have :

$$
\begin{gathered}
a_{n}=\beta_{1} a_{n-1}+\beta_{2} a_{n-2}+\cdots+\beta_{k} a_{n-k} ; \\
a_{0}=\gamma_{0}, a_{1}=\gamma_{1}, \ldots, a_{k-1}=\gamma_{k-1} .
\end{gathered}
$$

The idea is to look for solutions of the form

$$
a_{n}=r^{n}, \quad \text { where } r \text { is a constant. }
$$

6.3.3 Example. Consider the Fibonacci recurrence relation

$$
F_{n}=F_{n-1}+F_{n-2} ; \quad F_{0}=1, F_{1}=1 .
$$

We seek $r$ such that

$$
r^{n}=r^{n-1}+r^{n-2} ;
$$

that is,

$$
r^{n-2}\left(r^{2}-r-1\right)=0 .
$$

Since the solution $r=0$ is not acceptable, we solve

$$
r^{2}-r-1=0 .
$$

This equation is called the characteristic equation and we see that it has two solutions :

$$
r_{1,2}=\frac{1 \pm \sqrt{5}}{2} .
$$

Thus

$$
f_{1}(n)=\left(\frac{1+\sqrt{5}}{2}\right)^{n} \quad \text { and } \quad f_{2}(n)=\left(\frac{1-\sqrt{5}}{2}\right)^{n}
$$

are solutions of the given recurrence relation and so is

$$
F_{n}=C_{1} f_{1}(n)+C_{2} f_{2}(n)=C_{1}\left(\frac{1+\sqrt{5}}{2}\right)^{n}+C_{2}\left(\frac{1-\sqrt{5}}{2}\right)^{n}
$$

for any constants $C_{1}, C_{2}$. We therefore choose the constants $C_{1}$ and $C_{2}$ such that

$$
f(0)=1, \quad f(1)=1 .
$$

We need

$$
\left\{\begin{array}{l}
C_{1}+C_{2}=1 \\
\left(\frac{1+\sqrt{5}}{2}\right) C_{1}+\left(\frac{1-\sqrt{5}}{2}\right) C_{2}=1 .
\end{array}\right.
$$

Solving this system we obtain

$$
C_{1}=\frac{\sqrt{5}+1}{2 \sqrt{5}} \quad \text { and } \quad C_{2}=\frac{\sqrt{5}-1}{2 \sqrt{5}}
$$

and hence the desired solution is

$$
F_{n}=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}
$$

Note: In general, the associated characteristic equation has the form

$$
r^{k}-\beta_{1} r^{k-1}-\beta_{2} r^{k-2}-\cdots-\beta_{k-1} r-\beta_{k}=0,
$$

and its roots can be :

- real and distinct ;
- real and repeated ;
- complex.

Here, we shall deal only with the first two cases.
Distinct real roots : If $r_{1}, r_{2}, \ldots, r_{k}$ are distinct real roots of the characteristic equation, then for each $i \in[k]$ and each constant $C_{i}$, we have $C_{i} r_{i}^{n}$ as a solution (of the given recurrence relation) and it is easy to verify that

$$
a_{n}=C_{1} r_{1}^{n}+C_{2} r_{2}^{n}+\cdots+C_{k} r_{k}^{n}
$$

is also a solution. The presence of $k$ arbitrary constants $C_{1}, C_{2}, \ldots, C_{k}$ allows to find values for the $C_{i}$ such that the initial conditions are satisfied.

Repeated real roots : If a root $r$ is repeated $m$ times, then it is easy to verify that

$$
r^{n}, n r^{n}, n^{2} r^{n}, \ldots, n^{m-1} r^{n}
$$

are solutions and so also is

$$
a_{n}=C_{1} r^{n}+C_{2} n r^{n}+\cdots+C_{m} n^{m-1} r^{n}
$$

6.3.4 Example. Consider the linear recurrence relation

$$
a_{n+2}=4 a_{n+1}-4 a_{n} ; \quad a_{1}=1, a_{2}=3
$$

Its characteristic equation is

$$
r^{2}-4 r+4=0
$$

and the roots of this equation are $r_{1}=r_{2}=2$. Thus, for any constants $C_{1}, C_{2}$

$$
a_{n}=C_{1} 2^{n}+C_{2} n 2^{n}
$$

is a solution of our recurrence relation. We set

$$
\left\{\begin{aligned}
2 C_{1}+2 C_{2} & =1 \\
4 C_{1}+8 C_{2} & =3
\end{aligned}\right.
$$

and solve to obtain

$$
C_{1}=\frac{1}{4} \quad \text { and } \quad C_{2}=\frac{1}{4}
$$

The solution is therefore

$$
a_{n}=\frac{1}{4} 2^{n}+\frac{1}{4} n 2^{n}=\frac{(n+1) 2^{n}}{4}
$$

## The non-homogeneous case

We now investigate recurrence relations of the form

$$
a_{n}=\beta_{1} a_{n-1}+\beta_{2} a_{n-2}+\varphi(n)
$$

In other words, we are restricting ourselves to (non-homogeneous) linear recurrence relations of degree two. For such a recurrence relation, let us call

$$
a_{n}=\beta_{1} a_{n-1}+\beta_{2} a_{n-2}
$$

the associated homogeneous equation.
The following facts are easy to prove.
(1) If $f_{1}(n)$ and $f_{2}(n)$ are solutions, then

$$
f(n)=C_{1} f_{1}(n)+C_{2} f_{2}(n)
$$

is also a solution, for any constants $C_{1}$ and $C_{2}$.
(2) Suppose that $p(n)$ is a particular solution. Then $f(n)+p(n)$ is also a solution.

Based on these two simple facts one obtains an important result (given bellow without proof).
6.3.5 TheOrem. Consider the recurrence relation

$$
a_{n}=\beta_{1} a_{n-1}+\beta_{2} a_{n-2}+\varphi(n) ; \quad a_{0}=\gamma_{0}, a_{1}=\gamma_{1}
$$

Let $f_{1}(n)$ and $f_{2}(n)$ be solutions of the associated homogeneous equation such that $f_{2}(n)$ is not just a multiple of $f_{1}(n)$. Let $p(n)$ be a particular
solution. Then a sequence $\left(a_{n}\right)$ is a solution of the given recurrence relation if and only if

$$
a_{n}=C_{1} f_{1}(n)+C_{2} f_{2}(n)+p(n)
$$

where $C_{1}$ and $C_{2}$ are constants.

Note : (1) We call

$$
a_{n}=C_{1} f_{1}(n)+C_{2} f_{2}(n)+p(n)
$$

the general solution of the recurrence relation.
(2) Finding the particular solution $p(n)$ has to be done by inspection, since there is no general method available. If $\varphi(n)$ is a polynomial, we try for a polynomial $p(n)$ (of the same degree or higher). If $\varphi(n)$ is an exponential function, we try to find a similar function for $p(n)$. For example, if $\varphi(n)=3 \cdot 2^{n}$, we try for a function of the form $A \cdot 2^{n}$ or $(A n+B) \cdot 2^{n}$.
6.3.6 EXAMPLE. The recurrence relation

$$
s_{n}=2 s_{n-1}+1 ; \quad s_{1}=1
$$

has $f(n)=C 2^{n}$ as a solution of the associated homogeneous equation, for any constant $C$. Since $\varphi(n)=1$ is a polynomial of degree 0 , we seek a constant $A$ such that $p(n)=A$ is a particular solution. We obtain $A-2 A=1$ and so $A=-1$. Thus $p(n)=-1$ is a particular solution and so

$$
s_{n}=C 2^{n}-1
$$

is a solution of the given recurrence relation, for any constant $C$. If we set $1=C 2^{1}-1$ then $C=1$ and so

$$
s_{n}=2^{n}-1
$$

is the solution which satisfies the initial condition.
6.3.7 EXAMPLE. The recurrence relation

$$
a_{n}=-2 a_{n-1}-a_{n-2}+2^{n} ; \quad a_{1}=0, a_{2}=0
$$

has, as a solution to the associated homogeneous equation,

$$
f(n)=C_{1}(-1)^{n}+C_{2} n(-1)^{n} .
$$

To find a particular solution, we seek $k$ such that $p(n)=k 2^{n}$ is a solution.
We therefore require of $k$ that

$$
k 2^{n}+2 k 2^{n-1}+k 2^{n-2}=2^{n}
$$

and this means that

$$
k 2^{n-2}(4+4+1)=2^{n} .
$$

Thus

$$
k=\frac{2^{n}}{9 \cdot 2^{n-2}}=\frac{4}{9} .
$$

Thus

$$
a_{n}=C_{1}(-1)^{n}+C_{2} n(-1)^{n}+\frac{4}{9} \cdot 2^{n}
$$

is the general solution. We set

$$
\left\{\begin{array}{l}
-C_{1}-C_{2}+\frac{8}{9}=0 \\
C_{1}+2 C_{2}+\frac{16}{9}=0
\end{array}\right.
$$

and solve, to obtain $C_{1}=\frac{32}{9} \quad$ and $\quad C_{2}=-\frac{8}{3}$. Thus

$$
a_{n}=\frac{32}{9} \cdot(-1)^{n}-\frac{8}{3} \cdot n(-1)^{n}+\frac{4}{9} \cdot 2^{n}=\frac{(-1)^{n} 4}{9}\left(8-6 n+(-2)^{n}\right)
$$

is the solution of the given (complete) recurrence relation.
6.3.8 EXAMPLE. The recurrence relation

$$
a_{n}=-2 a_{n-1}+n+3 ; \quad a_{1}=\frac{5}{9}
$$

has $f(n)=C(-2)^{n}$ as solution to the associated homogeneous equation and, to find a particular solution, we try

$$
p(n)=A n+B
$$

We set

$$
(A n+B)+2(A(n-1)+B)=n+3
$$

Thus

$$
(3 A) n+(3 B-2 A)=1 n+3
$$

and, by comparing coefficients, we obtain

$$
3 A=1, \quad 3 B-2 A=3
$$

Thus $A=\frac{1}{3}$ and $B=\frac{11}{9}$. Therefore

$$
a_{n}=C(-2)^{n}+\frac{1}{3} \cdot n+\frac{11}{9}
$$

is the general solution. Setting

$$
-2 C+\frac{1}{3}+\frac{11}{9}=\frac{5}{9}
$$

yields $C=\frac{1}{2}$ and so

$$
a_{n}=\frac{1}{2} \cdot(-2)^{n}+\frac{1}{3} \cdot n+\frac{11}{9}
$$

is the solution which satisfies the given initial condition.

### 6.4 Exercises

Exercise 76 Show that each of the following sequences satisfies the given recurrence relation.
(a) $\left(a_{n}\right)_{n \geq 0}$, where $a_{n}=3 n+1 ; \quad a_{n}=a_{n-1}+3$ for $n \geq 1$.
(b) $\left(b_{n}\right)_{n \geq 0}$, where $b_{n}=5^{n} ; \quad b_{n}=5 b_{n-1}$ for $n \geq 1$.
(c) $\left(c_{n}\right)_{n \geq 0}$, where $c_{n}=2^{n}-1 ; \quad c_{n}=2 c_{n-1}+1 \quad$ for $n \geq 1$.
(d) $\left(d_{n}\right)_{n \geq 0}, \quad$ where $\quad d_{n}=\frac{(-1)^{n}}{n!} ; \quad d_{n}=-\frac{d_{n-1}}{n} \quad$ for $n \geq 1$.

Exercise 77 Describe each of the following sequences recursively. Include initial conditions.
(a) $a_{n}=5^{n}$.
(b) $b_{n}=1+2+3+\cdots+n$.
(c) $c_{n}=(-1)^{n}$.
(d) $\delta_{n}=\sqrt{2}$.
(e) $0,1,0,1,0,1, \ldots$
(f) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots$
(g) $0.1,0.11,0.111,0.1111, \ldots$.

Exercise 78 Each of the following sequences is defined recursively. Use iteration to guess an explicit formula for the sequence.
(a) $a_{n}=\frac{a_{n-1}}{1+a_{n-1}} \quad$ for $n \geq 1 ; \quad a_{0}=1$.
(b) $b_{n}=2 b_{n-1}+3$ for $n \geq 2 ; \quad b_{1}=2$.
(c) $c_{n}=c_{n-1}+2 n+1 \quad$ for $n \geq 1 ; \quad c_{0}=0$.
(d) $d_{n}=2^{n}-d_{n-1} \quad$ for $n \geq 1 ; \quad d_{0}=1$.
(e) $e_{n}=3 e_{n-1}+n$ for $n \geq 2 ; \quad e_{1}=1$.
(f) $f_{n}=f_{n-1}+n^{2}$ for $n \geq 2 ; \quad f_{1}=1$.
(g) $g_{n}=n g_{n-1} \quad$ for $n \geq 1 ; \quad g_{0}=5$.

## Exercise 79

(a) A sequence is defined recursively as follows:

$$
s_{n}=n-s_{n-1} \quad \text { for } n \geq 1 ; \quad s_{0}=0
$$

Use iteration to guess an explicit formula for the sequence, and then check by mathematical induction that the obtained formula is correct.
(b) A sequence is defined recursively as follows :

$$
t_{n}=t_{n-1} \cdot t_{n-2} \quad \text { for } n \geq 2 ; \quad t_{0}=t_{1}=2
$$

Use iteration to guess an explicit formula for the sequence, and then check by mathematical induction that the obtained formula is correct.

Exercise 80 Assume that the population of the world in 1995 is 7 billion and is growing $3 \%$ a year.
(a) Set up a recurrence relation for the population of the world $n$ years after 1995.
(b) Find an explicit formula for the population of the world $n$ years after 1995.
(c) What will the population of the world be in 2010 ?

## Exercise 81

(a) Find a recurrence relation for the number of bit strings of length $n$ that contain three consecutive 0s.
(b) What are the initial conditions ?
(c) How many bit strings of length seven contain three consecutive 0s?

Exercise 82 Messages are transmitted over a communications channel using two signals. The transmittal of one signal requires 1 microsecond, and the transmittal of the other signal requires 2 microseconds.
(a) Find a recurrence relation for the number of different messages consisting of sequences of these two signals, where each signal in the message is immediately followed by the next signal, that can be sent in $n$ microseconds.
(b) What are the initial conditions ?
(c) How many different messages can be sent in 10 microseconds using these two signals ?

Exercise 83 Suppose that inflation continues at $8 \%$ annually. (That is, an item that costs R 1.00 now will cost R 1.08 next year.) Let $R_{n}=$ the value (that is, the purchasing power) of one Rand after $n$ years.
(a) Find a recurrence relation for $R_{n}$.
(b) What is the value of R 1.00 after 20 years ?
(c) What is the value of R 1.00 after 80 years ?
(d) If inflation were to continue at $10 \%$ annually, find the value of R 1.00 after 20 years.
(e) If inflation were to continue at $10 \%$ annually, find the value of R 1.00 after 80 years.

Exercise 84 Solve the following recurrence relations.
(a) $a_{n}=5 a_{n-1}-6 a_{n-2}+2 n+1 ; \quad a_{1}=0, a_{2}=0$.
(b) $b_{n}=4 b_{n-1}-4 b_{n-2}+n^{2} ; \quad b_{1}=1, b_{2}=40$.
(c) $c_{n}=\frac{1}{4} c_{n-2} ; \quad c_{0}=1, c_{1}=0$.
(d) $d_{n}=d_{n-2}+10^{n} ; \quad d_{1}=\frac{1000}{99}, d_{2}=\frac{10198}{99}$.
(e) $e_{n}=e_{n-1}+n ; \quad e_{0}=1$.
(f) $L_{n}=L_{n-1}+L_{n-2} ; \quad L_{0}=2, L_{1}=1$
(g) $f_{n}=4 f_{n-1}-4 f_{n-2}+2^{n} ; \quad f_{1}=2, f_{2}=8$.
(h) $a_{n}+3 a_{n-1}=4 n^{2}-2 n+2^{n}$

## Exercise 85 (Compound Interest)

(a) The fractional increase in a quantity $y$, per unit of time, is $r$. Let $y=$ $y(n)$ denote the value of $y$ after $n$ units of time. Find an expression for $y=y(n)$.
(b) Money earns $i \%$ interest per year for $n$ years. Let $y=y(n)$ denote the value after $n$ years. Find an expression for $y=y(n)$.
(c) An initial deposit of $R 1000$ earns $10 \%$ interest per year for fifty years. Find the final value.
(d) An initial deposit of $R 1000$ earns $11 \%$ interest per year for fifty years. Find the final value. (Note the difference !)

Exercise 86 (Annuities : Periodic Fixed Deposits)
(a) The fractional increase in a quantity $y$, per unit time is $r$ and $y$ also increases by an increment $d$ per unit of time. Let $y=y(n)$ denote the value of $y$ after $n$ units of time. Find an expression for $y=y(n)$.
(b) A student, aged twenty, decides to quit smoking and estimates that this saves $R 1000$ per year. He makes an initial deposit of $R 1000$ and deposits $R 1000$ per year thereafter. Suppose that he earns $10 \%$ interest per year. Find the value of the annuity after fifty years.
(c) Compare the final amounts when the initial deposit is $R 1000, d=1000$, $n=50$ and $r$ is, in the one case $10 \%$ and in the other case $11 \%$. (There is a $69 \%$ difference in the final amount for a $1 \%$ difference in interest rate!)

Exercise 87 (Annuities : Periodic Increasing Deposits)
(a) The fractional increase in a quantity $y$, per unit time is $r$. After each unit of time, there is an increment $d$ to $y$ and the fractional increase in $d$, per unit of time is $i$. Let $y=y(n)$ denote the value of $y$ after $n$ units of time. Find an expression for $y=y(n)$.
(b) A student, aged twenty, decides to quit smoking and estimates that this saves $R 1000$ per year initially. He estimates that the price of cigarettes will increase at $15 \%$ per year so he makes an initial deposit of $R 1000$ and makes deposits which increase at $15 \%$ per year thereafter. Suppose that he earns $10 \%$ interest per year. Find the value of the annuity after fifty years.

## Exercise 88 (Amortisation)

(a) Let $y=y(n)$ denote the amount outstanding on a loan. So $y(0)$ is the initial loan. The loan is amortised by means of a sequence of equal payments. Let each payment be $d$. Find an expression for $y=y(n)$ if the interest rate is $r=i / 100$.
(b) Find the payment $d$ which is required to amortise a loan of $R 100000$ over 360 monthly payments if the interest rate is $1 \%$ per month.

## Exercise 89 (Population Problems)

(a) A population grows in such a way that the fractional change in population per period is constant. Let $p_{n}$ be the population after time period $n$. Show that there exists $r \in[0,1]$ such that

$$
p_{n+1}=(1+r) p_{n} .
$$

If the initial population is $p_{0}$, show that

$$
p_{n}=p_{0}(1+r)^{n} .
$$

(b) Suppose now that the population is culled, by an amount $d$, at the end of each time period. Show that :

$$
p_{n}=\left(p_{0}-\frac{d}{r}\right)(1+r)^{n}+\frac{d}{r} .
$$

Find the value of $d$ for which
i. the population vanishes ;
ii. the population remains stable ;
iii. the population still grows.

Exercise 90 (Drug Therapy)
Equal doses, $d$, of a substance are administered at equal time intervals. Let $s_{n}$ denote the amount of the substance in the blood at the beginning of period $n$. Suppose that $s_{1}=d$ and that the fractional decrease, per period, in the amount of the substance in the blood is $r$.
(a) Obtain an expression for $s_{n}$;
(b) Show that $s_{n} \rightarrow \frac{d}{r}$;
(c) If the dose $d=100 \mathrm{mg}$ and the substance decreases by $25 \%$ per day, show that $s_{n} \rightarrow 400$.

