# Chapter 8

# Determinants

#### **Topics** :

- 1. Determinants
- 2. Properties of determinants
- 3. Applications

Determinants are useful in further development of matrix theory and its applications. Throughout the 19<sup>th</sup> century determinants were considered the ultimate tool in linear algebra; recently, determinants have gone somewhat out of fashion. Nevertheless, it is still important to understand what a determinant is and to learn a few of its fundamental properties.

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### 8.1 Determinants

In section 7.3 we found a criterion for the invertibility of a  $2 \times 2$  matrix :

the matrix 
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
 is invertible if (and only if)  $ad - bc \neq 0$ .

**8.1.1** DEFINITION. The number ad - bc is called the **determinant** of the matrix A.

There are several common notations for determinants :

$$\det (A) = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

NOTE : The determinant is actually a *function* that associates with each square matrix (of order 2) the number  $\det(A)$ .

If the matrix A is invertible, then its inverse can be expressed in terms of the determinant :

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

It is natural to ask whether the concept of a determinant can be generalized to square matrices of arbitrary size. Can we assign a number  $\det(A)$  to any square matrix A (expressed in terms of the entries of A), such that A is invertible if (and only if)  $\det(A)$  ?

#### The determinant of a $3 \times 3$ matrix

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

be a  $3 \times 3$  matrix. The following formula for the *determinant* of A may be obtained (by means of geometric considerations or otherwise) :

 $\det\left(A\right) = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}.$ 

Since the formula for the determinant of a  $3 \times 3$  matrix is rather long, we may wonder how we can memorize it. Here is a convenient rule (stated by PIERRE F. SARRUS (1798-1861)) :

**Sarrus' Rule :** To find the determinant of a  $3 \times 3$  matrix A, write the first rows of A under A. Then multiply the entries along the six diagonals thus formed.

Add or subtract these diagonal products as shown in the diagram.

 $\det (A) = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$  $-a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}.$ 

8.1.2 EXAMPLE. Find

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

SOLUTION : We have

$$\det(A) = 1 \cdot 5 \cdot 10 + 4 \cdot 8 \cdot 3 + 2 \cdot 6 \cdot 7 - 3 \cdot 5 \cdot 7 - 6 \cdot 8 \cdot 1 - 10 \cdot 2 \cdot 4 = -3.$$

This matrix is invertible.

8.1.3 EXAMPLE. Find the determinant of the upper triangular matrix

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

SOLUTION : We find that det(A) = adf because all other contributions in Sarrus' formula are zero.

NOTE : The determinant of an upper (or lower) triangular  $3 \times 3$  matrix is the product of its diagonal entries.

#### The determinant of an $n \times n$ matrix

We may be tempted to define the determinant of an  $n \times n$  matrix by generalizing Sarrus' rule. For a  $4 \times 4$  matrix, a naïve generalization of Sarrus' rule produces the expression :

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a_{11}a_{22}a_{33}a_{44} + \dots + a_{14}a_{21}a_{32}a_{41} - a_{14}a_{23}a_{32}a_{41} - \dots - a_{13}a_{22}a_{31}a_{44} \quad (8 \text{ terms}).
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For example, for the *invertible* matrix

1	0	0	0
0	1	0	0
0	0	0	1
0	0	1	0

the expression given by a "generalization" of Sarrus' rule is 0. This shows that we cannot define the determinant by generalizing Sarrus' rule in this way : recall that the determinant of an invertible matrix must be nonzero.

We have a look for a more subtle structure in the formula

$$\det (A) = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$$
$$-a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

for the determinant of a  $3 \times 3$  matrix. Note that each of the six terms in this expression is a product of three factors involving exactly one entry from each row and column of the matrix. For lack of a better word, we call such a choice of a number in each row and column of a square matrix a **pattern** in the matrix. Observe that each pattern corresponds to a *permutation* on

3 elements. For example, the *diagonal pattern* – were we choose all diagonal entries  $a_{ii}$  – corresponds to the identity  $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ . Clearly, there are 3! = 6 such patterns.

When we compute the determinant of a  $3 \times 3$  matrix, the product associated with a pattern (and hence a permutation) is added if the permutation is even and is substracted if the permutation is odd. Using this observation as a guide, we now define the determinant of a general  $n \times n$  matrix.

**8.1.4** DEFINITION. For an  $n \times n$  matrix  $A = [a_{ij}]$ , the **determinant** of A is defined to be the number

$$\det(A) := \sum_{\alpha \in S_n} \operatorname{sgn}(\alpha) \ a_{1\alpha(1)} a_{2\alpha(2)} \cdots a_{n\alpha(n)}.$$

NOTE : The determinant of a non-square matrix is not defined.

**8.1.5** EXAMPLE. When A is a  $2 \times 2$  matrix, there are 2! = 2 patterns (permutations on 2 elements), namely (in cycle notation)  $\iota = (1)(2)$  and (1,2). So det (A) contains two terms :

$$\operatorname{sgn}(\iota) a_{11}a_{22}$$
 and  $\operatorname{sgn}((1,2)) a_{12}a_{21}$ .

Since sgn  $(\iota) = +1$  and sgn ((1, 2)) = -1, we obtain the familiar formula

$$\det (A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

**8.1.6** EXAMPLE. Find det(A) for

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

SOLUTION : The diagonal pattern makes the contribution  $2 \cdot 3 \cdot 5 \cdot 7 \cdot 9 = 1890$ . All other patterns contain at least one zero and will therefore make no contribution toward the determinant. We can conclude that

$$\det(A) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 9 = 1\,890.$$

NOTE : More generally, the determinant of a diagonal matrix is the product of the diagonal entries of the matrix.

8.1.7 EXAMPLE. Evaluate

$$\det \begin{bmatrix}
 1 & 2 & 3 & 4 & 5 \\
 0 & 2 & 3 & 4 & 5 \\
 0 & 0 & 3 & 4 & 5 \\
 0 & 0 & 0 & 4 & 5 \\
 0 & 0 & 0 & 0 & 5
 \end{bmatrix}$$

SOLUTION : Note that the matrix is upper triangular. The diagonal pattern makes the contribution  $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$ . Any pattern other than the diagonal pattern contains at least one entry below the diagonal, and the contribution the pattern makes to the determinant is therefore 0. We conclude that

$$\det(A) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$$

We can easily generalize this result :

**8.1.8** PROPOSITION. The determinant of an (upper or lower) triangular matrix is the product of the diagonal entries of the matrix.

# 8.2 Properties of determinants

The main goal of this section is to show that a square matrix of any size is invertible if (and only if) its determinant is nonzero. As we work toward this goal, we will discuss a number of other properties of the determinant that are of interest in their own right.

#### Determinant of the transpose

8.2.1 EXAMPLE. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \\ 2 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \end{bmatrix}.$$

Express det  $(A^T)$  in terms of det (A). You need not compute det (A).

SOLUTION: For each pattern in A we can consider the corresponding (transposed) pattern in  $A^T$ . The two patterns (in A and  $A^T$ ) - viewed as permutations on 5 elements – are inverse to each other. But a permutation and its inverse have the same signature, and thus the two patterns make the same contributions to the respective determinants. Since these observations apply to all patterns of A, we can conclude that det  $(A^T) = \det(A)$ .

Since we have not used any special properties of the matrix A in example above, we can state more generally :

**Property 1 :** If  $A^T$  is the transpose of the matrix A, then det $(A^T) = det(A)$ .

NOTE : Any property of the determinant expressed in terms of rows holds for the columns as well, and vice versa.

#### Linearity properties of the determinant

**8.2.2** EXAMPLE. Consider the matrix

$$B = \begin{bmatrix} 1 & x_1 + y_1 & 4 \\ 2 & x_2 + y_2 & 5 \\ 3 & x_3 + y_3 & 6 \end{bmatrix}.$$

Express  $\det(B)$  in terms of

$$\det \begin{bmatrix} 1 & x_1 & 4 \\ 2 & x_2 & 5 \\ 3 & x_3 & 6 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} 1 & y_1 & 4 \\ 2 & y_2 & 5 \\ 3 & y_3 & 6 \end{bmatrix}.$$

 ${\small Solution}: {\small \ We \ have }$ 

$$det (B) = 3(x_1 + y_1) - 6(x_2 + y_2) + 3(x_3 + y_3)$$
  
=  $(3x_1 - 6x_2 + 3x_3) + (3y_1 - 6y_2 + 3y_3)$   
=  $det \begin{bmatrix} 1 & x_1 & 4 \\ 2 & x_2 & 5 \\ 3 & x_3 & 6 \end{bmatrix} + det \begin{bmatrix} 1 & y_1 & 4 \\ 2 & y_2 & 5 \\ 3 & y_3 & 6 \end{bmatrix}.$ 

8.2.3 EXAMPLE. Consider the matrix

$$B = \begin{bmatrix} 1 & kx & 4 \\ 2 & ky & 5 \\ 3 & kz & 6 \end{bmatrix}.$$

Express  $\det(B)$  in terms of

$$\det \begin{bmatrix} 1 & x & 4 \\ 2 & y & 5 \\ 3 & z & 6 \end{bmatrix}.$$

Solution : We have

$$\det (B) = 3kx - 6ky + 3kz = k(3x - 6y + 3z) = k \cdot \det \begin{bmatrix} 1 & x & 4 \\ 2 & y & 5 \\ 3 & z & 6 \end{bmatrix}.$$

NOTE : The mapping

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \det \begin{bmatrix} 1 & x & 4 \\ 2 & y & 5 \\ 3 & z & 6 \end{bmatrix}$$

(satisfying the above two properties) is said to be *linear* (see section 10.1).

We can generalize :

**Property 2 :** Suppose that the matrices  $A_1, A_2$ , and B are identical, except for their  $j^{th}$  column, and that the  $j^{th}$  column of B is the sum of the  $j^{th}$  columns of  $A_1$  and  $A_2$ . Then

$$\det (B) = \det (A_1) + \det (A_2).$$

This result also holds if rows are involved instead of columns.

**Property 3**: Suppose that the matrices A and B are identical, except for their  $j^{th}$  column, and that the  $j^{th}$  column of B is k times the  $j^{th}$  column of A. Then

$$\det\left(B\right) = k \cdot \det\left(A\right).$$

This result also holds if rows are involved instead of columns.

#### Elementary row operations and determinants

Suppose we have to find the determinant of a  $20 \times 20$  matrix. Since there are  $20! \approx 2 \cdot 10^{18}$  patterns in this matrix, we would have to perform more than  $10^{19}$  multiplications to compute the determinant using DEFINITION 8.1.4. Even if a computer performed 1 billion multiplications a second, it would still take over 1000 years to carry out these computations. Clearly, we have to look for more efficient ways to compute the determinant.

So far, we have found Gaussian elimination (reduction) to be a powerful tool for solving numerical problems in *linear algebra*. If we could understand what happens to the determinant of a matrix as we *row-reduce* it, we could use Gaussian elimination to compute determinants as well.

We have to understand what happens to the determinant of a matrix as we perform the three elementary row operations : (a) swapping two rows, (b) multiplying a row by a scalar, and (c) adding a multiple of a row to another row. One can prove the following results.

**Property 4**: If B is obtained from A by a row swap, then

 $\det\left(B\right) = -\det\left(A\right).$ 

**Property 5 :** If B is obtained from A by multiplying a row of A by a scalar k, then

$$\det\left(B\right) = k \cdot \det\left(A\right).$$

**Property 6 :** If B is obtained from A by adding a multiple of a row of A to another row, then

$$\det\left(B\right) = \det\left(A\right).$$

NOTE : Analogous results hold for elementary column operations.

**8.2.4** EXAMPLE. If a matrix A has two equal rows, what can we say about det (A)?

SOLUTION : Swap the two equal rows and call the resulting matrix B. Since we have swapped two equal rows, we have B = A. Now

$$\det (A) = \det (B) = -\det (A)$$

so that

$$\det\left(A\right)=0.$$

Now that we understand how elementary row operations affect determinants, we can describe the relationship between the determinant of a matrix A and that of its reduced row echelon form rref (A).

Suppose that in the course of the row-reduction we swap rows s times and *divide* various rows by scalars  $k_1, k_2, \ldots, k_r$ . Then

det (rref (A)) = 
$$(-1)^s \frac{1}{k_1 k_2 \cdots k_r} \det (A)$$

or

$$\det (A) = (-1)^{s} k_1 k_2 \cdots k_r \det (\operatorname{rref} (A)).$$

Let us examine the cases when A is invertible and when it is not.

If A is invertible, then rref  $(A) = I_n$ , so that det (rref (A)) = 1, and

$$\det\left(A\right) = (-1)^{s} k_1 k_2 \cdots k_r.$$

Observe that this quantity is not 0 because all the scalars  $k_i$  are different from 0.

If A is not invertible, then rref (A) is an upper triangular matrix with some zeros on the diagonal, so that det (rref (A)) = 0 and det (A) = 0. We have established the following fundamental result :

**8.2.5** PROPOSITION. A square matrix A is invertible if and only if det  $(A) \neq 0$ .

If A is invertible, the discussion above also produces a convenient method to compute the determinant :

**Algorithm :** Consider an invertible matrix. Suppose you swap rows s times as you row-reduce A and you divide various rows by the scalars  $k_1, k_2, \ldots, k_r$ . Then

$$\det (A) = (-1)^s k_1 k_2 \cdots k_r \,.$$

NOTE: Here, it is not necessary to reduce A all the way to rref (A). It suffices to bring A into upper triangular form with 1's on the diagonal.

8.2.6 EXAMPLE. Evaluate

$$\begin{vmatrix} 0 & 2 & 4 & 6 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{vmatrix}.$$

Solution : We have

$$\begin{vmatrix} 0 & 2 & 4 & 6 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 6 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 1 \end{vmatrix}$$
$$= 2\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{vmatrix} = 2(-1)(-2)\begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 4.$$

We made two swaps and performed row divisions by 2, -1, -2, so that

 $\det (A) = (-1)^2 \cdot 2 \cdot (-1) \cdot (-2) = 4.$ 

#### The determinant of a product

Consider two  $n \times n$  matrices A and B. What is the relationship between  $\det(A)$  and  $\det(B)$ ?

First suppose that A is invertible. On can show that

$$\operatorname{rref} \begin{bmatrix} A & AB \end{bmatrix} = \begin{bmatrix} I_n & B \end{bmatrix}$$

Suppose we swap rows s times and divide rows by  $k_1, k_2, \ldots, k_r$  as we perform the elimination.

Considering the left and right "halves" of the matrices separately, we conclude that

$$\det (A) = (-1)^{s} k_1 k_2 \cdots k_r \det (I_n) = (-1)^{s} k_1 k_2 \cdots k_r$$

and

$$\det (AB) = (-1)^s k_1 k_2 \cdots k_r \det (B) = \det (A) \cdot \det (B).$$

Therefore,  $\det(AB) = \det(A) \cdot \det(B)$  when A is invertible. If A is not invertible, then neither is AB, so that  $\det(AB) = \det(A) \cdot \det(B) = 0$ .

We have obtained the following result :

**Property 7**: If A and B are square matrices, then

$$\det (AB) = \det (A) \cdot \det (B).$$

**8.2.7** EXAMPLE. If A is an invertible  $n \times n$  matrix, what is the relationship between det (A) and det  $(A^{-1})$ ?

SOLUTION : By definition of the inverse matrix, we have

$$AA^{-1} = I_n \,.$$

By taking the determinant of both sides, we find that

$$\det (AA^{-1}) = \det (A) \cdot \det (A^{-1}) = \det (I_n) = 1.$$

NOTE : If A is an *invertible* matrix, then

$$\det\left(A^{-1}\right) = \frac{1}{\det\left(A\right)} \, \cdot \,$$

**8.2.8** EXAMPLE. If S is an invertible  $n \times n$  matrix, and A an arbitrary  $n \times n$  matrix, what is the relationship between det (A) and det  $(S^{-1}AS)$ ? SOLUTION : We have

$$det (S^{-1}AS) = det (S^{-1}) \cdot det (A) \cdot det (S)$$
$$= (det (S))^{-1} \cdot det (A) \cdot det (S)$$
$$= det (A).$$

Thus,  $\det(S^{-1}AS) = \det(A)$ .

#### Laplace expansion

Recall the formula

 $\det\left(A\right) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$ 

for the determinat of a  $3 \times 3$  matrix. Collecting the two terms involving  $a_{11}$  and then those involving  $a_{21}$  and  $a_{31}$ , we can write :

$$det (A) = a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{21}(a_{32}a_{13} - a_{12}a_{33}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}).$$

Let us analyze the structure of this formula more closely. The terms  $(a_{22}a_{33} - a_{32}a_{23})$ ,  $(a_{32}a_{13} - a_{12}a_{33})$ , and  $(a_{12}a_{23} - a_{22}a_{13})$  can be thought of as the determinants of *submatrices* of A, as follows. The expression  $a_{22}a_{33} - a_{32}a_{23}$  is the determinant of the matrix we get when we omit the first row and thew first column of A; likewise for the other summands.

To state these observations more succintly, we introduce some terminology.

**8.2.9** DEFINITION. Let  $A = [a_{ij}]$  be an  $n \times n$  matrix. The (i, j) minor of A is the determinant  $M_{ij}$  of the  $(n-1) \times (n-1)$  submatrix that remains after deleting the  $i^{th}$  row and the  $j^{th}$  column of A. The (i, j) cofactor  $A_{ij}$  of A is defined to be

$$A_{ij} := (-1)^{i+j} M_{ij}.$$

8.2.10 EXAMPLE. Let

$$A = \begin{bmatrix} 5 & -2 & -3 \\ 4 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

Then

$$A_{11} = M_{11} = \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}, A_{12} = -M_{12} = -\begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}, A_{13} = M_{13} = \begin{vmatrix} 4 & 0 \\ 3 & -1 \end{vmatrix},$$
$$A_{21} = -M_{21} = -\begin{vmatrix} -2 & -3 \\ -1 & 2 \end{vmatrix}, A_{22} = M_{22} = \begin{vmatrix} 5 & -3 \\ 3 & 2 \end{vmatrix}, A_{23} = -M_{23} = -\begin{vmatrix} 5 & -2 \\ 3 & -1 \end{vmatrix},$$
$$A_{31} = M_{31} = \begin{vmatrix} -2 & -3 \\ 0 & 1 \end{vmatrix}, A_{32} = -M_{32} = -\begin{vmatrix} 5 & -3 \\ 4 & 1 \end{vmatrix}, A_{33} = M_{33} = \begin{vmatrix} 5 & -2 \\ 4 & 0 \end{vmatrix}.$$

We can now represent the determinant of a  $3 \times 3$  matrix more succinctly :

$$\det\left(A\right) = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}.$$

This representation of the determinant is called the **Laplace expansion of** det(A) **down the first column** (named after the French mathematician PIERRE-SIMON LAPLACE (1749-1827)).

Likewise, we can expand along the first row (since  $\det(A^T) = \det(A)$ ):

$$\det\left(A\right) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

In fact, we can expand along *any* row or down *any* column (we can verify this directly, or argue in terms of row and column swap).

**Laplace expansion :** The determinant det  $(A) = |a_{ij}|$  of an  $n \times n$  matrix  $A = [a_{ij}]$  can be computed by Laplace expansion along any row or down any column.

Expansion along the  $i^{th}$  row :

$$\det (A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}.$$

Expansion down the  $j^{th}$  column :

If

$$\det (A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \dots + a_{nj}A_{nj}.$$

**8.2.11** Example.

$$A = \begin{bmatrix} 5 & -2 & -3 \\ 4 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$

then

$$\det (A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$
  
=  $(5) \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} - (-2) \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix} + (-3) \begin{vmatrix} 4 & 0 \\ 3 & -1 \end{vmatrix}$   
=  $(5)(1) + (2)(5) - (3)(-4)$   
= 27.

8.2.12 EXAMPLE. To evaluate the determinant of

$$A = \left[ \begin{array}{rrrr} 7 & 6 & 0 \\ 9 & -3 & 2 \\ 4 & 5 & 0 \end{array} \right]$$

we expand along the third column because it has only a single nonzero entry. Thus

det 
$$(A) = -(2) \begin{vmatrix} 7 & 6 \\ 4 & 5 \end{vmatrix} = (-2)(35 - 24) = -22.$$

NOTE : Computing the determinant using Laplace expansion is a bit more efficient than using the definition of the determinant, but is a lot less efficient than Gaussian elimination.

# 8.3 Applications

### Cramer's rule

Suppose that we need to solve the linear system

$$Ax = b$$

where

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

We assume that the coefficient matrix A is invertible, so we know in advance that a unique solution x exists. The question is how to write x explicitly in terms of the coefficients  $a_{ij}$  and the constants  $b_i$ . In the following discussion we think of x as a fixed (though as yet unknown) column vector.

If we denote by  $a_1, a_2, \ldots, a_n$  the column vectors of the matrix A, then

$$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$$

The column vector b is expressed in terms of the entries  $x_1, x_2, \ldots, x_n$  of the solution vector x and the columns vectors of A by

$$b = \sum_{j=1}^{n} x_j a_j.$$

The *trick* for finding the  $i^{th}$  unknown  $x_i$  is to compute the determinant of the matrix

$$\begin{bmatrix} a_1 & \dots & b & \dots & a_n \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & b_1 & \dots & a_{1n} \\ a_{21} & \dots & b_2 & \dots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & b_n & \dots & a_{nn} \end{bmatrix}$$

that we obtain by replacing the  $i^{th}$  column  $a_i$  of A with the column vector b. We find that

$$\det \begin{bmatrix} a_1 & \dots & b & \dots & a_n \end{bmatrix} = \det \begin{bmatrix} a_1 & \dots & \sum_{j=1}^n x_j a_j & \dots & a_n \end{bmatrix}$$
$$= \sum_{j=1}^n \det \begin{bmatrix} a_1 & \dots & x_j a_j & \dots & a_n \end{bmatrix}$$
$$= \sum_{j=1}^n x_j \det \begin{bmatrix} a_1 & \dots & a_j & \dots & a_n \end{bmatrix}$$
$$= x_i \det \begin{bmatrix} a_1 & \dots & a_i & \dots & a_n \end{bmatrix}$$
$$= x_i \det (A).$$

We get the desired simple formula for  $x_i$  after we divide each side by det  $(A) \neq 0$ . Thus, we have obtained the following result :

**Cramer's Rule :** Consider the  $n \times n$  linear system

$$Ax = b$$

with

$$A = \begin{bmatrix} a_1 & \dots & a_i & \dots & a_n \end{bmatrix}.$$

If det  $(A) \neq 0$ , then the *i*<sup>th</sup> entry of the unique solution (vector) x is given by

$$x_{i} = \frac{\det \begin{bmatrix} a_{1} & \dots & b_{1} & \dots & a_{n} \end{bmatrix}}{\det (A)} = \frac{1}{\det (A)} \begin{vmatrix} a_{11} & \dots & b_{1} & \dots & a_{1n} \\ a_{21} & \dots & b_{2} & \dots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & b_{n} & \dots & a_{nn} \end{vmatrix}$$

where in the last expression the constant vector b replaces the  $i^{th}$  column vector  $a_i$  of A.

NOTE : This result is due to the Swiss mathematician GABRIEL CRAMER (1704-1752).

8.3.1 EXAMPLE. Use Cramer's rule to solve the system :

$$\begin{cases} x_1 + 4x_2 + 5x_3 = 2\\ 4x_1 + 2x_2 + 5x_3 = 3\\ -3x_1 + 3x_2 - x_3 = 1. \end{cases}$$

 ${\rm Solution}: \ {\rm We \ find \ that}$ 

$$\det(A) = \begin{vmatrix} 1 & 4 & 5 \\ 4 & 2 & 5 \\ -3 & 3 & -1 \end{vmatrix} = 29$$

and then

$$x_{1} = \frac{1}{29} \begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 5 \\ 1 & 3 & -1 \end{vmatrix} = \frac{33}{29}, \quad x_{2} = \frac{1}{29} \begin{vmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \\ -3 & 1 & -1 \end{vmatrix} = \frac{35}{29}$$
  
and  $x_{3} = \frac{1}{29} \begin{vmatrix} 1 & 4 & 2 \\ 4 & 2 & 3 \\ -3 & 3 & 1 \end{vmatrix} = -\frac{23}{29}.$ 

#### The adjoint formula for the inverse matrix

We now use Cramer's rule to develop an explicit formula for the inverse  $A^{-1}$  of the invertible matrix A. First we need to rewrite Cramer's rule more concisely. We have

$$x_{i} = \frac{1}{\det(A)} \left( b_{1}A_{1i} + b_{2}A_{2i} + \dots + b_{n}A_{ni} \right), \quad i = 1, 2, \dots, n$$

because the cofactor of  $b_k$  is simply the (k,i)-cofactor  $A_{ki}$  of A, and so

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det(A)} \begin{bmatrix} b_1 A_{11} + b_2 A_{21} + \dots + b_n A_{n1} \\ b_1 A_{12} + b_2 A_{22} + \dots + b_n A_{n2} \\ \vdots \\ b_1 A_{1n} + b_2 A_{2n} + \dots + b_n A_{nn} \end{bmatrix}$$
$$= \frac{1}{\det(A)} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$
$$= \frac{1}{\det(A)} \begin{bmatrix} A_{ij} \end{bmatrix}^T b.$$

NOTE : The transpose of the cofactor matrix of A is called the **adjoint matrix** of A and is denoted by

$$\operatorname{adj}(A) := \left[A_{ij}\right]^T.$$

With the aid of this notation, Cramer's rule can be written in the especially simple form

$$x = \frac{1}{\det(A)} \operatorname{adj}(A) b.$$

The fact that the formula above gives the *unique* solution  $x (= A^{-1}b)$  of Ax = b implies

$$A^{-1}b = \frac{1}{\det(A)} \operatorname{adj}(A) b$$

and thus

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

Therefore, we have proved the following result :

**8.3.2** PROPOSITION. (THE ADJOINT FORMULA FOR THE INVERSE MATRIX) The inverse of the invertible matrix A is given by the formula

$$A^{-1} = \frac{1}{\det\left(A\right)} \operatorname{adj}\left(A\right)$$

where  $\operatorname{adj}(A)$  is the adjoint matrix of A.

**8.3.3** EXAMPLE. Apply the adjoint formula to find the inverse of the matrix

$$A = \left[ \begin{array}{rrrr} 1 & 4 & 5 \\ 4 & 2 & 5 \\ -3 & 3 & -1 \end{array} \right].$$

Solution : First we calculate the cofactors of A :

$$A_{11} = + \begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} = -17, \quad A_{12} = - \begin{vmatrix} 4 & 5 \\ -3 & -1 \end{vmatrix} = -11, \quad A_{13} = + \begin{vmatrix} 4 & 2 \\ -3 & 3 \end{vmatrix} = 18,$$

$$A_{21} = -\begin{vmatrix} 4 & 5 \\ 3 & -1 \end{vmatrix} = 19, \quad A_{22} = +\begin{vmatrix} 1 & 5 \\ -3 & -1 \end{vmatrix} = 14, \quad A_{23} = -\begin{vmatrix} 1 & 4 \\ -3 & 3 \end{vmatrix} = -15,$$

$$A_{31} = + \begin{vmatrix} 4 & 5 \\ 2 & 5 \end{vmatrix} = 10, \quad A_{32} = - \begin{vmatrix} 1 & 5 \\ 4 & 5 \end{vmatrix} = 15, \quad A_{33} = + \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} = -14.$$

Thus the cofactor matrix of A is

$$[A_{ij}] = \begin{bmatrix} -17 & -11 & 18 \\ 19 & 14 & -15 \\ 10 & 15 & -14 \end{bmatrix}.$$

We next interchange rows and columns to obtain the adjoint matrix

$$\operatorname{adj}(A) = \begin{bmatrix} -17 & 19 & 10\\ -11 & 14 & 15\\ 18 & -15 & -14 \end{bmatrix}.$$

Finally, we divide by det(A) = 29 to get te inverse matrix

$$A^{-1} = \frac{1}{29} \begin{bmatrix} -17 & 19 & 10\\ -11 & 14 & 15\\ 18 & -15 & -14 \end{bmatrix}.$$

NOTE : Just like Cramer's rule, the adjoint formula for the inverse matrix is computationally inefficient and is therefore of more theoretical than practical importance. The Gaussian elimination should always be used to find inverses of  $4 \times 4$  and larger matrices.

# 8.4 Exercises

Exercise 106 Use the determinant to find out which matrices are invertible.

(a) 
$$\begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}$$
.  
(b)  $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ .  
(c)  $\begin{bmatrix} a & b & c \\ 0 & b & c \\ 0 & 0 & c \end{bmatrix}$ .  
(d)  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ .  
(e)  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \end{bmatrix}$ .

**Exercise 107** Find all (real) numbers  $\lambda$  such that the matrix  $A - \lambda I_n$  is not invertible.

(a) 
$$\begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix}$$
.  
(b) 
$$\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$
.

(c)  $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ . (d)  $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ . (e)  $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 8 & -2 \end{bmatrix}$ .

**Exercise 108** For which choices of  $\alpha \in \mathbb{R}$  is the matrix A invertible ?

	$\cos$	$s \alpha$	1	$-\sin \alpha$	
(a)		)	2	0	
	$\sin \alpha$		3	$\cos \alpha$	
	1	1	$\alpha$		
(b)	1	$\alpha$	$\alpha$	•	
	α	$\alpha$	$\alpha$		

**Exercise 109** Use (i) Gaussian elimination *and/or* (ii) Laplace expansion to evaluate the following determinants.

$$\begin{array}{c|ccccc} (a) & \left| \begin{array}{cccc} 4 & 0 & 6 \\ 5 & 0 & 8 \\ 7 & -4 & -9 \end{array} \right| \\ (b) & \left| \begin{array}{ccccc} 0 & 0 & 3 \\ 4 & 0 & 0 \\ 0 & 5 & 0 \end{array} \right| \\ (c) & \left| \begin{array}{ccccc} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{array} \right| \\ (d) & \left| \begin{array}{ccccc} 1 & 0 & 0 & 0 \\ 2 & 0 & 5 & 0 \\ 3 & 6 & 9 & 8 \\ 4 & 0 & 10 & 7 \end{array} \right| \\ \end{array}$$

	0	Ο	1	0	0		
	0	0	T	0	0		
	2	0	0	0	0		
(e)	0	0	0	3	0	•	
	0	0	0	0	4		
	0	5	0	0	0		
	.		0	11	٣		
	و	5 (	J	11	-0	)	0
	-2	2 4	4	13	6	;	$5 \mid$
(f)	(	) (	0	5	0	)	0
	7	7 (	6	-9	17	,	7
	(	) (	0	8	2	2	0

**Exercise 110** Evaluate each given determinant after first simplifying the computation by adding an appropriate multiple of some row or column to another.

(a)	$\begin{array}{c} 1\\ 2\\ 3\end{array}$	$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$	$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$		
(b)	-2	2 2 - 3	$3 \\ -3 \\ 2$	4 1 7	
(c)	3 0 6	-2 [ _2	2 5 : 4 :	5 17 12	
(d)	1 0 0 2	2 5 0 4	3 6 8 6	4 7 9 9	
(e)	2 ( ( 	2 ( ) [ ) ( 4 (	) 1 : ) )	0 11 5 0	-3 12 13 7

**Exercise 111** Show that det(A) = 0 without direct evaluation of the determinant.

(a) 
$$A = \begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ bc & ca & ab \end{bmatrix}$$
.  
(b)  $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{bmatrix}$ .

**Exercise 112** Evaluate the determinants :

(a)	ę	3 4	1	3		
		3 2	2	1	•	
	-;	3 2	2	4		
	3	2	_	3		
(b)	0	3		2		
	2	3	_	5		
	-2	2 5	5	4		
(c)	Ę	5 3	3	1		
, í	1	L	1	5		
		L	0		0	3
(1)	(	)	1	_	-2	0
(d)	-2	2	3	_	-2	3
	(	) -	-3		3	3
	1	-1	1	1	-1	
(a)	1	-1	1	3	2	
(e)	4	4 4	2	1	3	·
	3	ę	3	1	4	
(f)	3	1	_	2	1	
	1	1	_	3	2	
	2	0		2	3	.
	3	3		1	-3	

**Exercise 113** Let A be an  $n \times n$  matrix.

- (a) If det (A) = 3, what is det  $(A^T A)$  ?
- (b) If A is invertible, what can you say about the sign of det  $(A^T A)$ ?

**Exercise 114** If A is a matrix such that  $A^2 = A$ , show that det(A) = 0 or det(A) = 1.

Exercise 115 Prove or disprove.

- (a) If the matrix A is orthogonal, then  $det(A) = \pm 1$ .
- (b) If the  $3 \times 3$  matrix A is skew symmetric, then det (A) = 0.
- (c) If  $A^n = O$  for some positive integer n, then det (A) = 0.

Exercise 116 Use Cramer's rule to solve each of the following linear systems.

(a) 
$$\begin{cases} ax - by = 1 \\ bx + ay = 0 \end{cases}$$
;  
(b) 
$$\begin{cases} x_1 - 2x_2 + 2x_3 = 3 \\ 3x_1 + x_3 = -1 \end{cases}$$
;  
 $x_1 - x_2 + 2x_3 = 2$ ;  
(c) 
$$\begin{cases} x_1 + 4x_2 + 2x_3 = 3 \\ 4x_1 + 2x_2 + x_3 = 1 \end{cases}$$
;  
 $2x_1 - 2x_2 - 5x_3 = 1$ ;  
 $2x_1 - 2x_2 - 5x_3 = -1$ ;  
(d) 
$$\begin{cases} 2x_1 + 3x_2 - 5x_3 = 1 \\ 3x_2 + 2x_3 = -1 \\ 3x_1 + 2x_2 - 3x_3 = 1 \end{cases}$$

**Exercise 117** Use Cramer's rule to solve for x and y in terms of u and v:

- (a) u = 5x + 8y and v = 3x + 5y.
- (b)  $u = x \cos \theta y \sin \theta$  and  $v = x \sin \theta + y \cos \theta$ .

**Exercise 118** Consider the  $2 \times 2$  matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x \\ y \end{bmatrix}$$

where x and y denote the row vectors of B. Then the product AB can be written in the form

$$AB = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}.$$

Use this expression and the properties of the determinants to show that

$$\det (AB) = (ad - bc) \begin{vmatrix} x \\ y \end{vmatrix} = \det (A) \cdot \det (B).$$

**Exercise 119** Use the *adjoint formula* to find the inverse  $A^{-1}$  of each matrix A given below.

(a) 
$$\begin{bmatrix} -2 & 2 & -4 \\ 3 & 0 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$
  
(b) 
$$\begin{bmatrix} 3 & 5 & 2 \\ -2 & 3 & -4 \\ -5 & 0 & -5 \end{bmatrix}$$
  
(c) 
$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 120 TRUE or FALSE ?

- (a) If A is an  $n \times n$  matrix, then det  $(2A) = 2 \cdot \det(A)$ .
- (b) Suppose that A and B are  $n \times n$  matrices, and A is invertible. Then

$$\det\left(ABA^{-1}\right) = \det\left(B\right).$$

(c) If A is an  $n \times n$  matrix, then

$$\det\left(AA^{T}\right) = \det A^{T}A.$$

- (d) If all entries of a square matrix A are zeros and ones, then det(A) is 1, 0, or -1.
- (e) If A and B are  $n \times n$  matrices, then

$$\det (A+B) = \det (A) + \det (B).$$

(f) If all diagonal entries of a square matrix A are odd integers, and all other entries are even, then A is invertible.