Chapter 9

Vectors, Lines, and Planes

Topics :

A *vector* is usually defined as a “quantity having magnitude and direction”, such as the *velocity vector* of an object moving through space. It is helpful to represent a vector as an “arrow” attached to a point of space. Vectors can be added to one another and can also be multiplied by real numbers (often called *scalars* in this context). They provide a source of ideas for studying more abstract mathematical subjects, like linear algebra or modern geometry.

9.1 Vectors in the plane

Consider the *plane* Π of "elementary (read : high school) plane geometry". We draw a pair of perpendicular *lines* intersecting at a *point* O, called the **origin**. One of the lines, the x**-axis**, is usually taken in a "horizontal" position. The other line, the y**-axis**, is then taken in a "vertical" position. The x- and y-axes together are called **coordinate axes** and they form a **Cartesian coordinate system** on Π. We now choose a point on the x-axis to the right of O and a point on the y-axis above O to fix the *units of length* and *positive directions* on the coordinate axes. Frequently, these points are chosen so that they are both equidistant from O. With each point P in the plane we associate an ordered pair (x, y) of real numbers, its **coordinates**. Conversely, we can associate a point in the plane with each ordered pair of real numbers. Point P with coordinates (x, y) is denoted by $P(x, y)$ or, simply, by (x, y). Thus, *the plane* Π*, equipped with a Cartesian coordinate system, may be identified with the set* $\mathbb{R}^2$ *of all pairs of real numbers.*

Throughout, the set $\mathbb{R}^2$ will be referred to as *the* **Euclidean 2-space** or, simply, the **plane**.

NOTE : A **point** in the Euclidean 2-space is an ordered pair (x, y) of real numbers, and the **distance** between points (x_1, y_1) and (x_2, y_2) is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We are now going to introduce the concept of *(geometric) vector*.

NOTE : One can think of a vector as an *instruction to move*; the instruction makes sense wherever you are (in the plane), even if it may be rather difficult to carry out. Not every instruction to move is a vector; for an instruction to be a vector, it must specify movement *through the same distance* and *in the same direction* for every point.

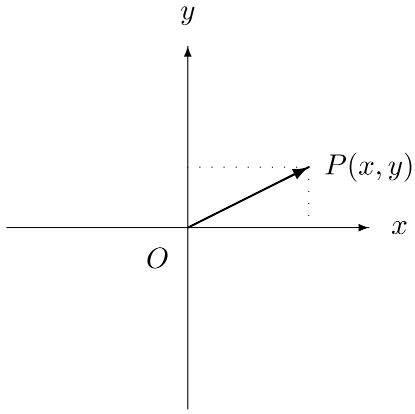
We make the following definition.

9.1.1 DEFINITION. A **vector** (in the plane) is a 2×1 matrix

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

where x and y are real numbers, called the **components** of $\vec{v}$.

With every vector $\vec{v}$ we can associate a *directed line segment*, with the initial point the origin and the terminal point $P(x,y)$. The directed line segment from O to P is denoted by $\overrightarrow{OP}$; O is called the **tail** and P the **head**.



Directed line segment : $\overrightarrow{OP}$.

NOTE : A directed line segment has a **direction**, indicated by an arrow pointing from O to P. The **magnitude** of a directed line segment is its length. Thus, *a directed line segment can be used to describe force, velocity, and acceleration.*

Conversely, with every directed line segment $\overrightarrow{OP}$, with tail $O(0,0)$ and head $P(x,y)$, we can associate the vector

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

9.1.2 DEFINITION. Two vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

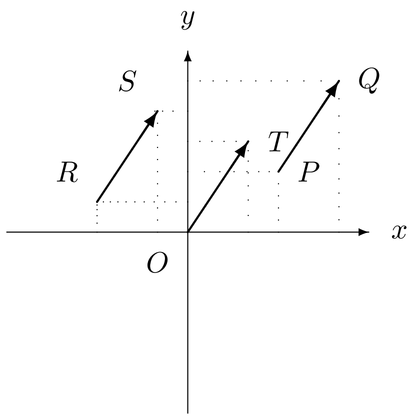
are said to be **equal** if $u_1 = v_1$ and $u_2 = v_2$. That is, two vectors are equal if their respective components are equal.

Frequently, in applications it is necessary to represent a vector $\vec{v}$ by a line segment $\overrightarrow{PQ}$ located at some point $P(x, y)$ (not the origin). In this case, if

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \text{then} \quad Q \quad \text{has coordinates} \quad (x + v_1, y + v_2).$$

9.1.3 Example. Consider the points $P(3, 2)$, $Q(5, 5)$, $R(-3, 1)$ and $S(-1, 4)$. The vectors (represented by) $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are equal, since they have their respective components equal. We write

$$\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \overrightarrow{RS}\ .$$



Equal vectors : $\overrightarrow{PQ} = \overrightarrow{RS}$ $(= \overrightarrow{OT})$.

With every vector

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

we can also associate the unique point $P(x, y)$; conversely, with every point $P(x, y)$ we associate the unique vector

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

This association is carried out by means of the directed line segment $\overrightarrow{OP}$, located at the origin. The directed line segment $\overrightarrow{OP}$ is one representation of a vector, sometimes denoted by $\vec{r}_P$ and called the **position vector** of the point P.

NOTE : The plane may be viewed both as *the set of all points* or *the set of all vectors* (in the plane).

Vector addition and scalar multiplication

9.1.4 DEFINITION. The **sum** of two vectors

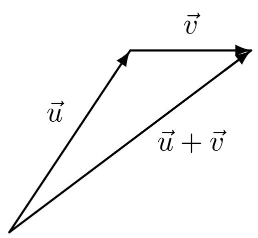
$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

is the vector

$$\vec{u} + \vec{v} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}.$$

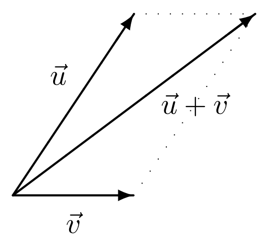
We can interpret vector addition geometrically as follows. We take a representative of $\vec{u}$, say $\overrightarrow{PQ}$, and then a representative of $\vec{v}$ starting from the terminal point of $\vec{u}$, say $\overrightarrow{QR}$. The sum $\vec{u} + \vec{v}$ is then the vector (represented by the directed line segment) $\overrightarrow{PR}$. Thus

$$\vec{u} + \vec{v} = \overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}\,.$$



Sum of two vectors : $\vec{u} + \vec{v}$.

We can also describe $\vec{u}+\vec{v}$ as the diagonal of the parallelogram defined by $\vec{u}$ and $\vec{v}$. This description of vector addition is sometimes called the **parallelogram rule**.



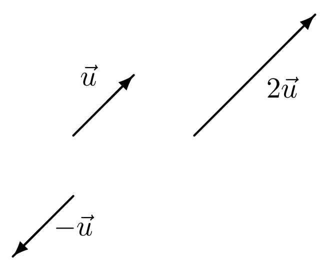
The parallelogram rule of vector addition.

9.1.5 Definition. If

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

is a vector and λ is a real number (scalar), then the **scalar multiple** of $\vec{u}$ by λ is the vector

$$\lambda\vec{u} := \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \end{bmatrix}.$$



Scalar multiples of a vector : $2\vec{u}$ and $-\vec{u}$.

The vector

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is called the **zero vector** and is denoted by $\vec{0}$. If $\vec{u}$ is a vector, it follows that

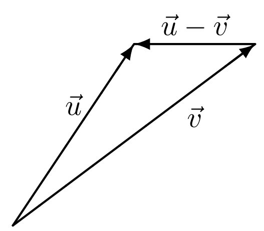
$$\vec{u} + \vec{0} = \vec{u}.$$

We can also show that

$$\vec{u} + (-1)\vec{u} = \vec{0}\,,$$

and we write $(-1)\vec{u}$ as $-\vec{u}$ and call it the **opposite** of $\vec{u}$. Moreover, we write $\vec{u} + (-1)\vec{v}$ as $\vec{u} - \vec{v}$ and call it the **difference** between $\vec{u}$ and $\vec{v}$.

NOTE : While vector addition gives one diagonal of a parallelogram, vector subtraction gives the other diagonal.



Difference between two vectors.

The following proposition summarizes the algebraic properties of vector addition and scalar multiplication of vectors.

9.1.6 PROPOSITION. *If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^2$ and r and s are scalars, then :*

(1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

(2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.

(3) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$.

(4) $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$.

(5) $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$.

(6) $(r + s)\vec{u} = r\vec{u} + s\vec{u}$.

(7) $r(s\vec{u}) = (rs)\vec{u}$.

(8) $1\vec{u} = \vec{u}$.

NOTE : The properties listed above may be summarize by saying that $\mathbb{R}^2$ *is a vector space* (over the field of real numbers).

Magnitude and distance

9.1.7 DEFINITION. The **length** (or the **magnitude**) of the vector

$$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$$

is defined to be the distance from the point (a, b) to the origin ; that is,

$$\boxed{\|\vec{v}\| := \sqrt{a^2 + b^2}\,.}$$

9.1.8 EXAMPLE. The length of the vector

$$\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

is

$$\|\vec{v}\| = \sqrt{3^2 + (-4)^2} = 5.$$

9.1.9 PROPOSITION. *If $\vec{u}$ and $\vec{v}$ are vectors, and r is a real number, then :*

(1) $\|\vec{u}\| \geq 0\,;$ $\|\vec{u}\| = 0$ *if and only if* $\vec{u} = 0$.

(2) $\|r\vec{u}\| = |r|\,\|\vec{u}\|$.

(3) $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ (*the* **triangle inequality**).

Proof : Exercise.

If

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

are vectors in $\mathbb{R}^2$, then the **distance** between $\vec{u}$ and $\vec{v}$ is defined as $\|\vec{u}-\vec{v}\|$. Thus

$$\|\vec{u}-\vec{v}\| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2}.$$

Note : This equation also gives the distance between the points (u_1, u_2) and (v_1, v_2).

9.1.10 Example. Compute the distance between the vectors

$$\vec{u} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Solution : The distance between $\vec{u}$ and $\vec{v}$ is

$$\|\vec{u}-\vec{v}\| = \sqrt{(-1-3)^2+(5-2)^2} = \sqrt{4^2+3^2} = 5.$$

Dot product and angle

9.1.11 Definition. Let

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

be vectors in $\mathbb{R}^2$. The **dot product** of $\vec{u}$ and $\vec{v}$ is defined as the *number*

$$\boxed{\vec{u} \bullet \vec{v} := u_1v_1 + u_2v_2.}$$

Note : The dot product is also called the **standard inner product** on $\mathbb{R}^2$.

9.1.12 Example. If

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

then

$$\vec{u} \bullet \vec{v} = (2)(4) + (3)(-2) = 2.$$

Note : (1) We can write the dot product of $\vec{u}$ and $\vec{v}$ in terms of matrix multiplication as $\vec{u}^T\vec{v}$, where we have ignored the brackets around the 1×1 matrix $\vec{u}^T\vec{v}$.

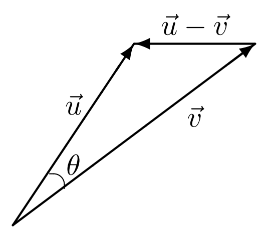
(2) If $\vec{v}$ is a vector in $\mathbb{R}^2$, then

$$\|\vec{v}\| = \sqrt{\vec{v} \bullet \vec{v}}.$$

Let us now consider the problem of determining the **angle** θ, $0 \leq \theta \leq \pi$, between two *nonzero* vectors in $\mathbb{R}^2$. Let

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

be two vectors in $\mathbb{R}^2$.



The angle between two vectors.

Using the *law of cosines*, we have

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta.$$

Hence

$$\begin{aligned}\cos\theta &= \frac{\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v} - \vec{u}\|^2}{2\|\vec{u}\|\|\vec{v}\|} \\ &= \frac{(u_1^2 + u_2^2) + (v_1^2 + v_2^2) - (v_1 - u_1)^2 - (v_2 - u_2)^2}{2\|\vec{u}\|\|\vec{v}\|} \\ &= \frac{u_1v_1 + u_2v_2}{\|\vec{u}\|\|\vec{v}\|} \\ &= \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\,\cdot\end{aligned}$$

That is,

$$\boxed{\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\,\cdot}$$

Note : The zero vector in $\mathbb{R}^2$ has no specific *direction.* The law of cosines expression above is true, for any angle θ, if $\vec{v} \neq \vec{0}$ and $\vec{u} = \vec{0}$. Thus, *the zero vector can be assigned any direction.*

9.1.13 Example. The angle θ between the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

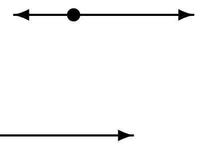
is determined by

$$\cos\theta = \frac{(1)(-1) + (0)(1)}{\sqrt{1^2 + 0^2}\,\sqrt{(-1)^2 + 1^2}} = -\frac{1}{\sqrt{2}}\,\cdot$$

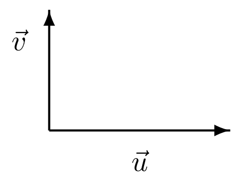
Since $0 \leq \theta \leq \pi$, it follows that $\theta = \frac{3\pi}{4}\cdot$

9.1.14 Definition. Two (nonzero) vectors $\vec{u}$ and $\vec{v}$ are

- **collinear** (or **parallel**) provided $\theta = 0$ or $\theta = \pi$.
- **orthogonal** (or **perpendicular**) provided $\theta = \frac{\pi}{2}$.



Collinear (or parallel) vectors.



Orthogonal vectors : $\vec{u} \perp \vec{v}$.

NOTE : (1) We regard the zero vector as both collinear with and orthogonal to *every* vector.

(2) If $\vec{v} \neq \vec{0}$, then vectors $\vec{u}$ and $\vec{v}$ are *collinear* $\iff$ $\vec{u} = r\,\vec{v}$ for some $r \in \mathbb{R}$. (See **Exercise 23** (b))

(3) Vectors $\vec{u}$ and $\vec{v}$ are *orthogonal* $\iff$ $\vec{u} \bullet \vec{v} = 0$.

9.1.15 EXAMPLE. The vectors

$$\vec{u} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

are orthogonal, since

$$\vec{u} \bullet \vec{v} = (2)(4) + (-4)(2) = 0.$$

Each of the properties of the dot product listed below is easy to establish.

9.1.16 PROPOSITION. *If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^2$, and r is a real number, then :*

(1) $\vec{u} \bullet \vec{u} \geq 0\,;\quad \vec{u} \bullet \vec{u} = 0$ *if and only if* $\vec{u} = \vec{0}$.

(2) $\vec{v} \bullet \vec{u} = \vec{u} \bullet \vec{v}$.

(3) $(\vec{u} + \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} + \vec{v} \bullet \vec{w}$.

(4) $(r\vec{u}) \bullet \vec{v} = r(\vec{u} \bullet \vec{v})$.

PROOF : Exercise.

A **unit vector** in $\mathbb{R}^2$ is a vector whose length is 1. If $\vec{v}$ is a nonzero vector, then the vector

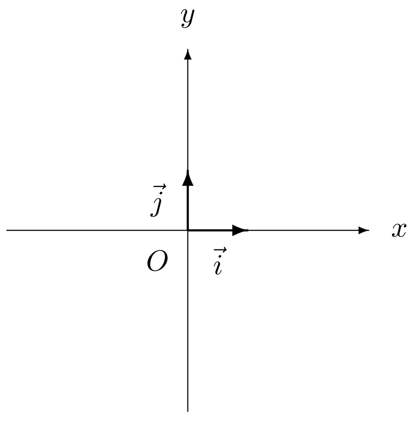
$$\frac{1}{\|\vec{v}\|}\vec{v}$$

is a unit vector (in the direction of $\vec{v}$).

There are two unit vectors in $\mathbb{R}^2$ that are of special importance. These are

$$\vec{\imath} := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{\jmath} := \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

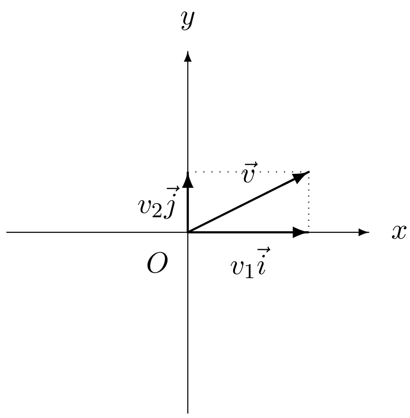
the unit vectors along the positive x- and y-axes. Observe that $\vec{\imath}$ and $\vec{\jmath}$ are orthogonal.



The standard unit vectors : $\vec{\imath}$ and $\vec{\jmath}$.

Note : Every vector in $\mathbb{R}^2$ can be written (uniquely) as a *linear combination* of the vectors $\vec{i}$ and $\vec{j}$; that is,

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_1\vec{i} + v_2\vec{j}.$$



Linear combination of two vectors : $\vec{v} = v_1\vec{i} + v_2\vec{j}$.

9.2 Vectors in space

The foregoing discussion of vectors in the plane can be generalized to vectors in space, as follows.

Consider the (three-dimensional) *space* Σ of "elementary (read : high school) solid geometry". We first fix a **Cartesian coordinate system** by choosing a *point*, called the **origin**, and three *lines*, called the **coordinate axes**, each passing through origin, so that each line is perpendicular to the other two. These lines are individually called the x-, y-, and z-**axes**. On each of these axes we choose a point fixing the *units of length* and *positive directions* on the coordinate axes. Frequently, these points are chosen so that they are both equidistant from the origin O. With each point P in space we associate an ordered triple (x, y, z) of real numbers, its **coordinates**. Conversely, we can associate a point in space with each ordered triple of real numbers. Point P with coordinates (x, y, z) is denoted by $P(x, y, z)$ or, simply, by (x, y, z). Thus, *the space* Σ*, equipped with a Cartesian coordinate system, may be identified with the set* $\mathbb{R}^3$ *of all triples of real numbers.*

Throughout this section, the set $\mathbb{R}^3$ will be referred to as *the* **Euclidean 3-space** or, simply, the **space**.

NOTE : A **point** in the Euclidean 3-space is an ordered triple (x, y, z) of real numbers, and the **distance** between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

We now introduce the concept of *vector* in space.

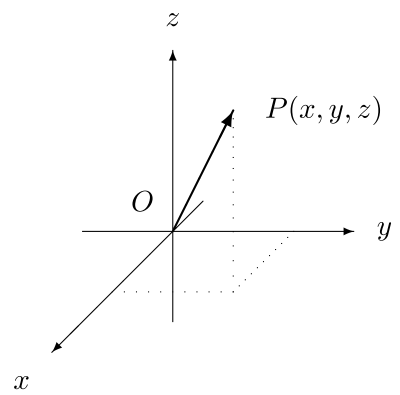
9.2.1 DEFINITION. A **vector** (in space) is a 3×1 matrix

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where $x, y,$ and z are real numbers, called the **components** of $\vec{v}$.

With every vector $\vec{v}$ we can associate a *directed line segment*, with the initial point the origin and the terminal point $P(x, y, z)$. The directed line segment from O to P is denoted by $\overrightarrow{OP}$; O is called the **tail** and P the **head**. Conversely, with every directed line segment $\overrightarrow{OP}$, with tail $O(0, 0, 0)$ and head $P(x, y, z)$, we can associate the vector

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$



Directed line segment : $\overrightarrow{OP}$.

9.2.2 Definition. Two vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

are said to be **equal** if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$. That is, two vectors are equal if their respective components are equal.

Frequently, in applications it is necessary to represent a vector $\vec{v}$ by a line segment $\overrightarrow{PQ}$ located at some point $P(x, y, z)$ (not the origin). In this case, if

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \text{then} \quad Q \quad \text{has coordinates} \quad (x + v_1, y + v_2, z + v_3).$$

9.2.3 Example. Consider the points $P(3, 2, 1)$, $Q(5, 5, 0)$, $R(-3, 1, 4)$ and $S(-1, 4, 3)$. The vectors (represented by) $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are equal, since they have their respective components equal. We write

$$\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \overrightarrow{RS} \, .$$

With every vector

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

we can also associate the unique point $P(x, y, z)$; conversely, with every point $P(x, y, z)$ we associate the unique vector

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

This association is carried out by means of the directed line segment $\overrightarrow{OP}$, located at the origin. The directed line segment $\overrightarrow{OP}$ is one representation of a vector, sometimes denoted by $\vec{r}_P$ and called the **position vector** of the point P.

Note : The space may be viewed both as *the set of all points* or *the set of all vectors* (in space).

Vector addition and scalar multiplication

9.2.4 Definition. The **sum** of two vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

is the vector

$$\vec{u} + \vec{v} := \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}.$$

The parallelogram rule, as a description (geometric interpretation) of vector addition remains valid.

9.2.5 Definition. If
$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$
is a vector and λ is a real number (scalar), then the **scalar multiple** of $\vec{u}$ by λ is the vector
$$\lambda\vec{u} := \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{bmatrix}.$$
The vector
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
is called the **zero vector** and is denoted by $\vec{0}$. If $\vec{u}$ is a vector, it follows that
$$\vec{u} + \vec{0} = \vec{u}.$$
Again, we can show that
$$\vec{u} + (-1)\vec{u} = \vec{0}$$
and we write $(-1)\vec{u}$ as $-\vec{u}$ and call it the **opposite** of $\vec{u}$. We write $\vec{u}+(-1)\vec{v}$ as $\vec{u} - \vec{v}$ and call it the **difference** between $\vec{u}$ and $\vec{v}$.

Note : While vector addition gives one diagonal of a parallelogram, vector subtraction gives the other diagonal.

The following proposition summarizes the algebraic properties of vector addition and scalar multiplication of vectors.

9.2.6 Proposition. *If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^3$ and r and s are scalars, then :*

(1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

(2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.

(3) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$.

(4) $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$.

(5) $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$.

(6) $(r + s)\vec{u} = r\vec{u} + s\vec{u}$.

(7) $r(s\vec{u}) = (rs)\vec{u}$.

(8) $1\vec{u} = \vec{u}$.

Note : The properties listed above may be summarize by saying that $\mathbb{R}^3$ *is a vector space* (over the field of real numbers).

Magnitude and distance

9.2.7 Definition. The **length** (or the **magnitude**) of the vector

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is defined to be the distance from the point (a, b, c) to the origin ; that is,

$$\boxed{\|\vec{v}\| := \sqrt{a^2 + b^2 + c^2}.}$$

9.2.8 Example. The length of the vector

$$\vec{v} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$$

is

$$\|\vec{v}\| = \sqrt{3^2 + (-4)^2 + 0^2} = 5.$$

9.2.9 Proposition. *If $\vec{u}$ and $\vec{v}$ are vectors, and r is a real number, then :*

(1) $\|\vec{u}\| \geq 0$; $\|\vec{u}\| = 0$ *if and only if* $\vec{u} = 0$.

(2) $\|r\vec{u}\| = |r|\,\|\vec{u}\|$.

(3) $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ (*the* **triangle inequality**).

PROOF : Exercise.

If

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

are vectors in $\mathbb{R}^3$, then the **distance** between $\vec{u}$ and $\vec{v}$ is defined as $\|\vec{u}-\vec{v}\|$. Thus

$$\|\vec{u}-\vec{v}\| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + (u_3-v_3)^2}.$$

NOTE : This equation also gives the distance between the points (u_1, u_2, u_3) and (v_1, v_2, v_3).

9.2.10 EXAMPLE. Compute the distance between the vectors

$$\vec{u} = \begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}.$$

SOLUTION : The distance between $\vec{u}$ and $\vec{v}$ is

$$\|\vec{u}-\vec{v}\| = \sqrt{(-1-3)^2 + (5-2)^2 + (-4+4)^2} = \sqrt{4^2+3^2+0^2} = 5.$$

Dot product and angle

9.2.11 DEFINITION. Let

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

be vectors in $\mathbb{R}^3$. The **dot product** of $\vec{u}$ and $\vec{v}$ is defined as the number

$$\boxed{\vec{u} \bullet \vec{v} := u_1v_1 + u_2v_2 + u_3v_3.}$$

Note : The dot product is also called the **standard inner product** on $\mathbb{R}^3$.

9.2.12 Example. If

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

then

$$\vec{u} \bullet \vec{v} = (2)(4) + (3)(-2) + (-1)(2) = 0.$$

Note : (1) We can write the dot product of $\vec{u}$ and $\vec{v}$ in terms of matrix multiplication as $\vec{u}^T \vec{v}$, where we have ignored the brackets around the 1×1 matrix $\vec{u}^T \vec{v}$.

(2) If $\vec{v}$ is a vector in $\mathbb{R}^3$, then

$$\|\vec{v}\| = \sqrt{\vec{v} \bullet \vec{v}}.$$

Let us now consider the problem of determining the **angle** θ, $0 \leq \theta \leq \pi$, between two *nonzero* vectors in $\mathbb{R}^3$. Let

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

be two vectors in $\mathbb{R}^3$. Using the *law of cosines*, we have

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos\theta\,.$$

Hence

$$\begin{aligned}
\cos\theta &= \frac{\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{v} - \vec{u}\|^2}{2\|\vec{u}\|\|\vec{v}\|} \\
&= \frac{(u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}{2\|\vec{u}\|\|\vec{v}\|} \\
&= \frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{\|\vec{u}\|\|\vec{v}\|} \\
&= \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\,.
\end{aligned}$$

That is,

$$\boxed{\cos\theta = \frac{\vec{u} \bullet \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\,.}$$

NOTE : The zero vector in $\mathbb{R}^2$ has no specific *direction*. The law of cosines expression above is true, for any angle θ, if $\vec{v} \neq \vec{0}$ and $\vec{u} = \vec{0}$. Thus, *the zero vector can be assigned any direction*.

9.2.13 EXAMPLE. The angle θ between the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

is determined by

$$\cos\theta = \frac{(1)(0) + (1)(1) + (0)(1)}{\sqrt{1^2 + 1^2 + 0^2}\,\sqrt{0^2 + 1^2 + 1^2}} = \frac{1}{2}\,.$$

Since $0 \leq \theta \leq \pi$, it follows that $\theta = \frac{\pi}{3}$.

9.2.14 DEFINITION. Two (nonzero) vectors $\vec{u}$ and $\vec{v}$ are

- **collinear** (or **parallel**) provided $\theta = 0$ or $\theta = \pi$.
- **orthogonal** (or **perpendicular**) provided $\theta = \frac{\pi}{2}$.

NOTE : (1) We regard the zero vector as both collinear with and orthogonal to *every* vector.

(2) If $\vec{v} \neq \vec{0}$, then vectors $\vec{u}$ and $\vec{v}$ are *collinear* $\iff$ $\vec{u} = r\,\vec{v}$ for some $r \in \mathbb{R}$. (See **Exercise 23** (b))

(3) Vectors $\vec{u}$ and $\vec{v}$ are *orthogonal* $\iff$ $\vec{u} \bullet \vec{v} = 0$.

9.2.15 PROPOSITION. *If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^2$, and r is a real number, then :*

(1) $\vec{u} \bullet \vec{u} \geq 0\,;$ $\vec{u} \bullet \vec{u} = 0$ *if and only if* $\vec{u} = \vec{0}$.

(2) $\vec{v} \bullet \vec{u} = \vec{u} \bullet \vec{v}$.

(3) $(\vec{u} + \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} + \vec{v} \bullet \vec{w}$.

(4) $(r\vec{u}) \bullet \vec{v}) = r(\vec{u} \bullet \vec{v})$.

Proof : Exercise.

A **unit vector** in $\mathbb{R}^3$ is a vector whose length is 1. If $\vec{v}$ is a nonzero vector, then the vector

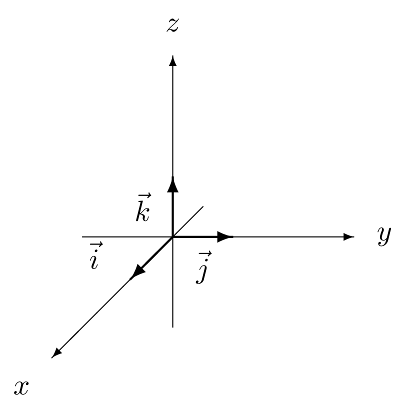
$$\frac{1}{\|\vec{v}\|}\vec{v}$$

is a unit vector (in the direction of $\vec{v}$).

There are three unit vectors in $\mathbb{R}^3$ that are of special importance. These are

$$\vec{i} := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

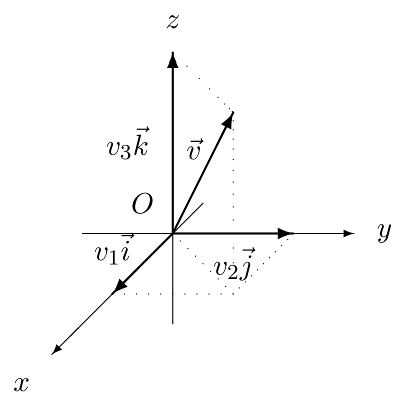
the unit vectors along the positive x-, y-, and z-axes. Observe that $\vec{i}$, $\vec{j}$, and $\vec{k}$ are mutually orthogonal.



The standard unit vectors : $\vec{i}$, $\vec{j}$, and $\vec{k}$.

NOTE : Every vector in $\mathbb{R}^3$ can be written (uniquely) as a *linear combination* of the vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$; that is,

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}\,.$$



Linear combination of three vectors : $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$.

Cross product

9.2.16 DEFINITION. Let

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

be vectors in $\mathbb{R}^3$. The **cross product** of $\vec{u}$ and $\vec{v}$ is defined to be the vector

$$\boxed{\vec{u} \times \vec{v} := \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}\,.}$$

The cross product is also called the **vector product**.

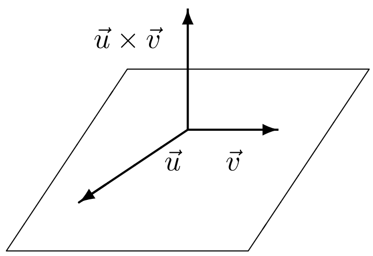
9.2.17 Example. Let $\vec{u} = 2\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{v} = 3\vec{i} - \vec{j} - 3\vec{k}$. Then

$$\vec{u} \times \vec{v} = -\vec{i} + 12\vec{j} - 5\vec{k}.$$

Note : (1) The cross product $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

(2) A common way of remembering the definition of the cross product $\vec{u} \times \vec{v}$ is to observe that it results from a *formal* expansion along the first row in the determinant

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$



The cross product of two vectors : $\vec{u} \times \vec{v}$.

9.2.18 Proposition. *If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors in $\mathbb{R}^3$, and r is a real number, then :*

(1) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

(2) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

(3) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$.

(4) $r(\vec{u} \times \vec{v}) = (r\vec{u}) \times \vec{v} = \vec{u} \times (r\vec{v})$.

Proof : Exercise.

One can show that (tedious computation)

$$\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\bullet\vec{v})^2.$$

Recall that

$$\vec{u}\bullet\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. Hence,

$$\|\vec{u}\times\vec{v}\|^2 = \|\vec{u}\|^2\,\|\vec{v}\|^2 - \|\vec{u}\|^2\,\|\vec{v}\|^2\cos^2\theta = \|\vec{u}\|^2\|\vec{v}\|^2(1-\cos^2\theta) = \|\vec{u}\|^2\,\|\vec{v}\|^2\sin^2\theta.$$

Taking square roots, we obtain

$$\boxed{\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\sin\theta.}$$

Observe that we do not have to write $|\sin\theta|$, since $\sin\theta$ is nonnegative for $0\leq\theta\leq\pi$.

NOTE : Vectors $\vec{u}$ and $\vec{v}$ (in $\mathbb{R}^3$) are *collinear* $\iff$ $\vec{u}\times\vec{v}=\vec{0}$.

Applications : area and volume

We now consider several applications of cross product.

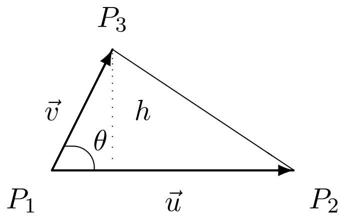
$\boxed{\text{A}}$ (Area of a Triangle) Consider a triangle with vertices P_1, P_2 and P_3. The *area* of this triangle is $\frac{1}{2}bh$, where b is the *base* and h is the *height*. If we take the segment between P_1 and P_2 to be the base and denote $\overrightarrow{P_1P_2}$ by the vector $\vec{u}$, then

$$b = \|\vec{u}\|.$$

Letting $\overrightarrow{P_1P_3}=\vec{v}$, we find that the height h is given by

$$h = \|\vec{v}\|\sin\theta$$

θ being the angle betweeen $\vec{u}$ and $\vec{v}$.



Area of a triangle : $\frac{1}{2}\|\vec{u}\times\vec{v}\|$.

Hence, *the area* A_t *of the triangle is*

$$\boxed{A_t = \tfrac{1}{2}\|\vec{u}\|\,\|\vec{v}\|\sin\theta = \tfrac{1}{2}\|\vec{u}\times\vec{v}\|.}$$

9.2.19 EXAMPLE. Find the area of the triangle with vertices

$$P_1(2,2,4),\quad P_2(-1,0,5),\quad \text{and}\quad P_3(3,4,3)\,.$$

SOLUTION : We have

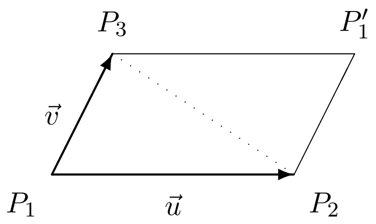
$$\vec{u} = \overrightarrow{P_1P_2} = -3\vec{i} - 2\vec{j} + \vec{k} \quad \text{and} \quad \vec{v} = \overrightarrow{P_1P_3} = \vec{i} + 2\vec{j} - \vec{k}.$$

Then

$$A_t = \frac{1}{2}\|(-3\vec{i} - 2\vec{j} + \vec{k})\times(\vec{i} + 2\vec{j} - \vec{k})\| = \frac{1}{2}\| - 2\vec{j} - 4\vec{k}\| = \sqrt{5}.$$

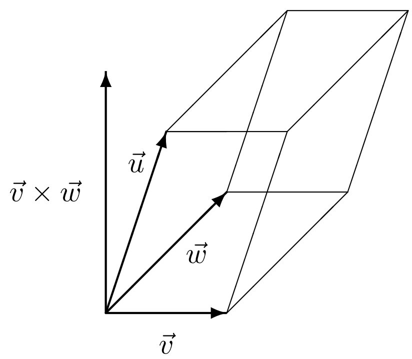
$\boxed{\text{B}}$ (Area of a Parallelogram) The area A_p of the parallelogram with adjacent sides $\vec{u}$ and $\vec{v}$ is $2A_t$, so

$$\boxed{A_p = \|\vec{u}\times\vec{v}\|.}$$



Area of a parallelogram : $\|\vec{u} \times \vec{v}\|$.

$\boxed{\text{C}}$ (Volume of a Parallelepiped) Consider the parallelepiped with a vertex at the origin and edges $\vec{u}$, $\vec{v}$, and $\vec{w}$. Then the volume V of the parallelepiped is the product of the area of the face containing $\vec{v}$ and $\vec{w}$ and the distance h from this face to the face parallel to it.



Volume of a parallelepiped : $|\vec{u} \bullet (\vec{v} \times \vec{w})|$.

Now

$$h = \|\vec{u}\| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v} \times \vec{w}$, and the area of the face determined by $\vec{v}$ and $\vec{w}$ is $\|\vec{v} \times \vec{w}\|$. Hence,

$$\boxed{V = \|\vec{v} \times \vec{w}\| \, \|\vec{u}\| \, |\cos\theta| = |\vec{u} \bullet (\vec{v} \times \vec{w})|.}$$

NOTE : The volume of the parallelepiped determined by the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$, can be expressed as follows

$$\text{volume} = \pm \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

9.2.20 EXAMPLE. Find the volume of the parallelepiped with a vertex at the origin and edges

$$\vec{u} = \vec{i} - 2\vec{j} + 3\vec{k}, \quad \vec{v} = \vec{i} + 3\vec{j} + \vec{k}, \quad \text{and} \quad \vec{w} = 2\vec{i} + \vec{j} + 2\vec{k}.$$

SOLUTION : We have

$$\vec{v} \times \vec{w} = 5\vec{i} - 5\vec{k}.$$

Hence, $\vec{u} \bullet (\vec{v} \times \vec{w}) = -10$, and thus the volume V is given by

$$V = |\vec{u} \bullet (\vec{v} \times \vec{w})| = |-10| = 10.$$

Alternatively, we have

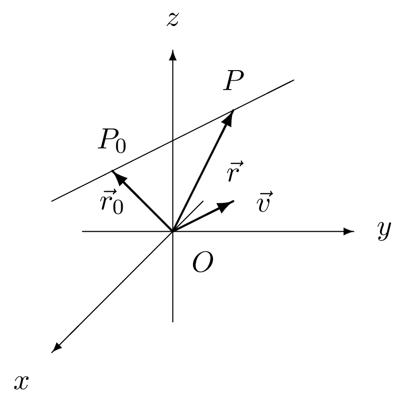
$$\text{volume} = \pm \begin{vmatrix} 1 & -2 & 3 \\ 1 & 3 & 1 \\ 2 & 1 & 2 \end{vmatrix} = \pm(-10) = 10.$$

9.3 Lines and planes

Lines

In elementary geometry a *straight line* in space is determined by any two points that lie on it. Here, we take the alternative approach that a *straight line in space is determined by a single point P_0 on it and a vector $\vec{v}$ that determines the direction of the line.*

9.3.1 DEFINITION. By the **straight line** $\mathcal{L}$ that passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the (nonzero) vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$ is meant the set of all points $P(x, y, z)$ in $\mathbb{R}^3$ such that the vector $\overrightarrow{P_0P} = \vec{r} - \vec{r}_0$ is collinear to $\vec{v}$. (Here, $\vec{r}$ and $\vec{r}_0$ stand for the position vectors of the points P and P_0, respectively.)



Vector equation of a line : $\vec{r} = \vec{r}_0 + t\vec{v}$.

Thus, the point P lies on $\mathcal{L}$ if and only if

$$\vec{r} - \vec{r}_0 = t\vec{v}, \quad \text{for some scalar } t;$$

that is, if and only if

$$\vec{r} = \vec{r}_0 + t\vec{v}, \quad t \in \mathbb{R}.$$

NOTE : We can visualize the point P as moving along the straight line $\mathcal{L}$, with $\vec{r}_0 + t\vec{v}$ being its location at "time" t.

The equation

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}, \quad t \in \mathbb{R}}$$

is a **vector equation** of the line $\mathcal{L}$.

By equating components of the vectors in this equation, we get the scalar equations

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct. \end{cases}$$

These are **parametric equations** (with **parameter** t) of the line $\mathcal{L}$.

Alternatively, one can write the equations

$$\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}\cdot}$$

These are called **symmetric equations** of the line $\mathcal{L}$.

9.3.2 EXAMPLE. Write parametric equations of the line $\mathcal{L}$ that passes through the points $P_1(1,2,2)$ and $P_2(3,-1,3)$.

SOLUTION : The *direction* of the line $\mathcal{L}$ is given by the vector $\vec{v} = \overrightarrow{P_1P_2} = 2\vec{i} - 3\vec{j} + \vec{k}$. With P_1 as the fixed point, we get the following parametric equations

$$x = 1 + 2t\,, \quad y = 2 - 3t\,, \quad z = 2 + t.$$

NOTE : If we take P_2 as the fixed point and $-2\vec{v} = -4\vec{i} + 6\vec{j} - 2\vec{k}$ as the direction vector, then we get different parametric equations

$$x = 3 - 4t\,, \quad y = -1 + 6t\,, \quad z = 3 - 2t.$$

Thus, the parametric equations of a line are *not* unique.

9.3.3 EXAMPLE. Determine whether the lines

$$\vec{r} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

intersect.

SOLUTION : We write parametric equations for the first line; that is,

$$x = 1 + 3t\,, \quad y = 1 + 2t\,, \quad z = 2 + t.$$

Now we determine whether there exists $t \in \mathbb{R}$ such that

$$\frac{1+3t-1}{2} = \frac{1+1t-2}{3} = \frac{2+t-3}{4}\cdot$$

The linear system

$$\begin{cases} 6t = 4t - 2 \\ 12t = 2t - 2 \end{cases}$$

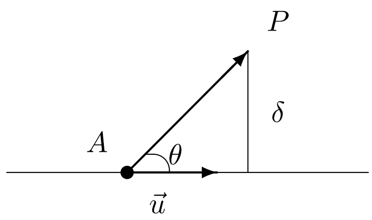
is clearly inconsistent (Why ?), and thus the given lines do not intersect.

9.3.4 Example. Find the *shortest* distance between the point $P(3, -1, 4)$ and the line given by

$$x = -2 + 3t\,, \quad y = -2t\,, \quad z = 1 + 4t.$$

Solution : First, we shall find a formula for the distance from a point P to a line $\mathcal{L}$.

Let $\vec{u}$ be the *direction vector* for $\mathcal{L}$ and A a point on the line. Let δ be the distance from P to the given line $\mathcal{L}$.



The shortest distance between a point and a line : $\frac{\|\overrightarrow{AP}\times\vec{u}\|}{\|\vec{u}\|}$.

Then

$$\delta = \|\overrightarrow{AP}\| \sin\theta\,,$$

where θ is the angle between $\vec{u}$ and $\overrightarrow{AP}$. We have

$$\|\vec{u}\|\,\|\overrightarrow{AP}\|\sin\theta = \|\vec{u}\times\overrightarrow{AP}\| = \|\overrightarrow{AP}\times\vec{u}\|.$$

Consequently,

$$\boxed{\delta = \| \overrightarrow{AP} \| \sin\theta = \frac{\| \overrightarrow{AP} \times \vec{u} \|}{\|\vec{u}\|} \cdot}$$

In our case, we have $\vec{u} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$ and to find a point on $\mathcal{L}$, let $t = 0$ and obtain $A(-2, 0, 1)$. Thus

$$\overrightarrow{AP} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} \quad \text{and} \quad \overrightarrow{AP} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -1 & 3 \\ 3 & -2 & 4 \end{vmatrix} = 2\vec{i} - 11\vec{j} - 7\vec{k} = \begin{bmatrix} 2 \\ -11 \\ -7 \end{bmatrix}.$$

Finally, we can find the distance to be

$$\delta = \frac{\| \overrightarrow{AP} \times \vec{u} \|}{\|\vec{u}\|} = \frac{\sqrt{174}}{\sqrt{29}} = \sqrt{6}\,.$$

Note : In the xy-plane, parametric equations of a line $\mathcal{L}$ take the form

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt. \end{cases}$$

Alternatively, one can write

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

or

$$y = y_0 + \frac{b}{a}(x - x_0)$$

or even

$$y = mx + n$$

where m is the *slope* of the line and n is the y-intercept. Furthermore, we observe that this "familiar" equation can be rewritten as

$$Ax + By + C = 0.$$

It is not difficult to show that the graph of such an equation, where $A, B, C \in \mathbb{R}$ (with A and B not all zero) is a straight line (in the plane) with slope $m = -\frac{A}{B} \cdot$

9.3.5 EXAMPLE. Show that the lines

$$(\mathcal{L}) \quad ax + by + c = 0 \quad \text{and} \quad (\mathcal{M}) \quad dx + ey + f = 0$$

are parallel if and only if $ae = bd$ and are perpendicular if and only if $ad+be = 0$.

SOLUTION : The direction vectors of $\mathcal{L}$ and $\mathcal{M}$ are

$$\vec{u} = \begin{bmatrix} -b \\ a \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} -e \\ d \end{bmatrix},$$

respectively. Then

$$\mathcal{L} \parallel \mathcal{M} \iff \vec{u}_1 = \lambda\,\vec{u}_2 \iff \vec{u}_1 \times \vec{u}_2 = \vec{0} \iff (ae-bd)\vec{k} = \vec{0} \iff ae = bd\,;$$

$$\mathcal{L} \perp \mathcal{M} \iff \vec{u}_1 \bullet \vec{u}_2 = 0 \iff ad + be = 0.$$

In particular, the lines

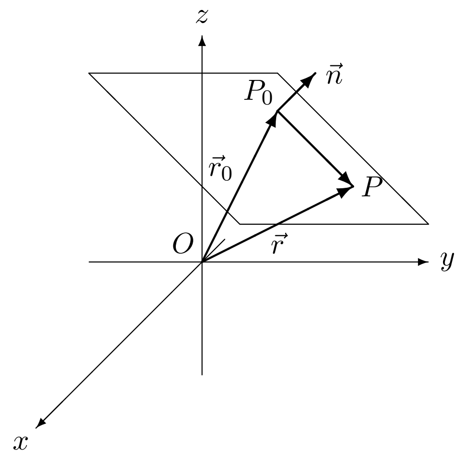
$$y = m_1 x + n_1 \quad \text{and} \quad y = m_2 x + n_2$$

are *parallel* if and only if $m_1 = m_2$ and are *perpendicular* if and only if $m_1 m_2 + 1 = 0$.

Planes

A *plane* in space is determined by any point that lies on it and any line through that point orthogonal to the plane. Alternatively, *a plane in space is determined by a single point* P_0 *on it and a vector* $\vec{n}$ *that is orthogonal to the plane.*

9.3.6 DEFINITION. By the **plane** α through the point $P_0(x_0, y_0, z_0)$ with **normal vector** $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ is meant the set of all points $P(x, y, z)$ in $\mathbb{R}^3$ such that the vectors $\overrightarrow{P_0P} = \vec{r} - \vec{r}_0$ and $\vec{n}$ are orthogonal. (Again, $\vec{r}$ and $\vec{r}_0$ stand for the position vectors of the points P and P_0, respectively.)



Vector equation of a plan : $\vec{n} \bullet (\vec{r} - \vec{r}_0) = 0$.

Thus, the point P lies on α if and only if

$$\boxed{\vec{n} \bullet (\vec{r} - \vec{r}_0) = 0\,.}$$

This is a **vector equation** of the plane α.

By substituting the components of the vectors involved in this equation, we get the scalar equation

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.}$$

This is a **point-normal equation** (or **standard equation**) of the plane α.

NOTE : The equation above can be rewritten as

$$ax + by + cz + d = 0.$$

It is not difficult to show that the graph of such an equation, where $a, b, c, d \in \mathbb{R}$ (with a, b, and c not all zero) is a plane with normal $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$.

9.3.7 EXAMPLE. Find an equation of the plane passing through the point $(3, 4, -3)$ and perpendicular to the vector $\vec{v} = 5\vec{i} - 2\vec{j} + 4\vec{k}$.

SOLUTION : We obtain an equation of the plane as

$$5(x - 3) - 2(y - 4) + 4(z + 3) = 0.$$

9.3.8 EXAMPLE. Find an equation of the plane passing through points $P_1(2,-2,1)$, $P_2(-1,0,3)$, and $P_3(5,-3,4)$.

SOLUTION : The (noncollinear) vectors $\overrightarrow{P_1P_2}= -3\vec{i}+2\vec{j}+3\vec{k}$ and $\overrightarrow{P_1P_3}= 3\vec{i}-\vec{j}+3\vec{k}$ lie in the plane, since the points P_1, P_2, and P_3 lie in the plane. The vector

$$\vec{v} =\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}= 8\vec{i}+15\vec{j}-3\vec{k}$$

is then perpendicular to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$, and is thus a normal to the plane. Using the vector $\vec{v}$ and the point $P_1(2,-2,1)$, we obtain

$$8(x-2)+15(y+2)-3(z-1)=0$$

as an equation of the plane.

NOTE : The general equation for a plane passing through three noncollinear points $P_i(x_i, y_i, z_i)$,
$i=1,2,3$ may be written in the form :

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

or, equivalently,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0.$$

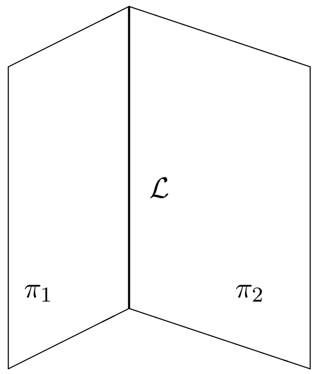
9.3.9 EXAMPLE. Find parametric equations of the line of intersection of the planes

$$(\pi_1) \quad 2x+3y-2z+4=0 \quad \text{and} \quad (\pi_2) \quad x-y+2z+3=0\,.$$

SOLUTION : Solving the linear system consisting of the equations of π_1 and π_2, we obtain (verify !)

$$x=-\frac{13}{5}-\frac{4}{5}t\,, \quad y=\frac{2}{5}+\frac{6}{5}t\,, \quad z=t$$

as parametric equations of the line $\mathcal{L}$ of intersection of the planes.



The line of intersection of two plans : $\mathcal{L} = \pi_1 \cap \pi_2$.

NOTE : (1) The line in the xy-plane, described by the (general) equation

$$ax + by + d = 0$$

may be viewed as the intersection of the planes

$$ax + by + cz + d = 0 \quad \text{and} \quad z = 0.$$

(2) The equations

$$x = y = 0$$

define a line (the z-axis). Alternative ways of describing the same line are, for instance :

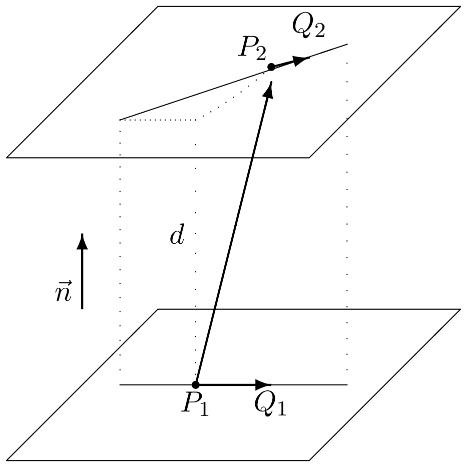
$$(1) \quad \vec{r} = t\vec{k},\ t \in \mathbb{R}\,; \quad (2) \quad x = 0,\ y = 0,\ z = t\,;\ \ t \in \mathbb{R}\,; \quad (3) \quad \frac{x}{0} = \frac{y}{0} = \frac{z}{1} \cdot$$

9.3.10 EXAMPLE. Find a formula for the *shortest* distance d between two skew lines $\mathcal{L}_1$ and $\mathcal{L}_2$.

SOLUTION : The lines are *skew* if they are not parallel and do not intersect. Choose points $P_1,\ Q_1$ on $\mathcal{L}_1$ and $P_2,\ Q_2$ on $\mathcal{L}_2$; so, the direction vectors for the lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are

$$\vec{v}_1 = \overrightarrow{P_1Q_1} \quad \text{and} \quad \vec{v}_2 = \overrightarrow{P_2Q_2}\,,$$

respectively.



The shortest distance between two skew lines : $\frac{|(\vec{v}_1\times\vec{v}_2)\bullet \overrightarrow{P_1P_2}|}{\|\vec{v}_1\times\vec{v}_2\|}$.

Then $\vec{v}_1 \times \vec{v}_2$ is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$ and hence a *unit vector* $\vec{n}$ orthogonal to both $\vec{v}_1$ and $\vec{v}_2$ is

$$\vec{n} = \frac{1}{\|\vec{v}_1 \times \vec{v}_2\|}\,(\vec{v}_1 \times \vec{v}_2)\ .$$

Let us consider planes through P_1 and P_2, respectively, each having normal vector $\vec{n}$. These planes are parallel and contain $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The distance d between the planes is measured along a line parallel to the common normal $\vec{n}$. It follows that d is the shortest distance between $\mathcal{L}_1$ and $\mathcal{L}_2$. So

$$\boxed{d = |\vec{n}\bullet \overrightarrow{P_1P_2}\,| = \frac{|(\vec{v}_1 \times \vec{v}_2)\bullet \overrightarrow{P_1P_2}\,|}{\|\vec{v}_1 \times \vec{v}_2\|}}$$

where $\vec{v}_1 = \overrightarrow{P_1Q_1}$ and $\vec{v}_2 = \overrightarrow{P_2Q_2}$ are the direction vectors for the lines $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

9.4 Exercises

Exercise 121

(a) Sketch a directed line segment representing each of the following vectors :

$$(1)\quad \vec{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix};\quad (2)\quad \vec{v} = \begin{bmatrix} -2 \\ -2 \end{bmatrix};\quad (3)\quad \vec{w} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}.$$

(b) Determine the head of the vector

$$\vec{u} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

whose tail is $(-3, 2)$. Make a sketch.

(c) Determine the tail of the vector

$$\vec{v} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

whose head is $(1, 2)$. Make a sketch.

Exercise 122 For what values of a and b are the vectors

$$\begin{bmatrix} a-b \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ a+b \end{bmatrix} \quad \text{equal ?}$$

Exercise 123 Compute $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$, $3\vec{u}-2\vec{v}$, $\|-2\vec{u}\|$, $\|\vec{u}+\vec{v}\|$ for

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Exercise 124 Prove that the vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

are collinear if and only if

$$\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = 0.$$

Exercise 125 Use the dot product to find the angle between the vectors $\vec{u}$ and $\vec{v}$.

(a) $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix};$

(b) $\vec{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}.$

Exercise 126

(a) Find r so that the vector $\vec{v} = \vec{i} + r\vec{j}$ is orthogonal to $\vec{w} = 2\vec{i} - \vec{j}$.

(b) Find k so that the vectors $k\vec{i} + 4\vec{j}$ and $2\vec{i} + 5\vec{j}$ are collinear.

Exercise 127 Use the fact that $|\cos\theta| \leq 1$ for all θ to show that the **Cauchy-Schwarz inequality**

$$|\vec{u} \bullet \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$$

holds for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

Exercise 128 Use the dot product to find the angle between $\vec{u}$ and $\vec{v}$.

(a) $\vec{u} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$;

(b) $\vec{u} = 2\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{v} = 3\vec{i} - \vec{j} - 3\vec{k}$.

Exercise 129 Compute

$$\vec{v} \times \vec{w}\,, \quad \vec{u} \times (\vec{v} \times \vec{w})\,, \quad (\vec{u} \bullet \vec{w})\vec{v} - (\vec{u} \bullet \vec{v})\vec{w}\,, \quad (\vec{u} \times \vec{v}) \bullet \vec{w}\,, \quad \vec{u} \bullet (\vec{v} \times \vec{w})$$

for

$$\vec{u} = \vec{i} + \vec{j} + \vec{k}\,, \quad \vec{v} = \vec{j} - \vec{k}\,, \quad \text{and} \quad \vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}.$$

Exercise 130 Use the cross product to find a nonzero vector $\vec{c}$ orthogonal to both of $\vec{a}$ and $\vec{b}$.

$$(a) \quad \vec{a} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}; \quad (b) \quad \vec{a} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix}.$$

Exercise 131 Find :

(a) the angles of the triangle with vertices

$$A(1,1,1)\,, \quad B(3,-2,3)\,, \quad \text{and} \quad C(3,4,6).$$

(b) the area of the triangle with vertices

$$P_1(1,-2,3),\ P_2(-3,1,4), \quad \text{and} \quad P_3(0,4,3).$$

(c) the area of the triangle with vertices P_1, P_2, and P_3, where

$$\overrightarrow{P_1P_2} = 2\vec{i} + 3\vec{j} - \vec{k} \quad \text{and} \quad \overrightarrow{P_1P_3} = \vec{i} + 2\vec{j} + 2\vec{k}.$$

(d) the area of the parallelogram with adjacent sides

$$\vec{u} = \vec{i} + 3\vec{j} - 2\vec{k} \quad \text{and} \quad \vec{v} = 3\vec{i} - \vec{j} - \vec{k}.$$

(e) the volume of the parallelepiped with the vertex at the origin and edges

$$\vec{u} = 2\vec{i} - \vec{j}\,, \quad \vec{v} = \vec{i} - 2\vec{j} - 2\vec{k} \quad \text{and} \quad \vec{w} = 3\vec{i} - \vec{j} + \vec{k}.$$

Exercise 132 Establish the **triangle inequality**

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

(for all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$) by first squaring both sides and then using the Cauchy-Schwarz inequality (see **Exercise 137**).

Exercise 133 Find the vector equation and parametric equations for the line passing through the points $(2, 1, 8)$ and $(4, 4, 12)$.

Exercise 134 Show that the line through the points $(2, -1, -5)$ and $(8, 8, 7)$ is parallel to the line through the points $(4, 2, -6)$ and $(8, 8, 2)$.

Exercise 135 Find the equation of the line through the point $(0, 2, -1)$ which is parallel to the line

$$x = 1 + 2t\,, \quad y = 3t\,, \quad z = 5 - 7t.$$

Exercise 136 Find the equation of the plane through the points

$$A(1, 0, -3)\,, \quad B(0, -2, -4)\,, \quad \text{and} \quad C(4, 1, 6).$$

Exercise 137 Find the equation of the plane which passes through the point $(6, 5, -2)$ and is parallel to the plane $x + y - z = 5$.

Exercise 138 Find the equation of the plane which is perpendicular to the line segment joining the points $(-3, 2, 1)$ and $(9, 4, 3)$, and which passes through the midpoint of the line segment.

Exercise 139 Find the point of intersection of the planes

$$(\pi_1) \quad x + 2y - z = 6\,, \quad (\pi_2) \quad 2x - y + 3z + 13 = 0\,, \quad (\pi_3) \quad 3x - 2y + 3z = -16.$$

Exercise 140 Find the equation of the plane passing through the point $(-3,2,-4)$ and the line of intersection of the planes

$$(\alpha) \quad 3x+y-5z+7=0 \quad \text{and} \quad (\beta) \quad x-2y+4z=3.$$

Exercise 141 Find the shortest distance between the lines

$$(\mathcal{L}_1) \quad \vec{r} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad (\mathcal{L}_2) \quad \vec{r} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Exercise 142 Find the point of intersection of the lines

$$(\mathcal{L}_1) \quad \vec{r} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \quad \text{and} \quad (\mathcal{L}_2) \quad \vec{r} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

Exercise 143 Find the line of intersection of the planes

$$(\alpha) \quad x+y-z=2 \quad \text{and} \quad (\beta) \quad (2\vec{i}-\vec{k}) \bullet \vec{r} = 2.$$

Exercise 144 Find the line which passes through the origin and intersects the plane

$$x+y+2z=6$$

orthogonally.

Exercise 145

(a) Find the distance from the point $(0,2,4)$ to the plane

$$x+2y+2z=3.$$

(b) Show that the distance from the point $P_0(x_0,y_0,z_0)$ to the plane with equation $ax+by+cz+d=0$ is

$$\delta = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

(c) Use the formula above to show that the distance between the two parallel planes

$$(\alpha) \quad ax+by+cz=d \quad \text{and} \quad (\beta) \quad ax+by+cz=e$$

is

$$\delta = \frac{|d-e|}{\sqrt{a^2+b^2+c^2}}.$$